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GENERALIZATION OF STEFFENSEN'S METHOD FOR OPERATOR EQUATIONS IN BANACH SPACE

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1. Introduction.

In this paper the Steffensen's method of solution of non-linear equations ([1], Appendix 5) is generalized for solution of non-linear equations in Banach space. Here I use the Schmidt's concept of the divided difference, introduced in [2(I)]; partly, I have made use of this work of his in methodological respect (in particular, paragraph 4), too.

Steffensen's method is an iterative method based on alternate performance of one step of the succesive approximation and one step of the method regula falsi. If we denote the initial approximation by x_0, then the iterative formula for the calculation of the roots of the equation $x = f(x)$ is either

$$x_{n+1} = f(x_n) + \delta f[f(x_n), x_n](x_{n+1} - x_n)$$

or

$$x_{n+1} = f[f(x_n)] + \delta f[f(x_n), x_n][x_{n+1} - f(x_n)],$$

where

$$\delta f[f(x_n), x_n] = \frac{f[f(x_n)] - f(x_n)}{f(x_n) - x_n} .$$

Both formulae are equivalent in the sense that they give the same sequence $\{x_n\}$ when beginning with the same x_0. In the generalization presented here, it is possible to solve the equation $x = Fx$ by the analogical iterations

(2,4) and (2,5) which are again equivalent in the same sense. Therefore, the sufficient conditions for the convergence of any of both sequences defined by the formulae (2,4) and (2,5) are sufficient even for the convergence of the other sequence. The formula (2,5) is simpler for the practical calculation. In spite of that, I shall deal further with formula (2,4), because in this way I have been successfull in obtaining less restrictive sufficient conditions for the convergence.

In the work [2(I)] J.W.Schmidt studies the solution of the equation $x = Fx$ by means of method applying the iterative process $x_{n+1} = Fx_n + \delta F(x_n, x_{n-1}) \cdot (x_{n+1} - x_n)$, calling it the Steffenson's method ([2(1)], method (2,9) on p.2; conditions of convergence stated in Theorem 4,1, on p.7). However, this process is quite different from the iterative process (2,5), being, essentially, a modification of the secant method ([1], Chapter 3, paragraph 9). Its convergence is of an other character than convergence of the process (2,5), as it is easy to see when compared the Schmidt's estimates of errors ([2(I)], (4,1)) with these contained in this paper. See also numerical example in paragraph 3.

The general results of this paper are presented in paragraph 2. Applications of the general theorems on systems of non-linear equations and on non-linear integral equations are stated in paragraphs 3 an 4.

2. Theorems of convergence and uniqueness

We shall use the following denotation: R is a Banach space, F a non-linear operator mapping R into R. The symbol $\delta F(u,v)$ will denote the divided difference of the operator F. This concept, introduced by Schmidt [2] under the title Steigung, is defined as follows. We shall say that the operator F has a divided difference $\delta F(u,v)$ in the space R, when there exist two non-negative numbers a, b such that for every two elements u, v from R there exists a linear bounded operator $\delta F(u,v)$ on R, satisfying the inequality

(2,1) $Fu - Fv = \delta F(u,v)(u-v),$

(2,2) $\|\delta F(u,v) - \delta F(v,w)\| \leq a\|u-w\| + b\|u-v\| + b\|v-w\| .$

Let an equation

(2,3) $x = Fx$

be given; to solve equation (2,3) we use the iterative processes

(2,4) $x_{n+1} = F^2x_n + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n)$ $(n = 0,1,2,\dots)$

(2,5) $x_{n+1} = Fx_n + \delta F(Fx_n, x_n)(x_{n+1} - x_n)$ $(n = 0,1,2,\dots)$

Lemma. Iterative processes (2,4) and (2,5) are equivalent in the following sense: Let x_0 be an arbitrary element from R. If the elements of either of the two sequences

$x_0, x_1, \dots, x_n$ defined by the process (2,4)

$x'_0, x'_1, \dots, x'_n,$ $(x'_0 = x_0)$ defined by the proces (2,5) are defined, then the ones of the other sequence are defined as well and the equalities $x_i = x'_i$, $i = 1,2,\dots,n$ hold.

Proof. The proof of this lemma may be achieved by means of full induction, as is easily seen from that, when subtracting the identity

$$\phi = F^2x_n - Fx_n - \delta F(Fx_n, x_n)(Fx_n - x_n)$$

from (2,4), we get (2,5).

Theorem 1. Let F be an operator which has the divided difference. Let the following conditions be fulfilled:

1) There exists a number $\lambda > 0$ such that ineuqality

(2,6) $\| Fu - Fv \| \leq \lambda \| u - v \|$

holds for two arbitrary elements u, v from R .

2) The inequality

(2,7) $\| \delta F(Fx_0, x_0) \| = d_0 < 1$

holds for the fixed element $x_0 \in R$.

3) The element x_1 is defined by (2,4) and there exists a real number t $(0 < t < 1)$ such that

(2,8) $$h = h(t) = \frac{[(a+b) + 2bt]t}{1-t} \|x_1 - x_0\| < 1$$

(2,9) $$d_0 + [(a+b)(1+\lambda) + 4b]\,[1 + \sigma(h)] \|x_1 - x_0\| \leq t < 1,$$

where

$$\sigma(h) = \sum_{k=1}^{\infty} h^{2^k - 1} \quad .$$

Then the equation (2,3) has a solution x^* in the sphere

(2,10) $D = \{x \in R,\ \|x - x_1\| \leq h[1 + \sigma(h^2)] \|x_1 - x_0\|\}$.

The sequences $\{x_n\}$ defined by equalities (2,4) or (2,5) converge in the norm of R to the solution x^* of (2,3) and the error $\|x^* - x_n\|$ of the approximation x_n satisfies

(2,11) $\|x^* - x_n\| \leq h^{2^{n-1}} [1 + \sigma(h^{2^n})] \|x_n - x_{n-1}\|, \ n = 1, 2, \dots ,$

(2,12) $\|x^* - x_n\| \leq h^{2^n - 1} [1 + \sigma(h^{2^n})] \|x_1 - x_0\|, \ n = 1, 2, \dots .$

<u>Proof</u>. Let us put

$$\|x_{n+1} - x_n\| = \kappa_n$$

$$\|\delta F(Fx_n, x_n)\| = d_n \qquad n = 0, 1, 2, \dots .$$

First of all, we shall show that the following inequalities

(2,13) $\| x_{n+1} - Fx_n \| \leq d_n \kappa_n$, (2,14) $\quad \| Fx_{n+1} - Fx_n \| \leq \lambda r_n \quad ,$

(2,15) $\quad \| x_n - Fx_n \| \leq d_n r_n + r_n \quad ,$

(2,16) $\quad \| Fx_{n+1} - x_{n+1} \| \leq \lambda r_n + d_n r_n$

are fulfilled.

We have

$$\| x_{n+1} - Fx_n \| = \| F^2 x_n - Fx_n + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n) \| =$$

$$= \| \delta F(Fx_n, x_n)(Fx_n - x_n) + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n) \| =$$

$$= \| \delta F(Fx_n, x_n)(x_{n+1} - x_n) \| \leq d_n r_n \; .$$

The correctness of inequalities (2,14),(2,15)and(2,16) can be easily verified.

We prove the following inequalities:

(2,17) $\quad r_{n+1} \leq d_{n+1} r_{n+1} + [(a+b) + 2 b d_n] d_n r_n^2 \quad ,$

(2,18) $\quad d_{n+1} \leq d_n + [(a+b)(1+\lambda) + 4 b d_n] r_n \quad . \quad n = 0, 1, 2, \ldots$

a) In the expression $r_{n+1} = \| x_{n+2} - x_{n+1} \|$, we replace x_{n+2} and x_{n+1} according to the formula (2,4); adding $- Fx_{n+1} + Fx_{n+1}$ and using formula (2,1) for the differences $F^2 x_{n+1} - Fx_{n+1}$, $Fx_{n+1} - F^2 x_n$, we get

$$r_{n+1} = \| \delta F(Fx_{n+1}, x_{n+1})(x_{n+2} - x_{n+1}) + [\delta F(x_{n+1}, Fx_n) - \delta F(Fx_n, x_n)](x_{n+1} - Fx_n) \| \, .$$

Using the inequalities (2,2),(2,13)and (2,15) we obtain (2,17).

b) By means of the triangle inequality, we get

$$d_{n+1} \leq d_n + \| \delta F(Fx_{n+1}, x_{n+1}) - \delta F(x_{n+1}, Fx_n) \| +$$

$$+ \| \delta F(x_{n+1}, Fx_n) - \delta F(Fx_n, x_n) \| \; .$$

By (2,2)

$$d_{n+1} \leq d_n + a \| Fx_{n+1} - Fx_n \| + b \| Fx_{n+1} - x_{n+1} \| + 2 b \|$$

$$\| x_{n+1} - Fx_n \| + b \| Fx_n - x_n \| + a r_n \; .$$

The formula (2,18) follows at once from (2,13),(2,14), (2,15) and (2,16).

