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FLOWS AND PERIODIC MOTIONS

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Essentially, this paper consists of the application of well-known fixed-point theorems, and others recently obtained in [4-6], to the existence problem of periodic solutipns in abstract flows.

Section 1 gives the necessary definitions. The main part is section 2. Here there appear, first, two rather general theorems, 9 and 10; it will be apparent that theorem 9, in some form or other, is well-known; theorem 10 was suggested in [7]. The remaining theorems 11,12,15,16 treat more special situations, possibly not covered by the preceding results. Section 3 then contains only notes and remarks, and its latter part may serve as a link between flow theory and dynamical system theory. Its presence at the conclusion of the paper was dictated by the wish not to intersperse the preliminaries to section 2 with details not absolutely necessary .

For integral $n > 0$, R^n denotes euclidean n-space, C^n its subset of points with non-negative integral coordinates, $(E^1)^\infty$ the Hilbert parallellepiped, S^n the n-sphere, all with their natural topology; the first two are also taken with their natural additive structure and partial order. P usually denotes a topological space; if triangulable, $\pi_q(P)$ is its q-th Betti number, and $\chi(P) = \Sigma(-1)^q \pi_q(P)$ its Euler characteristic. The composition of maps say f and g is usually denoted by $f \circ g$, so that $f \circ g(x) = f(g(x))$.

1. FLOWS.

<u>Convention 1</u>. A <u>semi-group</u> shall mean a topological quasi-ordered semi-group (in the usual sense) with unit element. (Also see section 3.)

Constructions such as "the semi-group R " will be preferred to the more correct (but, for our purposes, unnecessarily pedantic) "the semi-group $(R, +, \geqslant, t)$ " with R a set and $+$, $\geqslant$, t the semi-group, quasi-order and topology structures on R. In a similarly vague, but possibly obvious, sense we will say that a semi-group R is, e.g., a group, or is discrete; if the maximal relation on R is taken as the quasi-ordering (i.e. $\alpha \geqslant \beta$ always; this is indeed a quasi-order), then R will be termed <u>unordered.</u> Typical examples: R^n, C^n, R^n taken unordered. The unit of a semi-group R is often denoted by o, elements of R by lower-case Greek letters.

<u>Definition 2</u>. Let P be a topological space, R a semi-group. A <u>semi-flow</u> T <u>on</u> P <u>over</u> R is a mapping with properties 1^o – 3^o listed below.

1^o $T: \{[\alpha, \beta] \in R \times R : \alpha \geqslant \beta\} \times P \to P$ is continuous, in the induced topology. For fixed $\alpha \geqslant \beta$ in R, T defines a continuous map $P \to P$, standardly denoted as ${}_{\alpha}T_{\beta}$; using this notation we require further that

2^o ${}_{\alpha}T_{\alpha} = 1$, the identity map of P, for $\alpha \in R$,

3^o ${}_{\alpha}T_{\beta} \circ {}_{\beta}T_{\gamma} = {}_{\alpha}T_{\gamma}$ for $\alpha \geqslant \beta \geqslant \gamma$.

If R is unordered, T is called a <u>flow</u>.

Further terms: If R is discrete, T itself will be called <u>discrete</u>. If, for all $\alpha \geqslant \beta$, $\theta \geqslant o$,

$$ {}_{\alpha+\theta}T_{\beta+\theta} = {}_{\alpha}T_{\beta} \quad , $$

then the semi-flow T will be termed <u>stationary</u>. For fixed $x \in P$, T defines a continuous map $T_\sigma x : \{\alpha \in R : \alpha \geqslant \sigma\} \to P$, assuming the value ${}_\alpha T_\sigma x$ at $\alpha \geqslant \sigma$; this map will be called the <u>solution</u> (of T) through x.

Remark. Probably it is evident that a semi-flow is, essentially, a special type of covariant functor. Thus, let P, R be as in def. 3. Denote by $R^\wedge$ the category with objects $\alpha \in R$, morphisms $[\alpha, \beta] \in R \times R$ with $\alpha \geqslant \beta$, and composition

$$[\alpha, \beta]\,[\beta, \gamma] = [\alpha, \gamma]\,.$$

Let $P^\wedge$ be the category with P as sole object, and continuous maps $P \to P$ as morphisms. Then a semi-flow T on P over R defines a covariant $T^\wedge : R^\wedge \to P^\wedge$ by

$$T^\wedge[\alpha, \beta] = {}_\alpha T_\beta\ ;$$

conversely, a covariant functor $T^\wedge : R^\wedge \to P^\wedge$ similarly defines a discrete semi-flow on P over R (taken discrete).

