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THE RÔLE OF THE "FINITE CHARACTER PROPERTY" IN THE THEORY OF DEPENDENCE

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The purpose of this little note is to show some consequences of omitting the "finite character" axiom in an axiomatic dependence scheme. The note originated as a remark to one of Prof. R. Rado's problems mentioned in his lecture in the Conference on General Algebra in Warsaw, September 7-11,1964.

In order to avoid references to other papers we introduce, briefly, the basic concepts (in terms of the relation "an element depends on a set"). Let S be a set, $\mathfrak{P}S$ its power-set and $\rho \subseteq S \times \mathfrak{P}S$ a relation between elements and subsets of S. A subset $I \subseteq S$ is said to be ρ-independent if $[x, I \setminus (x)] \notin \rho$ for every $x \in I$; the family of all ρ-independent sets will be denoted by $\mathcal{I}_\rho$ ($\emptyset \in \mathcal{I}_\rho$ for any ρ). A relation ρ is called the dependence relation on S if it satisfies the following properties:

(I) $x \in X \rightarrow [x, X] \in \rho$ (incidence);

(E) $[x, X] \notin \rho \wedge [x, X \cup (y)] \in \rho \rightarrow [y, X \cup (x)] \in \rho$ (exchange);

(T) $[x, Y] \in \rho \wedge \forall y\, (y \in Y \rightarrow [y, X] \in \rho) \rightarrow [x, X] \in \rho$ (transitivity).

Let us remark that the property (T) together with (I) imply the following property (M) of a relation ρ

(M) $[x, Y] \in \rho \wedge Y \subseteq X \rightarrow [x, X] \in \rho$ (monotony).

Denote further by (E_k) and (T_k) the properties (E) and (T), respectively, restricted on $X \in \mathcal{I}_\rho$ and $Y \in \mathcal{I}_\rho$.

The following simple example of

$S_1 = (a, b, c)$ with $\rho_1 = (S_1 \times \mathcal{P} S_1) \setminus ([a,(b,c)],[b,(a,c)],[c,(a,b)])$ establishes the logical independence of (M) on (I), (E_k) and (T_k).

In paper [1], we have shown that all maximal ρ-independent sets (i.e. maximal elements of $\mathcal{J}_\rho$) have the same cardinality (the rank of S) if the relation ρ satisfies (I), (E_k), (T_k), (M) and

(F) $[x, X] \in \rho \rightarrow \exists F (F \subseteq X \wedge F$ finite $\wedge [x, F] \in \rho)$ (finite character) (i.e. ρ is a particular type of a GA-dependence relation introduced there). The main result of the present note reads that the same conclusion does not hold for a dependence relation ρ defined above. As a matter of fact, in this formulation the latter statement would be trivial; for, (I), (E) and (T) do not assure the existence of maximal elemets in $\mathcal{J}_\rho$ (this is a consequence of (F)), and the following example shows that no such elements may (in general) exist:

If S_2 is an infinite set and ρ_2 is defined by

$$[x, X] \in \rho_2 \leftrightarrow x \in X \quad \text{or} \quad X \quad \text{infinite},$$

then ρ_2 clearly satisfies (I), (E) and (T), and $\mathcal{J}_{\rho_2}$ being the family of all finite numbers of S_2 has no maximal elements.

To avoid this ambiguity in what follows we shall consider a dependence structure (S, ρ) as a pair of a set S and a dependence relation ρ with an additional property of

(B) $\mathcal{J}_\rho$ *has maximal elements* .

The main result reads then as follows.

<u>Theorem 1</u>. Let (S, ρ) be a dependence structure.

(i) If a maximal ρ-independent set is finite, then all are finite and have the same number of elements.

(ii) If a maximal ρ -independent set is infinite, then all are infinite.

It is evident that (ii) follows immediately from (i). The assertion (i) is the a consequence of the following two lemmas.

Lemma 1. Let ρ be a relation on S satisfying (I), (E_κ), (T_κ) and (M). Let M_1 and M_2 be two maximal ρ -independent sets and M_1 be finite. Then M_2 is finite, too.

Proof. Suppose, on the contrary, that M_2 is not finite. Let

$M_1 = (z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n)$, where $(z_1, z_2, \dots, z_m) = M_1 \cap M_2$; evidently $n \geq 1$. Let us choose n elements of $M_2 \setminus M_1$ and denote by M_2' the (infinite) set of all remaining elements of $M_2 \setminus M_1$:

$$M_2 = (z_1, z_2, \dots, z_m, y_1, y_2, \dots, y_n) \cup M_2' .$$

Since $M_2 \setminus (y_1)$ is no longer maximal (however, in view of (M), it is ρ -independent), there is an element $x_{i_1} \in M_1$ such that

$$[x_{i_1}, M_2 \setminus (y_1)] \notin \rho ;$$

for, otherwise

$[y_1, M_1] \in \rho$ and $\forall x_i (x_i \in M_2 \to [x_i, M_2 \setminus (y_1)] \in \rho$

would, in view of (T_κ), imply $[y_1, M_2 \setminus (y_1)] \in \rho$, a contradiction. Using (E_κ) together with (M), we can easily verify that

$$M_{21} = (z_1, z_2, \dots, z_m, x_{i_1}, y_2, \dots, y_n) \cup M_2' \in \mathcal{J}_\rho .$$