Now, by means of full induction we shall prove the following relations:

a) $d_n \le d_0 + [(a+b)(1+\lambda)+4b]\,[1+\sigma_{n-1}(h)]\, r_0 < t < 1$,

b) $r_n < h^{2^n-1} r_0$,

c) $x_n \in D$,

where $$\sigma_n(h) = \sum_{k=1}^{n} h^{2^k-1}$$

$$\sigma_0(h) = 0 \qquad \text{for } n \ge 1 .$$

1) Let us put $n = 1$, then we get for $n = 0$ from (2,18) and (2,7), (2,9) the inequality

$$d_1 \le d_0 + [(a+b)(1+\lambda)+4b]\; r_0 < t < 1 .$$

Hence, the inequality a) holds for $n = 1$.

Similarly, from (12,17) we have

$$r_1 \le d_1 r_1 + [(a+b) + 2bd_0]\, d_0 r_0^2 .$$

From (2,8) and in view of that the inequality $0 < d_1 < t < 1$ holds, we get

$$r_1 < \frac{[(a+b)+2bt]\,t}{1-t}\, r_0^2 = h r_0 .$$

2) Let the inequalities a), b) be fulfilled for n . According to (2,18), it follows that

$$d_{n+1} \le d_0 + [(a+b)(1+\lambda)+4b]\,[1+\sigma_{n-1}(h)]\, r_0 + $$
$$+ [(a+b)(1+\lambda)+4bd_n]\, h^{2^n-1} r_0 .$$

Because $d_n < 1$ and $\sigma_{n-1}(h) + h^{2^n-1} = \sigma_n(h) < \sigma(h)$, we obtain relations a) for the index $n+1$, considering supposition (2,9).

Because $d_{n+1} < t < 1$, it follows from (2,17) that

$$r_{n+1} < \frac{[(a+b)+2bt]t}{1-t} r_n^2 < \frac{[(a+b)+2bt]t}{1-t} r_0^2 h^{2^{n+1}-2}.$$

From (2,8) it follows that

$$r_{n+1} < h^{2^{n+1}-1} r_0 \quad .$$

Further, from b) we have

$$\|x_{n+1} - x_1\| \leq r_1 + r_2 + \cdots + r_n <$$
$$< h[1+\sigma_{n-1}(h^2)] r_0 < h[1+\sigma(h^2)] r_0 \, .$$

Therefore, $x_n \in D$ $\qquad n = 1, 2, \ldots$.

From the inequality

$$d_n = \|\delta F(Fx_n, x_n)\| < t < 1$$

it follows that the operators $I - \delta F(Fx_n, x_n)$, $n = 0, 1, \cdots$ have the inverse operators. Therefore, the sequence $\{x_n\}_0^\infty$ is defined by (2,4). Consequently, the inequality

$$\|x_{m+n} - x_n\| \leq r_{n+m-1} + r_{n+m-2} + \cdots + r_{n+1} + r_n <$$
$$< (1 + h^{2^{n+1}-2^n} + \ldots\ldots + h^{2^{n+m-2}-2^n} + h^{2^{n+m-1}-2^n}) h^{2^n-1} r_0 =$$
$$= h^{2^n-1}[1 + \sum_{k=1}^{m-1} (h^{2^n})^{2^k-1}] r_0 = h^{2^n-1}[1 + \sigma_{m-1}(h^{2^n})] r_0$$

holds for arbitrary $m > n \geq n_0 > 1$.

From here it follows that $\{x_m\}$ is a fundamental sequence. R_* being a complete space, the sequence $\{x_m\}$ possesses the limit element x^* . Of course, $x^* \in D$.

We shall prove the inequality (2,11). Let us denote

$$q = \frac{h}{r_0} = \frac{[(a+b)+2bt]t}{1-t} \quad .$$

Because $d_n < t < 1$ for arbitrary n , we have

(2,19) $$r_{n+1} \leq q \, r_n^2 \quad .$$

From here it is easy to show that

(2,20) $$r_{n+k} \leq q^{2^k-1} r_n^{2^k} \qquad n = 0, 1, \cdots, \quad k = 1, 2, \ldots \quad .$$

Then it follows

$$\| x_{n+m+1} - x_n \| \le r_{n+m} + \dots + r_n \le$$
$$\le (q^{2^m-1} r_n^{2^m} + q^{2^{m-1}-1} r_n^{2^{m-1}} + \dots + q r_n^2 + r_n) = [1+\sigma_m (q r_n)] r_n ,$$

$$n = 0, 1, 2, \dots , \qquad m = 1, 2, \dots .$$

In view of the relations

$$q r_n \le q h^{2^n-1} r_0 = h^{2^n} ,$$
$$r_n \le q r_{n-1}^2 \le q r_0 h^{2^{n-1}-1} r_{n-1} = h^{2^{n-1}} r_{n-1}$$

and the above inequality, it follows that

$$\| x_{n+m+1} - x_n \| \le h^{2^{n-1}} [1+\sigma_m (h^{2^n})] r_{n-1} .$$

Hence for $m \to \infty$ we obtain the estimate (2,11).

The proof of the estimate (2,12) from (2,11) using the inequality b) $r_{n-1} \le h^{2^{n-1}-1} r_0$ is obvious.

We shall prove that x^* satisfies the equation (2,3). First of all we have

$$\| x^* - F x^* \| \le \| x^* - x_n \| + \| x_n - F x^* \| .$$

In the expression $\| x_n - F x^* \|$, we replace x_n according to the formula (2,4); we add $-F x_{n-1} + F x_{n-1}$ and we use formula (2,1) for the difference $F^2 x_{n-1} - F x_{n-1}$. We get

$$\| x_n - F x^* \| \le d_{n-1} \| x_n - x_{n-1} \| + \| F x_{n-1} - F x^* \| .$$

Because $x_{n-1} \in D$, $x^* \in D$ and $d_{n-1} < t < 1$, we obtain by (2,6)

$$\| x^* - F x^* \| \le \| x^* - x_n \| + t \| x_n - x_{n-1} \| + \lambda \| x_{n-1} - x^* \| .$$

Hence, for $n \to \infty$, we get $\| x^* - F x^* \| \le 0$.

Therefore

$$x^* = F x^* .$$

This completes the proof.

<u>Theorem 2</u>. If the assumptions of Theorem 1 are fulfilled

and if $0 < \lambda < 1$ holds, then the equation (2,3) has a unique solution in the sphere D .

Further the inequalities

(2,21) $$\|x^* - x_n\| \leq \frac{[(a+b)+2bt]t}{1-\lambda}\|x_n - x_{n-1}\|^2, \quad n = 1, 2, \dots,$$

(2,22) $$\|x^* - x_n\| \leq \frac{[(a+b)+2bt]t}{1-\lambda} h^{2^n-2}\|x_1 - x_0\|, \; n = 1, 2, \dots .$$

hold.

Proof. Let us assume that the equation (2,3) has two different solutions x^*, $\tilde{x}$ in the sphere D . Then we get

$$\|x^* - \tilde{x}\| = \|Fx^* - F\tilde{x}\| \leq \lambda\|x^* - \tilde{x}\| < \|x^* - \tilde{x}\|.$$

This is a contradiction showing that $x^* = \tilde{x}$.