<u>Example 3</u>. In a Banach space P, let

$$\frac{dx}{d\theta} = A(\theta)x \qquad (x \in P,\ \theta \in R^1)$$

be a (homogeneous linear) differential equation with $A(\theta)$ a linear bounded operator $P \to P$ depending continuously on θ.
For α, β in R^1 let $U(\alpha, \beta)$ be the corresponding resolvent operator [10, p.150]; then ${}_\alpha T_\beta = U(\alpha, \beta)$ defines a flow on P over R^1 (necessarily taken unordered).

Slightly more generally, let

(1) $$\frac{dx}{d\theta} = f(x, \theta)$$

be a differential equation with $f : P \times R^1 \to P$ continuous, and with global existence and unicity of solutions, and continuous dependence of solutions on initial data (if P is finite-dimensional, the latter condition follows from the preceding). Take any $x \in P$, $\alpha, \beta \in R^1$, and determine the unique solution y of (1) which satisfies $y(\beta) = x$; then set

$$ {}_{\alpha}T_{\beta}\, x = y(\alpha) . $$

Obviously this defines a flow on P over R^1; flows of this type may be termed <u>differential</u>. It may be noted that it is stationary iff f is independent of θ.

There are many other interesting and natural examples of flows, e.g. in ergodic theory (e.g.[9], or[2, chap.XVI]); (see also dynamical systems in section 3). However, example 3 is to be considered as the fundamental one for the purposes of the present paper.

<u>Lemma 4.</u> If T is a flow on P over R, then every ${}_{\alpha}T_{\beta}$ is a homeomorphism $P \approx P$ and

$$ {}_{\alpha}T_{\beta}^{-1} = {}_{\beta}T_{\alpha} . $$

(Proof: ${}_{\alpha}T_{\beta} \circ {}_{\beta}T_{\alpha} = {}_{\alpha}T_{\alpha} = 1$, ${}_{\beta}T_{\alpha} \circ {}_{\alpha}T_{\beta} = 1$.)

<u>Definition 5</u>. Let T be a semi-flow on P over R, and assume given a $\tau \geq \sigma$ in R. Then T is said to <u>admit</u> the period τ if, for all $\alpha \geq \sigma$,

$$ {}_{\alpha+\tau}T_{\sigma} = {}_{\alpha}T_{\sigma} \circ {}_{\tau}T_{\sigma} . \tag{2} $$

<u>Examples 6</u>. Every semi-flow admits the period σ. A stationary semi-flow T admits all periods $\tau \geq \sigma$:

$$ {}_{\alpha}T_{\sigma} \circ {}_{\tau}T_{\sigma} = {}_{\alpha+\tau}T_{\tau} \circ {}_{\tau}T_{\sigma} = {}_{\alpha+\tau}T_{\sigma} . $$

As a partial converse, a flow admitting all periods is stationary: using lemma 4,

$$ {}_{\alpha}T_{\beta} = {}_{\alpha}T_{\sigma}\, ({}_{\beta}T_{\sigma})^{-1} $$

for all α, β, so that

$$ {}_{\alpha+\theta}T_{\beta+\theta} = {}_{\alpha+\theta}T_{\sigma} \circ ({}_{\beta+\theta}T_{\sigma})^{-1} = {}_{\alpha}T_{\sigma} \circ {}_{\theta}T_{\sigma} \circ ({}_{\beta}T_{\sigma} \circ {}_{\theta}T_{\sigma})^{-1} = $$
$$ = {}_{\alpha}T_{\sigma} \circ {}_{\theta}T_{\sigma} \circ {}_{\sigma}T_{\theta} \circ {}_{\sigma}T_{\beta} = {}_{\alpha}T_{\beta} \ . $$

A differential flow (cf.(1), example 3) admits a period $\tau \geq \sigma$ iff, for each fixed $x \in P$, $f(x, \theta)$ has period τ in θ. (This may be proved by showing that the latter condition is equivalent to: $y(\theta+\tau)$ is a solution of (1) whenever $y(\theta)$ is.) In the first case of example 3 this is, of course, the familiar Floquet's theorem (e.g. [12,III,§ 2]).

