Otomar Hájek
Critical points of abstract dynamical systems

Commentationes Mathematicae Universitatis Carolinae, Vol. 5 (1964), No. 3, 121--124

Persistent URL: http://dml.cz/dmlcz/104966

Commentationes Mathematicae Universitatis Carolinae
5, 3 (1964)

CRITICAL POINTS OF ABSTRACT DYNAMICAL SYSTEMS

O. H Á J E K, Praha

Existence of critical points of local dynamical systems on triangulable spaces with nonvanishing Euler characteristic.

In a classical paper, Poincaré established the existence of critical points in analytical dynamical systems on closed surfaces with genus $\neq 1$ (i.e. with Euler characteristic $\neq 0$). The proof proceeded by the "index method", concluding with the proposition that the Euler characteristic equals the sum of indices of essential critical points. Later, Stiefel [4] showed that a differentiable manifold has the property that every vector field on it has at least one singularity, if and only if its Euler characteristic is nonzero. This applies directly to differential dynamical systems of class C^1 on such manifolds. In the present paper, abstract dynamical systems are considered; for these, there is no vector field available, and one is forced to adopt other methods; also, it seems natural to drop the differentiability condition on the manifold.

A <u>local dynamical system</u> on a topological space T is a continuous mapping τ from a subset of $T \times E^1$ to T with properties $1^o - 3^o$ listed below (E^1 is euclidean 1-space; the value of τ at $(x, \theta) \in T \times E^1$ is denoted by $x \tau \theta$)

1^o $x \tau 0 = x$,

2^o $(x \tau \theta_1) \tau \theta_2 = x \tau (\theta_1 + \theta_2)$ if $x \tau \theta_1$ and either side is defined,

3^o domain τ is open.

Axiom 1° may be replaced by the formally weaker requirement that τ maps onto T ; this follows from 2° easily. From 1° and 3° , to every $x \in T$ there is a neighborhood U of x and an open interval $I \subset E^1$ containing 0 , such that τ is defined in $U \times I$.

If 3° is replaced by domain $\tau = T \times E^1$, the local system is called an <u>abstract dynamical system</u> (cf. [3, p.346],[2]; this notion is not new, the remaining are). If 3° is weakened to domain τ is open in $T \times \langle 0, +\infty)$, the resulting object T, τ may be termed a <u>unilateral</u> local system. The obvious interpretations of these notions are as follows:

(i) abstract dynamical systems - autonomous systems of differential equations with unicity and prolongability of solutions,

(ii) local dynamical systems - autonomous systems of diff. equations with local existence and unicity of solutions,

(iii) unilateral local systems - systems as in (ii), but considered only on (positively) invariant sets,i.e. on subsets of T mapped into themselves by the positive semitrajectories.

Henceforth, assume given a local dynamical system τ on a space T . An $x_0 \in T$ will be said to have period $\theta_0 \in E^1$ if $x_0 = x_0 \tau \theta_0$; obviously then it also has periods $k\theta_0$ for all integers k . The notion of critical points is related to this: $x_0 \in T$ is called <u>critical</u> if $x_0 = x_0 \tau \theta$ for all $\theta \in E^1$. (The modification of these definitions for unilateral systems is perhaps obvious.) There is a useful variation of the conditions for criticality:

<u>Lemma 1</u>. If $x_0 \in T$ has arbitrarily small nonzero periods, it is critical.

Proof. This is quite standard. Assume $x_0 = x_0 \tau \theta_n$, $0 < \theta_n \to \to 0$. Take any $\theta > 0$ such that $x_0 \tau \lambda$ is defined for $0 \leq \lambda \leq \theta$. Define inductively integers $k_1, k_2, \dots$ as follows: let k_n be the integral part of

$$\frac{1}{\theta_n}\left(\theta - \sum_1^{n-1} k_i \theta_i\right)$$

(with $\sum_1^0 = 0$), and set $\lambda_n = \sum_1^n k_i \theta_i$, the n-th partial sum. Then

$$0 \leq \theta - \lambda_n < \theta_n \to 0 ,$$

and since λ_n is an integral combination of periods, $x_0 = x_0 \tau \lambda_n$. By continuity, therefore, $x_0 = x_0 \tau \theta$, for all the θ described above. However, from 3^o we have that $x_0 = x_0 \tau \theta = (x_0 \tau \theta) \tau 0 = x_0 \tau 2\theta$ is defined, and thus so is $x_0 \tau 3\theta$, etc. Thus, finally, $x_0 = x_0 \tau \theta$ for all θ , as was to be proved.

Lemma 2. Assume that either T is compact or τ is an abstract dynamical system; define $f_\theta : T \to T$ by $f_\theta x = x \tau \theta$. Then f_θ is homotopic to the identity map of T .

Proof. This is trivial: the homotopy is $f_{\lambda\theta}$, $0 \leq \lambda \leq 1$, and f_0 is the identity by 1^o . In case τ is local and T compact, we shall only show that f_θ is defined for small θ' s .

$T \times (0)$ is compact and covered by sets $U \times I$, where U is open in T and $0 \in I$ is an open interval, and τ is defined in $U \times I$. Take a finite cover $U_i \times I_i$; then τ is defined in $\bigcup U_i \times I_i \supset (\bigcup U_i) \times \bigcap I_i = T \times J$ where J is an open interval containing 0 ; then, finally, f_θ is defined for $\theta \in J$.

Theorem. Let τ be a local dynamical system in a triangulable space T with Euler characteristic $\chi(T) \neq 0$. Then τ

has a critical point.

Proof. Define maps $f_\theta: T \to T$ as in lemma 2, for say $0 < \theta < < \varepsilon$. Since $\chi(T) \neq 0$ and f_θ is homotopic to the identity, f_θ has a fixed point [1, ch.XVII, 1-43]. Thus for $\theta = 2^{-n}$ there exist $x_n \in T$ with

$$x_n = x_n \tau\, 2^{-n} .$$

Take any integer $m > 0$. For $n \geq m$, $2^{-m} = 2^{n-m} . 2^n$ is an integral multiple of the period 2^{-n} of x_n, so that

$$x_n = x_n \tau\, 2^{-m}.$$

Now T is compact, so that there is accumulation point x_0 of the x_n's ; and then by continuity

$$x_0 = x_0 \tau\, 2^{-m} .$$

Thus x_0 has arbitrarily small periods 2^{-m}; from lemma 1, x_0 is critical.

Corollary. Let τ be a local dynamical system on S^2 ; then there is at least one critical point. If the system has a cycle C , there is at least one critical point in both components of $S^2 - C$.

The theorem also holds for unilateral systems.

R e f e r e n c e s :

[1] ALEXANDROV P.S., Combinatorial Topology (in Russian), Gostechizdat, Moscow-Leningrad, 1949.

[2] HÁJEK O., Sections of dynamical systems in E^2, to appear in Czech.Math.Journ.

[3] NIEMYCKIĬ V.V., STĚPANOV V.V., Qualitative theory of differential equations (in Russian), 2nd ed., Gostechizdat, Moscow-Leningrad, 195 .

[4] STIEFEL E., Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, Comment. Math.Helv.,8(1936), 3-51.