We shall now prove the estimate (2,21). Let $n \geq 1$. We replace x^* in the expression $\|x^* - x_n\|$ by Fx^* and we replace x_n according to (2,4); adding $-Fx_n + Fx_n$ and using (2,1) for the difference $Fx_n - F^2x_{n-1}$,we get

(2,23)
$$\|x^* - x_n\| = \|Fx^* - Fx_n + [\delta F(x_n, Fx_{n-1}) - \delta F(Fx_{n-1}, x_{n-1})](x_n - Fx_{n-1})\|.$$

We shall use the triangle inequality for the norm of the difference in the square brackets we shall use the inequality (2,2) and for the difference $Fx^* - Fx_n$ (2,6).

After a slight modification we obtain

$$\|x^* - x_n\| \leq \lambda\|x^* - x_n\| + [(a+b)+2bt]t\|x_n - x_{n-1}\|^2,$$

Because $0 < \lambda < 1$, it follows that

$$\|x^* - x_n\| \leq \frac{[(a+b)+2bt]t}{1-\lambda}\|x_n - x_{n-1}\|^2 .$$

From the inequality $\|x_n - x_{n-1}\| \leq h^{2^{n-1}-1}\|x_1 - x_0\|$ and (2,21), we get the estimate (2,22).

Theorem 3. Let F be an operator which has the divided difference. Let the following conditions be fulfilled:

1) The inequality

$$\| \delta F(Fx_0, x_0) \| = d_0 < 1, \tag{2,24}$$

holds for the fixed element $x_0 \in R$.

2) The element x_1 is defined by (2,4) and there exists a real number t $(0 < t < 1)$ such that

$$h = h(t) = \frac{[(a+b)+2bt]t}{1-t} \| x_1 - x_0 \| < 1 , \tag{2,25}$$

$$d_0 + (a+b)(a+7b)[1+\sigma(h^2)] \| x_1 - x_0 \|^2 + \\ + 2(a+3b)[1+\sigma(h)] \| x_1 - x_0 \| \le t < 1 . \tag{2,26}$$

Then the equation (2,3) has a solution x^* in the sphere

$$D = \{ x \in R , \; \| x - x_1 \| \le h[1+\sigma(h^2)] \| x_1 - x_0 \| \} .$$

The sequences $\{x_n\}$ defined by formulae (2,4) or (2,5) are convergent in the norm of R to the solution x^* of (2,3) and the error $\| x^* - x_n \|$ of the approximation x_n satisfies

$$\| x^* - x_n \| \le h^{2^{n-1}} [1+\sigma(h^{2^n})] \| x_n - x_{n-1} \| , \tag{2,27}$$

$$\| x^* - x_n \| \le h^{2^n - 1} [1+\sigma(h^{2^n})] \| x_1 - x_0 \| , \tag{2,28}$$

$$\| x^* - x_n \| \le \frac{\{ h^{2^n}(a+b)[1+\sigma(h^{2^n})]^2 + [(a+b)+2bt]t \}}{1-t} \| x_n - x_{n-1} \|^2 , \tag{2,29}$$

$$\| x^* - x_n \| \le \frac{h^{2^{n+1}-2}(a+b)[1+\sigma(h^{2^n})]}{1-t} \| x_1 - x_0 \|^2 + h^{2^n - 1} \| x_1 - x_0 \| . \tag{2,30}$$

$$n = 1, 2, \cdots .$$

<u>Proof</u>. The main idea of the proof is the same as in Theorem 1. The formulae

$$r_{n+1} \le d_{n+1} r_{n+1} + [(a+b)+2bd_n] d_n r_n^2 , \tag{2,31}$$

$$d_{n+1} \le d_n + 2(a+3b) r_n + (a+b)(a+7b) r_n^2 , \quad n = 0, 1, 2, \dots \tag{2,32}$$

play here the rôle of formulae (2,17) and (2,18).

The first formula is the same as formula (2,17), the second one will be proved in the following way.

By means of the triangle inequality, and using the inequalities (2,2), (2,13) and

$$\|Fx_{n+1}-x_{n+1}\| \leq \|Fx_{n+1}-Fx_n\|+\|Fx_n-x_{n+1}\|, \ \|Fx_n-x_n\| \leq \|Fx_n-x_{n+1}\|+ \\ +\|x_{n+1}-x_n\| \ ,$$

we get

$$d_{n+1} \leq d_n+\|\delta F(Fx_{n+1}, x_{n+1})-\delta F(x_{n+1}, Fx_n)\|+ \\ +\|\delta F(x_{n+1}, Fx_n)-\delta F(Fx_n, x_n)\| \leq d_n+(a+b+4bd_n)r_n+(a+b)\| \ . \\ \|Fx_{n+1}-F_n\| \ .$$

Now

$$\|Fx_{n+1}-Fx_n\| \leq \|\delta F(x_{n+1}, x_n)-\delta F(x_n, Fx_n)+\delta F(x_n, Fx_n)- \\ -\delta F(Fx_n, x_n)+\delta F(Fx_n, x_n)\| r_n \leq [\|\delta F(x_{n+1}, x_n)- \\ -\delta F(x_n, Fx_n)\|+\|\delta F(x_n, Fx_n)-\delta F(Fx_n, \dot{x}_n)\|+d_n] r_n \leq \\ \leq [(a+3b)d_n r_n+4br_n+d_n] r_n \ .$$

From the above inequality we obtain (2,32).

We shall now prove that the relations

(2,33) a) $d_n \leq d_0+(a+b)(a+7b)[1+\sigma_{n-1}(h^2)]r_0^2+ \\ +2(a+3b)[1+\sigma_{n-1}(h)]r_0 < t < 1,$

(2,34) b) $r_n \leq h^{2^n-1} r_0 \ ,$

c) $x_n \in D$

hold for $n = 1, 2, \ldots$. For $n = 1$ it is evident.

Let us suppose that a), b) hold for n . According to (2,32) we have

$$d_{n+1} \leq d_0+(a+b)(a+7b)[1+\sigma_{n-1}(h^2)]r_0^2+2(a+3b)[1+\sigma_{n-1}(h)]r_0+ \\ +2(a+3b)h^{2^n-1}r_0+(a+b)(a+7b)h^{2^{n+1}-2}r_0^2 \leq d_0+(a+b)(a+7b), \\ [1+\sigma_n(h^2)]r_0^2+2(a+3b)[1+\sigma_n(h)]r_0 < t < 1 \ .$$

The proof of all the assertions of theorem except that $Fx^* = x^*$ and except the proofs of the inequalities (2,29) and (2,30) will be made in the same manner as in Theorem 1.

We shall now prove that the limit x^* of the sequence $\{x_n\}$ satisfies the equation (2,3). We have

$$\|Fx^* - x^*\| \leq \|Fx^* - Fx_n\| + \|Fx_n - x_{n+1}\| + \|x_{n+1} - x^*\| \leq$$

$$\leq \|\delta F(x^*, x_n)\| \; \|x^* - x_n\| + t\,\|x_{n+1} - x_n\| + \|x_{n+1} - x^*\|.$$

Evidently, it suffices to show that $\|\delta F(x^*, x_n)\|$ is bounded in the sphere D. But it follows that the inequality

$$\|\delta F(x^*, x_n)\| \leq \|\delta F(x^*, x_n) - \delta F(x_n, Fx_n)\| + \|\delta F(x_n, Fx_n) -$$

$$- \delta F(Fx_n, x_n)\| + d_n \leq a\|x^* - Fx_n\| + b\|x^* - x_n\| + 3b\|x_n - Fx_n\| +$$

$$+ t \leq t + a\|x^* - x_{n+1}\| + b\|x^* - x_n\| + [(a+3b)t + 3b] \,.$$

$$\|x_{n+1} - x_n\|$$ holds.