Lemma 7. Let T be a semi-flow on P over R, admitting a period $\tau \geq \sigma$. Then T also admits all periods $n\tau$, $n \geq 0$ integral, and

$$ (3) \qquad {}_{n\tau}T_{\sigma} = {}_{\tau}T_{\sigma}^{\,n} \ . $$

If R is an unordered (topological) group, then this holds for all integers n without restriction.

(Proof.) Using (2) thrice one obtains

$$ {}_{\alpha+2\tau}T_{\sigma} = {}_{(\alpha+\tau)+\tau}T_{\sigma} = {}_{\alpha+\tau}T_{\sigma} \circ {}_{\tau}T_{\sigma} = {}_{\alpha}T_{\sigma} \circ {}_{\tau}T_{\sigma} \circ {}_{\tau}T_{\sigma} = {}_{\alpha}T_{\sigma} \circ {}_{2\tau}T_{\sigma} \ , $$

and by induction,

$$ (4) \qquad {}_{\alpha+n\tau}T_{\sigma} = {}_{\alpha}T_{\sigma} \circ {}_{n\tau}T_{\sigma} \qquad \text{for} \quad n \geq 0 \ . $$

Hence, for $\alpha = \tau$,

$$ {}_{(n+1)\tau}T_{\sigma} = {}_{\tau}T_{\sigma} \circ {}_{n\tau}T_{\sigma} \ , $$

so that, by induction, ${}_{n\tau}T_{\sigma} \circ {}_{\tau}T_{\sigma}^{\,n}$. with (4) this completes the proof of the first statement.

Finally, if R is an unordered group, then from (4)

$$ (5) \qquad {}_{\alpha}T_{\sigma} = {}_{(\alpha-n\tau)+n\tau}T_{\sigma} = {}_{\alpha-n\tau}T_{\sigma} \circ {}_{n\tau}T_{\sigma} \ , $$

and in particular ${}_{-n\tau}T_{\sigma} = {}_{n\tau}T_{\sigma}^{-1}$ (cf. lemma 4). Thus from (5),

$$ {}_{\alpha-n\tau}T_{\sigma} = {}_{\alpha}T_{\sigma} \circ {}_{n\tau}T_{\sigma}^{-1} = {}_{\alpha}T_{\sigma} \circ {}_{-n\tau}T_{\sigma} \ , $$

as was to be shown.

Remark. It may be remarked that for flows, property (3) is equivalent with stationarity of the "sampled" flow on P over C^1, defined by

$$ {}_{m\tau}T_{n\tau} \qquad (m \geq n \text{ in } C^1) . $$

2. PERIODIC SOLUTIONS.

Throughout this section P denotes a topological space and R a semi-group (cf. convention 1 and section 3).

As may be expected, a solution $T_\sigma x$ is called τ-*periodic* (T a semi-flow on P over R, $x \in P$, $\tau \geq \sigma$) if

$$ {}_{\theta+\tau}T_\sigma x = {}_\theta T_\sigma x \qquad \text{for all } \theta \geq \sigma . $$

(This is current usage if $R = R^1$ is taken unordered; if $R = R^1$ with natural order, the term is so used in Laplace transform theory.) Obviously, a τ-periodic solution is $n\tau$-periodic for all integers $n \geq 0$.

The main tool used to obtain conditions for existence of periodic solutions is the following

Proposition 8. Let T be a semi-flow on P over R admitting a period $\tau \geq \sigma$. For $x \in P$, the solution $T_\sigma x$ is τ-periodic iff x is a fixed point of ${}_\tau T_\sigma : P \to P$.

(Proof.) This is direct verification. If $T_\sigma x$ is τ-periodic, then ${}_{\theta+\tau}T_\sigma x = {}_\theta T_\sigma x$ for all $\theta \geq \sigma$; for $\theta = \sigma$ one obtains ${}_\tau T_\sigma x = x$, a fixed point of ${}_\tau T_\sigma$. Conversely, if ${}_\tau T_\sigma x = x$, then

$$ {}_{\theta+\tau}T_\sigma x = {}_\theta T_\sigma \circ {}_\tau T_\sigma x = {}_\theta T_\sigma x , $$

i.e., $T_\sigma x$ is τ-periodic.