Now, there is another element $x_{i_2} \in M_1$ such that

$$[x_{i_2}, M_{21} \setminus (y_2)] \notin \rho ;$$

this follows again from the fact that M_1 is maximal (and hence, $[y_2, M_1] \in \rho$). Thus

$$M_2 = (z_1, z_2, \dots, z_m, x_{i_1}, x_{i_2}, \dots, x_{i_n}, y_3, \dots, y_n) \cup M_2' \in \mathcal{J}_\rho .$$

Proceeding in this manner we reach in n steps the following ρ -independent set

$$M_{2n} = (z_1, z_2, \dots, z_m, x_{i_1}, x_{i_2}, \dots, x_{i_n}) \cup M_2' = M_1 \cup M_2' .$$

Hence, we get a contradiction of the maximality of M_1. The proof of Lemma 1 is completed.

The latter proof can be readily extended to finite sets M_1 and M_2 and we get thus

Lemma 2. Let ρ be a relation on S satisfying (I), (E_k), (T_k) and (M). If M_1 and M_2 are two finite maximal ρ-independent sets, then they have the same number of elements.

P r o o f . Since both

$$card(M_1) \geq card(M_2) \quad \text{and} \quad card(M_1) \leq card(M_2),$$

Lemma 2 immediately follows.

The following theorem shows that (ii) of Theorem 1 cannot be strengthened.

Theorem 2. Let $(\alpha_\gamma)_{\gamma \in \Gamma}$ be a family of infinite cardinal numbers. Then there exists a dependence structure with a family $(M_\gamma)_{\gamma \in \Gamma}$ of maximal independent sets such that

$$card(M_\gamma) = \alpha_\gamma \quad \text{for each } \gamma \in \Gamma .$$

P r o o f . Consider a family $(S_\gamma)_{\gamma \in \Gamma}$ of mutually disjoint sets such that

$$card(S_\gamma) = \alpha_\gamma \text{ for each } \gamma \in \Gamma ,$$

and denote by S_0 the union of these sets $S_0 = \bigcup_{\gamma \in \Gamma} S$. Define the relation $\rho_0 \subseteq S_0 \times \mathcal{P} S_0$ on S_0 in the following way: For $x \in S_0$ and $X \subseteq S_0$,

$(*)$ $[x, X] \in \rho_0 \leftrightarrow x \in X$ or, for a certain $\gamma_0 \in \Gamma$,

$X = (S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}$, where $F_{\gamma_0} \subseteq S_{\gamma_0}$ is finite,

$A_{\gamma_0} \subseteq \bigcup_{\substack{\gamma \in \Gamma \\ \gamma \neq \gamma_0}} S_\gamma$ and $card(A_{\gamma_0}) \geq card(F_{\gamma_0})$.

It can be easily seen that, besides (I), also (T) is satisfied by this relation ρ_o . Now, prove the validity of (E) for ρ_o . Thus, let $x \in S_o$, $y \in S_o$ and $X \subseteq S_o$ be such that

$(**)$ $[x, X] \notin \rho_o$ and $[x, X \cup (y)] \in \rho_o$.

Then, $x \notin X$. The conclusion $[y, X \cup (x)] \in \rho_o$ is trivial for $x = y$; suppose, therefore, that $x \neq y$. The assumption $(**)$ implies that

$X \cup (y) = (S_{\gamma_o} \setminus F_{\gamma_o}) \cup A_{\gamma_o}$ with $card\,(F_{\gamma_o}) = card\,(A_{\gamma_o}) < \aleph_o$.

for a suitable $\gamma_o \in \Gamma$. We have to consider four (in fact, very similar) cases:

(i) $y \in S_{\gamma_o}$, $x \in S_{\gamma_o}$,i.e. $y \in S_{\gamma_o} \setminus F_{\gamma_o}$, $x \in F_{\gamma_o}$; then, evidently, $[y, (S_{\gamma_o} \setminus [(F_{\gamma_o} \cup (y)) \setminus (x)]) \cup A_{\gamma_o}] \in \rho_o$;

(ii) $y \in S_{\gamma_o}$, $x \in S_{\gamma_o}$,i.e. $y \in S_{\gamma_o} \setminus F_{\gamma_o}$, $x \notin S_{\gamma_o} \cup A_{\gamma_o}$; then,

$[y, (S_{\gamma_o} \setminus [F_{\gamma_o} \cup (y)]) \cup A_{\gamma_o} \cup (x)] \in \rho_o$;

(iii) $y \notin S_{\gamma_o}$, $x \notin S_{\gamma_o}$, i.e. $y \in A_{\gamma_o}$, $x \in F_{\gamma_o}$; then,

$[y, (S_{\gamma_o} \setminus [F_{\gamma_o} \setminus (x)]) \cup (A_{\gamma_o} \setminus (y))] \in \rho_o$;

(iv) $y \notin S_{\gamma_o}$, $x \notin S_{\gamma_o}$, i.e. $y \in A_{\gamma_o}$, $x \notin S_{\gamma_o} \cup A_{\gamma_o}$; then,

$[y, (S_{\gamma_o} \setminus F_{\gamma_o}) \cup ([A_{\gamma_o} \setminus (y)] \setminus (x))] \in \rho_o$.