We shall prove estimates (2,29) and (2,30). From (2,4),(2,1), (2,2) and from the inequality $d_n < t < 1 \quad (n = 1, 2, \dots)$, it follows

$$\|x^* - x_n\| = \|Fx^* - F^2x_{n-1} - \delta F(Fx_{n-1}, x_{n-1})(x_n - Fx_{n-1})\| =$$

$$= \|Fx^* - Fx_n + Fx_n - F^2x_{n-1} - \delta F(Fx_{n-1}, x_{n-1})(x_n - Fx_{n-1})\| \leq$$

$$(2,35) \quad \leq \|\delta F(x^*, x_n)\| \; \|x^* - x_n\| + \|\delta F(x_n, Fx_{n-1}) -$$

$$- \delta F(Fx_{n-1}, x_{n-1})\| \; \|x_n - Fx_{n-1}\| \leq$$

$$\leq \|\delta F(x^*, x_n)\| \; \|x^* - x_n\| + [(a+b) + 2bt\,\|x_n - x_{n-1}\|^2 .$$

Because the inequality

$$\|\delta F(x^*, x_n)\| \leq \|\delta F(x^*, x_n) - \delta F(x_n, Fx_{n-1})\| + \|\delta F(x_n, Fx_{n-1}) - \delta F(Fx_{n-1}, x_{n-1})\| + d_{n-1}$$

holds, then from (2,2),(2,4) it follows that

$$(2,36) \quad \|\delta F(x^*, x_n)\| \leq d_{n-1} + (a+b)\|x^* - x_n\| + [(a+b)t + b] \,.$$

$$\|x_n - x_{n-1}\| \,.$$

Considering (2,35), we get

$$\|x^*-x_n\| \leq \{[(a+3b)t+a+b]\|x_n-x_{n-1}\|+d_{n-1}\}\|x^*-x_n\|+$$
$$+(a+b)\|x^*-x_n\|^2+[(a+b)+2bt]t\|x_n-x_{n-1}\|^2 .$$

From estimate (2,27) in the second term. $\|x^*-x_n\|^2$ is not larger than $h^{2^n}[1+\sigma(h^{2^n})]^2\|x_n-x_{n-1}\|^2$; then, we get

$$\|x^*-x_n\| \leq \{[(a+3b)t+a+b]\|x_n-x_{n-1}\|+d_{n-1}\}\|x^*-x_n\|+$$

(2,37)

$$+\{h^{2^n}(a+b)[1+\sigma(h^{2^n})]^2+[(a+b)+2bt]t\}\|x_n-x_{n-1}\|^2 .$$

We shall now prove that the inequality

$$[(a+3b)t+a+b]\|x_n-x_{n-1}\|+d_{n-1}<t<1 \quad (n=1,2,\dots)$$

holds.

In fact, from (2,33) and (2,34) it follows that for $n\geq 1$

$$[(a+3b)t+a+b]\|x_n-x_{n-1}\|+d_{n-1}\leq 2(a+2b)h^{2^{n-1}-1}\cdot$$
$$\|x_1-x_0\|+d_0+(a+b)(a+7b)[1+\sigma_{n-2}(h^2)]\|x_1-x_0\|^2+$$
$$+2(a+3b)[1+\sigma_{n-2}(h)]\cdot\|x_1-x_0\|\leq d_0+(a+b)(a+7b)$$
$$[1+\sigma_{n-2}(h^2)]\|x_1-x_0\|^2+2(a+3b)h^{2^{n-1}-1}\|x_1-x_0\|+$$
$$+2(a+3b)[1+\sigma_{n-2}(h)]\|x_1-x_0\|\leq d_0+(a+b)(a+7b)[1+\sigma_{n-2}(h^2)]\cdot$$
$$\|x_1-x_0\|^2+2(a+3b)[1+\sigma_{n-1}(h)]\|x_1-x_0\|<t<1 .$$

From (2,37) we obtain the estimate (2,29):

$$\|x^*-x_n\|\leq\frac{\{h^{2^n}(a+b)[1+\sigma(h^{2^n})]^2+[(a+b)+2bt]t\}}{1-t}\|x_n-x_{n-1}\|^2 .$$

From (2,29) and (2,34) we get the estimate (2,30):

$$\|x^*-x_n\|\leq\frac{h^{2^{n+1}-2}(a+b)[1+\sigma(h^{2^n})]^2}{1-t}\|x_1-x_0\|^2+h^{2^n-1}\|x_1-x_0\| .$$

The proof is complete.

<u>Theorem 4</u>. Let F be an operator which has the divided

difference. Let the following conditions be fulfilled:

1) The inequality

(2,24) $$\| \delta F(Fx_0, x_0) \| = d_0 < 1$$

holds for the fixed element $x_0 \in R$.

2) The element x_1 is defined by (2,4), and there exists a real number t $(0 < t < 1)$ such that

(2,25) $$h = h(t) = \frac{[(a+b)+2bt]t}{1-t} \| x_1 - x_0 < 1 ,$$

(2,38) $$d_0 + (a+b)(a+7b)[1+\sigma(h^2)] \| x_1 - x_0 \|^2 + \\ + 2(a+3b) \sum_{k=0}^{\infty} h^k \| x_1 - x_0 \| \leq t < 1 .$$

Then the equation (2,3) has a solution x^* in the sphere

$$D = \{ x \in R, \| x - x_1 \| \leq h [1 + \sigma(h^2)] \| x_1 - x_0 \| \} .$$

The sequences $\{x_n\}$ defined by formulae (2,4) or (2,5) are convergent in the norm of R to the solution x^* of (2,3) and the error $\| x^* - x_n \|$ of the approximation x_n satisfies

(2,27) $$\| x^* - x_n \| \leq h^{2^{n-1}} [1 + \sigma(h^{2^n})] \| x_n - x_{n-1} \| ,$$

(2,28) $$\| x^* - x_n \| \leq h^{2^n - 1} [1 + \sigma(h^{2^n})] \| x_1 - x_0 \| ,$$

(2,39) $$\| x^* - x_n \| \leq \frac{[(a+b)+2bt]t}{1-t} \| x_n - x_{n-1} \|^2 ,$$

(2,40) $$\| x^* - x_n \| \leq h^{2^n - 1} \| x_1 - x_0 \| \qquad n = 1, 2, \ldots .$$

The solution x^* is unique in the sphere D' defined by

$$D' = \{ x \in R, \| x - x_2 \| \leq h^3 [1 + \sigma(h^4)] \| x_1 - x_0 \| \} \subset D .$$

<u>Proof</u>. The proofs of all the assertions of our theorem except the proofs of inequalities (2,39),(2,40) and of the uniqueness are the same as in Theorem 3.

Now, we shall prove inequalities (2,39) and (2,40). According to (2,35) and (2,36), we obtain

$$\|x^*-x_n\| \le \{[(a+3b)t+a+b]\,\|x_n-x_{n-1}\|+(a+b)\|x^*-x_n\|+$$

$$(2,41)\qquad +d_{n-1}\}\cdot\|x^*-x_n\|+[(a+b)+2bt]\,t\,\|x_n-x_{n-1}\|^2 .$$

From (2,27) and (2,41) it follows that

$$(2,42)\qquad \|x^*-x_n\| \le \{[(a+3b)t+a+b]\,\|x_n-x_{n-1}\|\ \frac{+d_{n-1}}{+(a+b)h^{2^{n-1}}}\cdot$$

$$\cdot[1+\sigma(h^{2^n})]\,\|x_n-x_{n-1}\|\}\,\|x^*-x_n\|+[(a+b)+2bt]\,t\,\|x_n-x_{n-1}\|^2 .$$

Now, we shall prove that the inequality

$$\{(a+3b)t+(a+b)[1+h^{2^{n-1}}(1+\sigma(h^{2^n}))]\}\,\|x_n-x_{n-1}\|+d_{n-1}\le t<1$$

holds for $n \ge 1$.