Proposition 8 will be applied, without further reference, in reading off existence of periodic solutions from various fixed-point theorems. In each pair of theorems 9-10, 11-12, 15-16 there appear similar results under varied assumptions on the semi-flow and its carrier space.

Theorem 9. Let T be a semi-flow on P over R, admitting a period $\tau \geqslant \sigma$. If there exists an $X \subset P$ which is a retract of $(E^1)^\infty$ and has ${}_\tau T_\sigma X \subset X$, then there exists a τ-periodic solution.

(Proof.) Partialised ${}_\tau T_\sigma : X \to X$ is continuous; apply the Schauder-Tichonov fixed-point theorem [11, p.263].

Note that the conclusion obtains, in particular, if P itself is a retract of $(E^1)^\infty$.

Theorem 10. Let T be a semi-flow on P over R, admitting a period $\tau \geqslant 0$; assume that P is triangulable with $\chi(P) \neq 0$, and that $\{\theta \in R : \theta \geqslant \sigma\}$ is connected.

(Proof.) Denote by $J(f)$ the Lefschetz invariant of a continuous map $f : P \to P$ (cf.[1, p.598], or [4]). By assumption, ${}_\theta T_\sigma$ depends continuously on $\theta \geqslant \sigma$; from [4, lemma 7] it then follows that $J({}_\theta T_\sigma)$ also varies continously with θ. Since $J(f)$ is integer-valued and $\{\theta \in R : \theta \geqslant \sigma\}$ connected, $J({}_\theta T_\sigma)$ is constant. Therefore

$$J({}_\theta T_\sigma) = J({}_\sigma T_\sigma) = J(1) = \chi(P) \neq 0 .$$

By the Lefschetz-Hopf fixed-point theorem, there exists a τ-periodic solution of T.

Remark. Theorem 10 applies a fortiori if R is arcwise connected, e.g. for $R = R^1$. In this case the proof may be simplified, omitting all reference to [4] and lemma 17, as follows: use the assumed path from σ to τ in R to show that ${}_\sigma T_\sigma = 1$ is homotopic to ${}_\tau T_\sigma$; then again $J({}_\tau T_\sigma) = J(1) \neq 0$. This was the idea of [7, theorem].

Theorem 11. Let T be a flow on P over R, admitting a period $\tau \geqslant \sigma$; and assume that P is triangulable with

$1 \leq n \leq \sum \pi_q (P)$.

(Proof.) From lemma 4, ${}_{\tau}T_{\sigma}$ is now a homeomorphism $P \approx P$; apply corollary 5 of [6].

<u>Theorem 12</u>. Let T be a semi-flow on P over R , admitting a period $\tau \geq \sigma$; assume that $P \neq \emptyset$ is non-odd. Then there exists an $n\tau$ -periodic solution with $1 \leq n \leq$ $\leq \sum \pi_q (P) = \chi (P)$.

(Proof: [5, theorem 2].)

Remark. Non-oddness is a concept introduced in [5, definition] : P is non-odd if $\pi_{2q+1} (P) = 0$ for all q , i.e. if all odd-dimensional homology groups are periodic. In particular, then, each semi-flow on S^{2n} admitting a period $\tau \geq \sigma \in R$ has a 2τ -periodic solution.

Before presenting the next two theorems, it will be necessary to introduce and illustrate another concept. A continuous map $F : P \to P$ will be termed a <u>symmetry</u> of P if $F^2 = 1$; necessarily, then, $F : P \approx P$ homeomorphically.

<u>Definition 13</u>. Let F be a symmetry of P , and T a semi-flow on P over R . Then T will be termed F <u>-symmetric</u> if each ${}_{\alpha}T_{\beta}$ commutes with F .

<u>Example 14</u>. Let T be a differential flow on a Banach space P over R^1, defined by a differential equation (1) as in example 3 . Also, let F be a linear symmetry of P. Then T is F -symmetric iff $F f (x, \theta) = f (F x , \theta)$ for all $x \in P$, $\theta \in R^1$ (hint: show that Fy is a solution of (1) iff y is). E.g. the flow described by $dx / d\theta = A(\theta) x$ is F -symmetric for F defined by $Fx = -x$.