Thus, (E) holds for ρ_o .

Moreover, since, for any element $x \in S_\gamma$, $S_\gamma \setminus (x)$ is not of the form described in $(*)$, S_γ is ρ_o -independent for each $\gamma \in \Gamma$. Also, $S_\gamma = (S_\gamma \setminus \emptyset) \cup \emptyset$ is maximal for each $\gamma \in \Gamma$, hence, the last condition (B) is satisfied for ρ_o and, thus, (S_o , ρ_o) is a dependence structure (in the sense of this note).

This completes the proof, for the existence of maximal ρ_o -independent sets with prescribed cardinalities has also been established (take e.g. $M_\gamma = S_\gamma$).

R e m a r k . As a matter of fact, referring back to the dependence structure (S_o , ρ_o) constructed in the proof of

Theorem 2, all sets of the form

(* * *) $(S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}$ with $A_{\gamma_0} \cap S_{\gamma_0} = \emptyset$ and $card(F_{\gamma_0}) = card(A_{\gamma_0}) < \aleph_0$ are maximal and ρ_0-independent. Evidently,

$$card((S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}) = card(S_{\gamma_0}) = \alpha_{\gamma_0}$$

On the other hand, any maximal ρ_0-independent set of this structure is of the form (* * *). For, any maximal set must necessarily be of the form (*) and any maximal ρ_0-independent set must, moreover, satisfy the last condition on cardinalities in (* * *). Thus, the cardinality of an arbitrary maximal ρ_0-independent set of (S_0, ρ_0) is equal to one of the numbers

Finally, let us remark that the maximal ρ_0-independent sets of (S_0, ρ_0) satisfy also the conditions denoted in [2] by $(\tilde{B}'_{2f})$ and $(\tilde{B}_{2f})$:

(B'_{2f}) For any two maximal independent sets M_1 and M_2 and any finite subset $M'_1 \subseteq M_1 \setminus M_2$ there exists a subset $M'_2 \subseteq M_2 \setminus M_1$ of the same number of elements such that $(M_1 \setminus M'_1) \cup M'_2$ is a maximal independent set.

$(\tilde{B}'_{2f})$ For any two maximal independent sets M_1 and M_2 and any finite subset $M'_1 \subseteq M_1 \setminus M_2$ there exists a subset $M'_2 \subseteq M_2 \setminus M_1$ of the same number of elements such that $M'_1 \cup (M_2 \setminus M'_2)$ is a maximal independent set.

Both properties suffice to be proved for single-point subsets M'_1 and M'_2 (the properties (B'_2) and $(\tilde{B}'_2)$ in [2]); the proof involves several simple cases to be considered and is left to the reader. Thus, the example of the dependence structure (S_0, ρ_0) in the proof of Theorem 2 shows that the assumption of the finite character property

(B_3) If every finite subset of a set X is a subset of a suitable maximal independent set, then X is a subset of a maximal independent set. was essential in § 5 of [2].

In order to show also the logical independence of (B_3) on the stronger properties (B'_{2q}) and $(\tilde{B}'_{2q})$ of [2], consider the following simple example (S_*, ρ_*) of a dependence structure:

$$S_* = S_1 \cup S_2 \quad \text{with } S_1 \cap S_2 = \emptyset,\ card(S_1) \geq \aleph_0,\ card(S_2) > \aleph_0 \cdot$$

and

$$[x, X] \in \rho_* \longleftrightarrow x \in X \quad \text{or} \quad card(X \cap S_2) \geq \aleph_0 \quad \text{or}$$

$$X = (S_1 \setminus F_1) \cup A_2 \text{ with } F_1 \text{ finite and } card(F_1) \leq card(A_2).$$

It is a matter of routine to check that ρ_* satisfies (I), (E) and (T), that all maximal ρ_*-independent sets are of the form

$$M = (S_1 \setminus F_1) \cup A_2 \quad \text{with } card(F_1) = card(A_2) < \aleph_0$$

and that they satisfy the properties (B'_{2q}) and $(\tilde{B}'_{2q})$ (which reduce to (B'_{2f}) and $(\tilde{B}'_{2f})$, respectively). All maximal ρ_*-independent sets have thus the same cardinality $(= card(S_1))$ – a fact which follows, in general, from the property (B'_{2q}). However, it turns out that (B_3) is not fulfilled:

Let T be a countable subset of S_2 and $(F_{1n})_{n \geq 1}$ a family of subsets of S_1 such that

$$card(F_{1n}) = n \quad \text{for every } n \geq 1 \cdot$$

Then, for any finite subset F_2 of T there is a natral number n (the number of elements of F_2) such that

$$(S_1 \setminus F_{1n}) \cup F_2$$

is a maximal ρ_*-independent set. But, there is no maximal

ρ_* -independent set containing the set T.

References

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