In fact, from formulae (2,33) and (2,34) it follows that for $n \ge 1$

$$\{(a+3b)t+(a+b)[1+h^{2^{n-1}}(1+\sigma(h^{2^n}))]\}\,\|x_n-x_{n-1}\|+$$

$$+d_{n-1}\le d_0+(a+b)(a+7b)[1+\sigma_{n-2}(h^2)]\,\|x_1-x_0\|^2+$$

$$+2(a+3b)[1+\sigma_{n-2}(h)]\cdot\|x_1-x_0\|+$$

$$+\{(a+3b)+(a+b)[1+h^{2^{n-1}}(1+\sigma(h^{2^n}))]\}\,h^{2^{n-1}-1}\|x_1-x_0\| .$$

Since $h<1$, we have

$$1+\sigma_{n-2}(h)\le\sum_{k=0}^{n-2}h^k ,\quad 1+\sigma(h^{2^n})\le\sum_{k=0}^{\infty}h^k=\frac{1}{1-h} .$$

Using these inequalities, we obtain

$$\{(a+3b)t+(a+b)[1+h^{2^{n-1}}(1+\sigma(h^{2^n}))]\}\,\|x_n-x_{n-1}\|+d_{n-1}\le$$

$$\le d_0+(a+b)(a+7b)[1+\sigma_{n-2}(h^2)]\,\|x_1-x_0\|^2+2(a+3b)\sum_{k=0}^{n-2}h^k\cdot$$

$$\cdot\|x_1-x_0\|+2(a+3b)h^{2^{n-1}-1}\cdot\|x_1-x_0\|+2(a+3b)h^{2^{n-1}}\sum_{k=0}^{\infty}h^k\cdot$$

$$\|x_1-x_0\|\le d_0+(a+b)(a+7b)[1+\sigma_{n-2}(h^2)]\,\|x_1-x_0\|^2+$$

$$+2(a+3b)\sum_{k=0}^{\infty}h^k\cdot\|x_1-x_0\|<t<1 .$$

From inequality (2,42) we get estimate (2,39),

$$\|x^*-x_n\|\le\frac{[(a+b)+2bt]\,t}{1-t}\,\|x_n-x_{n-1}\|^2 .$$

From (2,39) and (2,34) it is possible to get estimate (2,40).

Now, we shall prove the uniqueness of the solution of the equation (2,3) in the sphere D'. From (2,28) it follows that $x^* \in D'$. It is easy to show that $D' \subset D$. Assuming that x^*, $\tilde{x}$ are two different solutions of (2,3) in the sphere D', $x^* = \lim_{n \to \infty} x_n$, we shall show that $x^* = \tilde{x}$.

Now, we replace x^* in the formulae (2,35) and (2,36) by $\tilde{x}$. We are right in doing so, as these formulae are based upon (2,1), (2,2), (2,4) only. We get

$$\begin{aligned} (2,43) \qquad & \| \tilde{x} - x_n \| \leq \| \delta F(\tilde{x}, x_n) \| \, \| \tilde{x} - x_n \| + \\ & + [(a+b) + 2bt] t \, \| x_n - x_{n-1} \|^2 , \end{aligned}$$

$$\begin{aligned} (2,44) \qquad & \| \delta F(\tilde{x}, x_n) \| \leq d_{n-1} + (a+b) \| \tilde{x} - x_n \| + \\ & + [(a+3b)t + a + b] \, \| x_n - x_{n-1} \| . \end{aligned}$$

From (2,44) it follows that

$$\begin{aligned} (2,45) \qquad & \| \delta F(\tilde{x}, x_n) \| \leq d_{n-1} + (a+b) \| \tilde{x} - x_{n-1} \| + \\ & + [(a+3b)t + 2a + 2b] \, \| x_n - x_{n-1} \| . \end{aligned}$$

We shall prove that the relations

a) $$\| \delta F(\tilde{x}, x_n) \| < t < 1 ,$$

b) $$\| \tilde{x} - x_n \| \leq \frac{[(a+b) + 2bt] t}{1-t} \| x_n - x_{n-1} \|^2 \leq h^{2^n - 1} \| x_1 - x_0 \| .$$

hold for $n \geq 3$.

First of all, we shall prove that these assertions are correct for $n = 3$.

a) We assume that $\tilde{x} \in D'$, then we have the inequality $\| \tilde{x} - x_2 \| \leq h^3 [(1 + \sigma(h^4)] \, \| x_1 - x_0 \| \}$. According to this inequality and (2,33),(2,34), we obtain

$$\begin{aligned} & \| \delta F(\tilde{x}, x_3) \| \leq d_0 + (a+b)(a+7b)[1+h^2] \, \| x_1 - x_0 \|^2 + \\ & + 2(a+3b)(1+h) \| x_1 - x_0 \| + (a+b) h^3 [1 + \sigma(h^4)] \| x_1 - x_0 \| + \\ & + [(a+3b)t + 2a + 2b] h^3 \, \| x_1 - x_0 \| . \end{aligned}$$

From a simple estimation of the last two terms it follows that
$$\| \delta F(\tilde{x}, x_3) \| \le d_0 + (a+b)(a+7b)(1+h^2)\| x_1 - x_0 \|^2 + $$
$$+ 2(a+3b) \cdot \sum_{k=0}^{\infty} h^k \| x_1 - x_0 \| < t < 1 .$$

b) From the inequality a) just proved and from (2,43) it follows that
$$\| \tilde{x} - x_3 \| \le \frac{[(a+b)+2bt]\,t}{1 - \| \delta F(\tilde{x}, x_3) \|} \| x_3 - x_2 \|^2 \le$$
$$\le \frac{[(a+b)+2bt]\,t}{1-t} \| x_3 - x_2 \|^2 \le h^7 \| x_1 - x_0 \| .$$

Now, let the inequalities a) and b) hold for $n > 3$.

a) According to (2,45)
$$\| \delta F(\tilde{x}, x_{n+1}) \| \le d_n + (a+b) \| \tilde{x} - x_n \| + [(a+3b)t + 2a + 2b] \cdot \| x_{n+1} - x_n \| .$$
From the inequality b) and inequalities for d_n and $\| x_{n+1} - x_n \|$ we obtain
$$\| \delta F(\tilde{x}, x_{n+1}) \| \le d_0 + (a+b)(a+7b)[1+\sigma_{n-1}(h^2)] \cdot$$
$$\| x_1 - x_0 \|^2 + 2(a+3b)[1+\sigma_{n-1}(h)] \| x_1 - x_0 \| + (a+b) h^{2^n-1} \cdot$$
$$\| x_1 - x_0 \| + [(a+3b)t + 2a + 2b] h^{2^n-1} \| x_1 - x_0 \| .$$

Because $[1+\sigma_{n-1}(h)] \le \sum_{k=0}^{n-1} h^k$ and $0 < t < 1$,

it is evident that
$$(a+b) h^{2^n-1} \| x_1 - x_0 \| + [(a+3b)t + 2a + 2b] h^{2^n-1} \cdot$$
$$\| x_1 - x_0 \| \le 2(a+3b) h^n \| x_1 - x_0 \| + 2(a+3b) h^{n+1} \cdot$$
$$\| x_1 - x_0 \| = 2(a+3b)(h^n + h^{n+1}) \| x_1 - x_0 \| ;$$

then we get
$$\| \delta F(\tilde{x}, x_{n+1}) \| \le d_0 + (a+b)(a+7b)[1+\sigma_{n-1}(h^2)] \cdot$$
$$\| x_1 - x_0 \|^2 + 2(a+3b) \sum_{k=0}^{n+1} h^k \| x_1 - x_0 \| < t < 1 .$$

b) From a) it follows easily, that

$$\| \tilde{x} - x_{n+1} \| \leq \frac{[(a+b)+2bt]\,t}{1 - \| \delta F(\tilde{x}, x_{n+1}) \|} \, \| x_{n+1} - x_n \|^2 \leq$$

$$\leq \frac{[(a+b)+2bt]\,t}{1-t} \, \| x_{n+1} - x_n \|^2 \leq h^{2^{n+1}-1} \| x_1 - x_0 \| \ .$$

Thus we have proved the validity of the inequality b) for all the $n \geq 3$.

From this and by the limiting process for $n \to \infty$, we get at once

$$\lim_{n \to \infty} x_n = \tilde{x} \ .$$

As $x^* = \lim_{n \to \infty} x_n$, the limit being unique, we conclude that $\tilde{x} = x^*$.

Remark. If in theorems 1 - 4 we replace everywhere the series $\sigma(h)$, $\sigma(h^2)$, ... by the series $\sum_{k=1}^{\infty} h^k$, $\sum_{k=1}^{\infty} (h^2)^k$, ... and the sums $\sigma_n(h)$, $\sigma_n(h^2)$, ... by the sums $\sum_{k=1}^{n} h^k$, $\sum_{k=1}^{n} (h^2)^k$, ... , then theorems 1 - 4 hold again, evidently.

3. Application on systems of non-linear equations

Application of the method to the solving of systems of non-linear equations in the space R_n.