Physical systems with n degrees of freedom are often described by differential equations such as

$$\frac{d^2x}{d\theta^2} = f(x, \frac{dx}{d\theta}, \theta) \qquad (x \in R^n, \theta \in R^1).$$

These may be "reduced" to systems of type (1) by a familiar device,

(6) $$\frac{dx}{d\theta} = p, \quad \frac{dp}{d\theta} = f(x, p, \theta)$$

with $[x, p] \in R^{2n}$. If, as sometimes happen,

$$f(-x, p, \theta) = -f(x, p, \theta), \quad ([x, p, \theta] \in R^{2n+1})$$

then (under the appropriate conditions on f) (6) defines a flow on R^{2n} over R^1; this flow is then F-symmetric for F defined by

$$F[x, p] = [-x, p].$$

<u>Theorem 15</u>. Let F be a symmetry of P, and T an F-symmetric semi-flow on P over R, admitting a period $\tau \geq \sigma$. If there exists a subset $X \subset P$ with X a retract of $(E^1)^\infty$ and

(7) $${}_{\tau}T_{\sigma} X \subset FX,$$

then there exists a 2τ-periodic solution of T.

(Proof.) Recall that $F = F^{-1}$. Partialised $F \circ {}_{\tau}T_{\sigma} : X \to X$, so that (Schauder-Tichonov) there is a fixed point x of $F \circ {}_{\tau}T_{\sigma}$. Then also ${}_{\tau}T_{\sigma} x = Fx$, and, using lemma 7 and F-symmetry,

$${}_{2\tau}T_{\sigma} x = {}_{\tau}T_{\sigma} \circ {}_{\tau}T_{\sigma} x = {}_{\tau}T_{\sigma} \circ Fx = F \circ {}_{\tau}T_{\sigma} x = x$$

is a fixed point of ${}_{2\tau}T_{\sigma}$.

Remarks. This is an abstract form of the Poincaré symmetry principle for dynamical systems in R^2 [12, p.145]. Obviously (7) is satisfied if $X = P$, i.e. if P itself is a retract of $(E^1)^\infty$.

<u>Theorem 16</u>. Let F be a symmetry of S^{2n}, T an

F-symmetric semi-flow on S^{2n} over R, admitting a period $\tau \geqslant \sigma$. If F has no fixed-point, then T has at least two 2τ-periodic solutions.

(Proof.) From theorem 11, T has at least one 2τ-periodic solution. These are in $1-1$ correspondence with the fixed points x of ${}_{2\tau}T_\sigma$; from F-symmetry there then follows ${}_{2\tau}T_\sigma \cdot Fx = Fx$, so that $Fx = x$ if there is only one 2τ-periodic solution. This contradicts the assumption on F and concludes the proof.

Remarks. The assertion may also be formulated thus: either there is at least one non-constant 2τ-periodic solution, or there are at least two constant solutions. In the case that F is a negative symmetry (i.e. degree $F = -1$), the existence of one 2τ-periodic solution also follows from [5, theorem 3].

3. ADDENDA.

For definiteness in convention 1, a semi-group means some

$$(R , + , \geq , t)$$

where R is a set and $+, \geqslant, t$ are structures as follows.

The $+$ is a semi-group operator, i.e. a binary associative operator on R ; there exists a unit $\sigma \in R$ ($\alpha + \sigma = \sigma + \alpha = \alpha$ always). For integral $n > 0$ and $\alpha \in R$ we write

$$n\alpha = \alpha + \alpha + \cdots + \alpha \ (n \text{ terms}), \quad 0\alpha = \sigma .$$

The $\geqslant$ is a quasi-order in R , i.e. a reflexive and transitive relation (laxly speaking, a partial ordering less the anti-symmetry condition [2,I,§ 4]). The advantage is that a single formulation serves for both the interesting cases, of $\geqslant$ a partial order, and also of $\geqslant$ the maximal relation on R ($\alpha \geqslant \beta$ always); in the latter case the semi-group was termed unordered. Lastly, t is a topology on R.

We require, further, these compatibility conditions:

(i) $\alpha \geqslant \beta$ and $\alpha' \geqslant \beta'$ implies $\alpha + \alpha' \geqslant \beta + \beta'$;

(ii) $+$ is continuous, considered as a map $R \times R \to R$ (in the induced topology);

(iii) the set $\{[\alpha, \beta] : \alpha \geqslant \beta\}$ is closed in $R \times R$.

Since exchange of coordinates is a homeomorphism of $R \times R$, $\{[\alpha, \beta] : \beta \geqslant \alpha\}$ is also closed.

Lemma 17. Let R be a partially ordered semi-group. Then

1° R is a Hausdorff space,

2° if R is connedted and $\{\theta : \theta > o\}$ open, then $R_+ = \{\theta : \theta \geqslant \sigma\}$ is connected;

3° for $\alpha \geqslant \sigma$ the set $\{n\alpha\}_{n \in C^+}$ is discrete.

(Proof.) The diagonal in $R \times R$ is the intersection of closed sets

$$\{[\alpha, \beta] : \alpha \geqslant \beta\}, \quad \{[\alpha, \beta] : \beta \geqslant \alpha\},$$

and hence is also closed. Thus the Bourbaki conditbn is satisfied and one has 1° (cf. theorem 13 in [2, chap.IV]).

Next, assume R_+ is not connected. Since it is closed, as a section of $\{[\alpha, \beta] : \alpha \geqslant \beta\}$ over σ, there exists a non-trivial decomposition into closed sets,

$$R_+ = A \cup B, \qquad \sigma \in B.$$

Set $C = R - R_+$, so that one has the decomposition

$$R = A \cup (B \cup C).$$

As R is connected, to obtain a contradiction it suffices to show that $A \cap \overline{C} = \emptyset$. Assume $\gamma_i \in C$, $\gamma_i \to \gamma \in A$. Since

$\sigma \in B$, $\gamma > 0$ and hence is in the open set $\{\theta : \theta > \sigma\}$; then $\gamma_i > \sigma$ for some i, contradicting $\gamma_i \in C \subset R - R_+$. This proves 2^o.

For 3^o, assume $k_n \to +\infty$, $k_n\alpha \to \kappa\alpha$ with k_n, $\kappa \in C^+$, $\alpha \geq \sigma$. Take any $s \geq \kappa$; then $k_n\alpha \geq s\alpha$ for large n, and hence

$$\kappa\alpha \leftarrow k_n\alpha \geq s\alpha \geq \kappa\alpha .$$

Therefore $s\alpha = \kappa\alpha$ for all $s \geq \kappa$, and $\{n\alpha\}_{n \in C^+}$ is discrete.

Definition 18. Let P be a topological space, R a semi-group. A continuous map $\tau : P \times \{\theta \in R : \theta \geq \sigma\} \to P$ (to be written as a binary operator) with the properties

$$x \tau \sigma = x , \qquad (x \tau \theta) \tau \theta' = x \tau (\theta' + \theta)$$

(for all $x \in P$, $\theta \geq \sigma \leq \theta'$) is called a semi-dynamical system on P over R; and, if R is unordered, a dynamical system on P over R. (For the case $R = R^1$ see "unilateral" in [7], and "global semi-dynamical" in [8].)

Lemma 19. A stationary (semi-) flow T defines a (semi-) dynamical system τ (both on P over R) by

$$x \tau \theta = {}_{\theta}T_{\sigma} x \quad \text{for } x \in P, \quad \theta \geq \sigma .$$

If R is a group then every (semi-) dynamical system τ defines a stationary (semi-) flow T, both on P over R, by

$${}_{\alpha}T_{\beta} x = x \tau (\alpha - \beta) \quad \text{for } x \in P, \quad \alpha \geq \beta .$$

(Proof: direct verification).

On passing to a different space, even non-stationary flows may be described in terms of dynamical systems:

Lemma 20. If T is a (semi-) flow on P over R, then

(8) $$[x, \alpha] \tau \theta = [{}_{\theta+\alpha}T_{\alpha} x, \ \theta + \alpha]$$

($x \in P, \alpha \in R, \theta \geq \sigma$ defines a (semi-) dynamical system τ

on $P \times R$ over R ; the solution ${}_\theta T_\sigma x$ is then the projection of $[x, \sigma] \tau \theta$.

(Proof: direct verification)

In this connection, P is sometimes called the <u>phase space</u> of T, and $P \times R$ its <u>solution space</u>. The semi-dynamical system defined by (θ) is somewhat singular; thus, if $R = R^1$ then there are no critical points nor cycles (in fact, $P \times (0)$ is a section generating $P \times R^1$).

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