Let R_n be an Euclidean space. Let the system of non-linear equations

$$x_i = f_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{3,1}$$

be given, where $f_i(x_1, x_2, \ldots, x_n)$ has continuous partial derivatives of second order in R_n. Using the notation

$$x = (x_1, x_2, \ldots, x_n),$$

$$Fx = F(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), \ldots f_n(x_1, \ldots x_n)),$$

we can write the system (3,1) in the form

(3,2) $$x = Fx ,$$

where F is a non-linear operator defined on R_n .

Let us introduce on R_n the norm

$$\| x_n \| = \max_{j=1,\dots,n} |x_j| ;$$

then the norm of the linear operator A given by the matrix of real numbers (a_{ij}) is

$$\| A \| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}| .$$

The operator F has the divided difference $\delta F(x,y)$ defined in R_n :

$$\delta F(x,y) = \begin{pmatrix} \delta f_1(x_1 y_1, x_2, \dots x_n), \dots, \delta f_1(y_1, \dots, y_{j-1}, x_j y_j, x_{j+1}, \dots, x_n), \\ \dots, \delta f_1(y_1, \dots, y_{n-1}, x_n y_n) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \delta f_i(x_1 y_1, x_2, \dots, x_n), \dots, \delta f_i(y_1, \dots, y_{j-1}, x_j y_j, x_{j+1}, \dots, x_n) \\ \dots, \delta f_i(y_1, \dots, y_{n-1}, x_n y_n) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \delta f_n(x_1 y_1, x_2, \dots, x_n), \dots, \delta f_n(y_1, \dots, y_{j-1}, x_j y_j, x_{j+1}, \dots, x_n), \\ \dots, \delta f_n(y_1, \dots, y_{n-1}, x_n y_n) \end{pmatrix},$$

where

(3,3) $$\begin{cases} \delta f_i(y_1, \dots, y_{j-1}, x_j y_j, x_{j+1}, \dots, x_n) \overset{df}{=} \dfrac{1}{x_j - y_j} [f_i(y_1, \dots, y_{j-1}, \\ \qquad x_j, x_{j+1}, \dots, x_n) - f_i(y_1, \dots, y_{j-1}, y_j, x_{j+1}, \dots, x_n)], \\ a \geq \dfrac{1}{2} \max\limits_{i=1,2,\dots,n} \sum\limits_{j=1}^{n} f_{jj}^{i} , \\ b \geq \max\limits_{i=1,2,\dots,n} \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{j-1} f_{ik}^{i} = \max\limits_{i=1,\dots,n} \sum\limits_{j=1}^{n} \sum\limits_{k=j+1}^{n} f_{jk}^{i} . \end{cases}$$

(See the Schmidt's paper [2(II)].)

Let us solve the equation (3,2) by iterative process

(3,4) $$x^{(n+1)} = F^2 x^{(n)} + \delta F(Fx^{(n)}, x^{(n)})(x^{(n+1)} - Fx^{(n)}) .$$

For the chosen special space and operator F, theorem 4 (paragraph 2) gives as a special case

Theorem 1: 1) Let the condition

$$\max_{i=1,\dots,m}\sum_{j=1}^{n}|\delta f_i(x_1^{(0)},\dots,x_{j-1}^{(0)},x_j^{(0)}w_j,w_{j+1},\dots,w_n)| = d_0 < 1$$

hold for the fixed element $x^{(0)} = (x_1^{(0)},\dots,x_n^{(0)}) \in R_n$, where $Fx^{(0)} = (w_1,\dots,w_n)$.

2) Denote by $x^{(1)} = (x_1^{(1)},\dots,x_n^{(1)})$ the element defined to the element $x^{(0)}$ by (3,4). Let there exist a real number t $(0 < t < 1)$ such that the inequalities

$$h = h(t) = \frac{[(a+b)+2bt]t}{1-t}\max_{j=1,\dots,n}|x_j^{(1)} - x_j^{(0)}| < 1 ,$$

$$d_0 + \frac{(a+b)(a+7b)}{1-h^2}\left(\max_{j=1,\dots,n}|x_j^{(1)}-x_j^{(0)}|\right)^2 + \frac{2(a+3b)}{1-h}\max_{j=1,\dots,n}|x_j^{(1)}-x_j^{(0)}| \le t < 1,$$

hold, where a and b are constants as in (3,3).

Then the equation (3,2) has a solution x^* in the sphere

$$D = \{x \in R_n,\ \|x-x^{(1)}\| = \max_{j=1,\dots,n}|x_j - x_j^{(1)}| \le \frac{h}{1-h^2}\max_{j=1,2,\dots,n}|x_j^{(1)} - x_j^{(0)}|\} .$$

The sequences $\{x_n\}$ defined by (3,4) are convergent to the solution x^* of (3,2) (i.e. $\max_i |x_i^{(n)} - x_i^*| \xrightarrow[n\to\infty]{} 0$).

The following estimates hold:

$$(3,5)\quad \|x^* - x^{(n)}\| = \max_{j=1,\dots,n}|x_j^* - x_j^{(n)}| \le \frac{h^{2^{n-1}}}{1-h^2}\max_{j=1,\dots,n}|x_j^{(n)} - x_j^{(n-1)}| ,$$

$$(3,6)\quad \|x^* - x^{(n)}\| = \max_{j=1,\dots,n}|x_j^* - x_j^{(n)}| \le \frac{h^{2^{n}-1}}{1-h^2}\max_{j=1,\dots,n}|x_j^{(1)} - x_j^{(0)}| ,$$

$$(3,7)\quad \|x^* - x^{(n)}\| = \max_{j=1,\dots,n}|x_j^* - x_j^{(n)}| \le \frac{[(a+b)+2bt]t}{1-t}\left(\max_{j=1,\dots,n}|x_j^{(n)} - x_j^{(n-1)}|\right)^2 ,$$

$$(3,8)\quad \|x^* - x^{(n)}\| = \max_{j=1,\dots,n}|x_j^* - x_j^{(n)}| \le h^{2^n-1}\max_{j=1,\dots,n}|x_j^{(1)} - x_j^{(0)}| .$$

Numerical example. Let us solve the non-linear system of algebraic equations

$$(3,9)\qquad \begin{cases} x_1 = x_1^2 + x_2^2 - 11 \\ x_2 = (x_1+1)^2 + 2(x_2-1)^2 - 14 \end{cases}$$

which has the solution $x_1 = 2$, $x_2 = 3$. Let us find an approximate solution by means of Schmidt's method ([2(I)], method (2,9) and Theorem 4,1) and by means of our Theorem 1'. Moreover, we shall choose for R_2:

$$R_2 = (-\infty, +\infty) \times (-\infty, +\infty) .$$

Since this system does not satisfy neither the first condition of Schmidt's Theorem nor the first condition of Theorem 1', we shall construct an equivalent system, for which both conditions will be fulfilled:

We have the equivalent system with (3,9)

$$(3,10)\qquad \begin{cases} x_1 = x_1 + \alpha(x_1 - x_1^2 - x_2^2 + 11) + \beta[x_2 - (x_1+1)^2 - \\ \qquad\qquad - 2(x_2-1)^2 + 14] \\ x_2 = x_2 + \gamma(x_1 - x_1^2 - x_2^2 + 11) + \delta[x_2 - (x_1+1)^2 - \\ \qquad\qquad - 2(x_2-1)^2 + 14] \end{cases} ,$$

where

$$\begin{cases} \alpha = -0.53003 \\ \beta = 0.43710 \\ \gamma = 0.43711 \\ \delta = -0.23060 \end{cases} \qquad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = -0.06884 \neq 0 .$$

Let us calculate according to (3,3); we get $a = 0.4371$, $b = 0$.

Solution based upon Theorem 1':

Let us choose $x^{(0)} = (2.2\,,\ 3.2)$. According to (3,10) we obtain $Fx^{(0)} = (2.0075444,\ 3.0054652)$. Conditions of Theorem 1' are fulfilled:

1) $\|\delta F(Fx^{(0)}, x^{(0)})\| = d_0 \leq 0.062987943 < 1$,

2) For $t = 0.255\,728\,423$ hold

$$h = h(t) = \frac{[(a+b)+2bt]\,t}{1-t}\|x^{(1)}-x^{(0)}\| = 0,03 < 1,$$

$$d_0 + \frac{(a+b)(a+7b)}{1-h^2}\|x^{(1)}-x^{(0)}\|^2 + \frac{2(a+3b)}{1-h}\|x^{(1)}-x^{(0)}\| \leq$$
$$\leq 0.250\,642\,826 < t < 1.$$

Then

$$D = \{x \in R_2,\ \|x - x^{(1)}\| \leq 0.005\,997\,978\}.$$

From (3,7) we get estimates

$$\|x^* - x^{(1)}\| \leq 0.005\,992\,584,$$

$$\|x^* - x^{(2)}\| \leq 0.000\,000\,013.$$

Actual errors

$$\|x^* - x^{(1)}\| = 0.000\,291\,280,$$

$$\|x^* - x^{(2)}\| = 0.000\,000\,001.$$

Solution based upon Schmidt's Theorem:

When using Schmidt's method, we must choose the first approximation $x^{(0)}$ and the second one $x^{(1)}$. They are mutually independent, their choise being limited by the conditions 2) only (see below). Let us choose again $x^{(0)} = (2.2,\ 3.2)$. As $x^{(1)}$ we choose the first approximation which was computed by use of our Theorem 1'. Conditions of Schmidt's Theorem 4.1 are fulfilled:

1) $\|\delta F(x^{(1)}, x^{(0)})\| = d_1 \leq 0.065\,446\,778$.

2) For $t = 0.16$ hold:

$$h = h(t) = \frac{1}{1-t}[a\|x^{(2)}-x^{(0)}\| + b\|x^{(2)}-x^{(1)}\| + b\|x^{(1)}-x^{(0)}\|] =$$
$$= 0.104\,066\,612 < 1,$$

$$d_1 + (a+b)\|x^{(1)}-x^{(0)}\| + \frac{2(a+b)}{1-h}\|x^{(2)}-x^{(1)}\| \leq 0.153\,033\,911 < t < 1,$$

$$2^{*}\| x^{(2)} - x^{(1)} \| \leq \| x^{(2)} - x^{(1)} \| ,$$

$$0.000\,564\,046 < 0.199\,990\,743 \quad .$$

3) $\mathcal{D} = \{ y \in R_2 \; \| y - x^{(2)} \| \leq 0.000\,034\,585 \}$.

We have estimate:

$$\| x^{*} - x^{(2)} \| \leq 0.000\,034\,585 \quad .$$

Actual error:

$$\| x^{*} - x^{(2)} \| = 0.000\,015\,714 \quad .$$

T a b l e

	Theorem 1´		Schmidt´s Theorem	
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	2.2	3.2	2.2	3.2
1	2.000 247 197	3.000 291 280	2.000 247 197 x)	3.000 291 280 x)
2	2.000 000 000	3.000 000 001	2.000 015 714	3.000 009 257

x) Not computed, but chosen.

4. Application on non-linear integral equations

For the practical solving of integral equations it is, in general, necessary to use an approximative method replacing the integral equation by a system of algebraic equations. If we solve this system by use of the methods given in paragraph 2, we get the sequence of the approximate solutions of this system, namely the sequence of vectors $\overline{x}_n$. We then construct an approximate solution of the integral equation $x_n'(s)$ from the vector $\overline{x}_n$. Now, it is necessary to know how to estimate the proximity of $x_n'(s)$ is to the solution x^{*}.

I deal with the problem in this paragraph.

Let X, $\bar{X}$ be two Banach spaces. Let the space $\bar{X}$ be linearly isomorphic with the subspace $X' \subset X$ and isomorphism be realized by the linear operator φ_0. φ_0 maps X' into $\bar{X}$ and has a linear inverse operator φ_0^{-1}. Let there exist a linear operator φ mapping X into $\bar{X}$ and in the space X' $\varphi = \varphi_0$.

Let a non-linear equation

(4,1) $\qquad x = Fx$

be given, where $x \in X$ and the non-linear operator F maps X into X. Let the conditions of Theorem 2 be fulfilled (paragraph 2). From Theorem 1 and 2, the equation (4,1) has a unique solution x^* in the sphere D, which is defined by the formula

$$D = \{x \in X, \quad \|x - x_1\| \leq h\,[1 + \delta(h^2)]\,\|x_1 - x_0\|\} \subset X.$$

Let an approximate equation

(4,2) $\qquad \bar{x} = \bar{F}\bar{x}$

be given where $x \in \bar{X}$ and $\bar{F}$ is a non-linear operator mapping $\bar{X}$ into $\bar{X}$ and having the divided difference $\delta\bar{F}(\bar{u},\bar{v})$ mapping $\bar{X}$ into $\bar{X}$, so that following relations hold:

(4,3) $\quad \bar{F}\bar{u} - \bar{F}\bar{v} = \delta\bar{F}(\bar{u},\bar{v})(\bar{u}-\bar{v}) \qquad \bar{u},\bar{v} \in \bar{X}$

(4,4) $$\|\delta\bar{F}(\bar{u},\bar{v}) - \delta\bar{F}(\bar{v},\bar{w})\| \leq \bar{a}\,\|\bar{u}-\bar{w}\| + b\,\|\bar{u}-\bar{v}\| + b\,\|\bar{v}-\bar{w}\|, \qquad \bar{u},\bar{v},\bar{w} \in \bar{X};$$

we shall further suppose that

1) $\|\bar{F}\bar{u} - \bar{F}\bar{v}\| \leq \lambda\,\|\bar{u}-\bar{v}\|$, where $0 < \lambda < 1$, $\bar{u},\bar{v} \in \bar{X}$;

then $\qquad \|\delta\bar{F}(\bar{F}\varphi x_0, \varphi x_0)\| = \bar{\alpha}_0 < 1$.

2) The element $\bar{x}_1$ is defined by the formula

$$\bar{x}_1 = \bar{F}^2\bar{x}_0 + \delta\bar{F}(\bar{F}\bar{x}_0,\bar{x}_0)(\bar{x}_1 - \bar{F}\bar{x}_0), \qquad \bar{x}_0 = \varphi x_0 ;$$

there exists a real number $\bar{t}$ $(0 < \bar{t} < 1)$ such that the inequalities

$$\bar{h} = \bar{h}(\bar{t}) = \frac{[(\bar{a}+\bar{b})+2\bar{b}\bar{t}]\bar{t}}{1-\bar{t}} \| \bar{x}_1 - \bar{x}_0 \| < 1 ,$$

$$\bar{d}_0 + [(\bar{a}+\bar{b})(1+\lambda)+4\bar{b}][1+\sigma(\bar{h})] \| \bar{x}_1 - \bar{x}_0 \| \leq \bar{t} < 1$$

hold.

According to suppositions 1) and 2), from Theorems 1 and 2 (paragraph 2) the equation (4,2) has a unique solution $\bar{x}^*$ in the sphere $\bar{D}$ defined by the formula

$$\bar{D} = \{ \bar{x} \in \bar{X},\ \| \bar{x} - \bar{x}_1 \| \leq \bar{h}[1+\sigma(\bar{h}^2)] \| \bar{x}_1 - \bar{x}_0 \| \} \subset \bar{X} .$$

Then the sequence $\{\bar{x}_m\}$ defined by the formula

$$\bar{x}_m = \bar{F}^2 \bar{x}_{m-1} + \delta \bar{F}(\bar{F}\bar{x}_{m-1}, \bar{x}_{m-1})(\bar{x}_m - \bar{F}\bar{x}_{m-1}) , \tag{4,5}$$

belongs to the sphere $\bar{D}$ and converges to $\bar{x}^*$. There hold the estimates:

$$\| \bar{x}^* - \bar{x}_m \| \leq \bar{h}^{2^{m-1}} [1+\sigma(\bar{h}^{2^m})] \| \bar{x}_m - \bar{x}_{m-1} \| , \tag{4,6}$$

$$\| \bar{x}^* - \bar{x}_m \| \leq \bar{h}^{2^m-1} [1+\sigma(\bar{h}^{2^m})] \| \bar{x}_1 - \bar{x}_0 \| , \tag{4,7}$$

$$\| \bar{x}^* - \bar{x}_m \| \leq \frac{[(\bar{a}+\bar{b})+2\bar{b}\bar{t}]\bar{t}}{1-\lambda} \| \bar{x}_m - \bar{x}_{m-1} \|^2 , \tag{4,8}$$

$$\| \bar{x}^* - \bar{x}_m \| \leq \frac{[(\bar{a}+\bar{b})+2\bar{b}\bar{t}]\bar{t}}{1-\lambda} \bar{h}^{2^m-2} \| \bar{x}_1 - \bar{x}_0 \|^2 . \tag{4,9}$$

Let the operator φ_0^{-1} mapping $\bar{X}$ into X' satisfy the inequality

3) $\| \varphi_0^{-1} \bar{u} - \varphi_0^{-1} \bar{v} \| \leq \mu \| \bar{u} - \bar{v} \| \qquad \bar{u}, \bar{v} \in \bar{D}$.

Finally, let the inequality

4) $\| F \varphi_0^{-1} \bar{x}^* - \varphi_0^{-1} \bar{F} \bar{x}^* \| \leq \varepsilon$

hold; then

$$\| x^* - \varphi_0^{-1} x^* \| \leq \| F x^* - F\varphi_0^{-1} \bar{x}^* \| + \| F\varphi_0^{-1} \bar{x}^* - \varphi_0^{-1} \bar{F} \bar{x}^* \| \leq$$
$$\leq \lambda \| x^* - \varphi_0^{-1} x^* \| + \varepsilon ,$$

or

(4,10) $$\| x^* - \varphi_0^{-1} \bar{x}^* \| \leq \frac{1}{1-\lambda} \varepsilon .$$

From (4,10) and supposition 3) it follows that

$$\| x^* - \varphi_0^{-1} \bar{x}_m \| \leq \| x^* - \varphi_0^{-1} \bar{x}^* \| + \| \varphi_0^{-1} \bar{x}^* - \varphi_0^{-1} \bar{x}_m \| \leq$$
$$\leq \frac{1}{1-\lambda} \varepsilon + \mu \| \bar{x}^* - \bar{x}_m \| .$$

Considering (4,6) - (4,9), we obtain estimates:

(4,11) $$\| x^* - \varphi_0^{-1} \bar{x}_m \| \leq \frac{1}{1-\lambda} \varepsilon + \mu \bar{h}^{2^{m-1}} [1 + \sigma(\bar{h}^{2^m})] \| \bar{x}_m - \bar{x}_{m-1} \| ,$$

(4,12) $$\| x^* - \varphi_0^{-1} \bar{x}_m \| \leq \frac{1}{1-\lambda} \varepsilon + \mu \bar{h}^{2^{m-1}} [1 + \sigma(\bar{h}^{2^m})] \| \bar{x}_1 - \bar{x}_0 \| ,$$

(4,13) $$\| x^* - \varphi_0^{-1} \bar{x}_m \| \leq \frac{1}{1-\lambda} \{ \mu [\bar{a} + \bar{b}) + 2 \bar{b} \bar{t}] \bar{t} \| \bar{x}_m - \bar{x}_{m-1} \|^2 + \varepsilon \} ,$$

(4,14) $$\| x^* - \varphi_0^{-1} \bar{x}_m \| \leq \frac{1}{1-\lambda} \{ \mu [(\bar{a} + \bar{b}) + 2 \bar{b} \bar{t}] \bar{t} \bar{h}^{2^m - 2} \| \bar{x}_1 - \bar{x}_0 \|^2 + \varepsilon \} .$$

We shall now make use of the precedent considerations on integral operator.

Let F be an operator defined in the space $C \langle 0,1 \rangle$ of functions $x(s)$, continuous on the interval $\langle \alpha, \beta \rangle$ (norm: $\| x \| = \max_{0 \leq s \leq 1} | x(s) |$) by the formula

$$F x(s) = \int_0^1 f(s, t, x(t))\, dt ,$$

where the function $f(s, t, u)$ is defined in the region

$$\Omega = \{ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ -\infty < u < +\infty \} .$$

We assume that in Ω $f(s, t, u)$ is a continuous function of all three variables and has continuous partial derivatives of the second order with respect to u .

The operator F maps the space $C\langle 0,1\rangle$ into $C\langle 0,1\rangle$ and has in this space the devided difference

$$\delta F(u,v)x(s)=\int_0^1 \delta f(s,t,u(t)v(t))x(t)dt\,,$$

where

$$\delta f(s,t,u(t)v(t))=\begin{cases}\dfrac{f(s,t,u(t))-f(s,t,v(t))}{u(t)-v(t)} & \text{for } u(t)\neq v(t),\\[2ex] f'_u(s,t,u(t)) & \text{for } u(t)=v(t).\end{cases}$$

$\delta F(u,v)$ satisfies the relations (2,1), (2,2), where

$$a\geq \max_{0\leq s\leq 1}\int_0^1 k(s,t)dt, \qquad b=0,$$

$$k(s,t)\geq \sup_{y\in C\langle 0,1\rangle}|f''_{yy}(s,t,y(t))|$$

(see Schmidt's paper [2(II)], paragraph 7).

The supposition (A). Let us suppose that there exists a region Ω',

$$\Omega'=\{0\leq s\leq 1,\ 0\leq t\leq 1,\ p\leq u\leq q\},$$

and a number λ $(0<\lambda<1)$ such that the relation

$$|f(s,t,u)-f(s,t,v)|\leq\lambda|u-v|$$

holds for two arbitrary points $(s,t,u)\in\Omega'$, $(s,t,v)\in\Omega'$. If we denote

$$\mathcal{L}=\mathop{E}_{u}\{u(s)\in C\langle 0,1\rangle,\ p\leq u(s)\leq q\},$$

then, of course,

$$\|Fu-Fv\|\leq\lambda\|u-v\| \qquad \text{for } u,v\in\mathcal{L}.$$

From the suppositions of Theorem 2, these concerning the existence of $\delta F(u,v)$ and the existence of the number λ being already fulfilled, we shall suppose, that the remaining postulates are satisfied, i.e. the following ones: there exists an element $x_0\in\mathcal{L}$ and a number t $(0<t<1)$

$$\|\delta F(Fx_0, x_0)\| = d_0 < 1 ,$$

$$h = \frac{at}{1-t} \|x_1 - x_0\| < 1 ,$$

$$d_0 + a(1+\lambda)[1+\sigma(h)]\|x_1 - x_0\| \leq t < 1 ,$$

where x_1 is defined by the formula

$$x_1 = F^2 x_0 + \delta F(Fx_0, x_0)(x_1 - Fx_0) .$$

Then there exists in D

$$D = \{x \in C\langle 0,1\rangle, \|x - x_1\| \leq h[1+\sigma(h^2)]\|x_1 - x_0\|\} \subset \mathcal{L} ,$$

unique solution x^* of the equation

(4,15) $$x(s) = \int_0^1 f(s,t,x(t))\,dt ,$$

x^* being the limit of the sequence $\{x_n\}_0^\infty$ defined by the formula

(4,16) $$x_{n+1} = F^2 x_n + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n) .$$

Simultaneously, we are considering the system

(4,17) $$x_i = \sum_{k=1}^{N} A_k f(t_i, t_k, x_k), \qquad i = 1, 2, \dots, N ,$$

where

$$t_k = \frac{k-1}{N}, \qquad A_k = \frac{1}{N} ,$$

which we get from (4,15) by using a quadrature formula. Let us denote the vector $(x_1, \dots, x_N)$ by the symbol $\bar{x}^{(N)}$ or $\bar{x}$. We shall omit to write the N when N is of a constant value. We can write (4,17) in the form

$$\bar{x} = \bar{F}\bar{x} ,$$

where $\bar{F}$ denotes the operator

$$\bar{F}\bar{x} = \{\sum_{k=0}^{N} A_k f(t_i, t_k, x_k)\}_1^N$$

mapping m_N into m_N.

According to the preceding paragraph there exists in the space m_N (the norm: $\|\bar{x}\| = \max_k |x_k|$) the divided difference $\delta\bar{F}(\bar{u}, \bar{v})$ which is the matrix operator given by the matrix

$$(B_{ik}) = (A_k \delta f(t_i, t_k, u_k, v_k)).$$

The relations (4,3) and (4,4) hold, where

$$\bar{a} \geq \frac{1}{2} \max_i \sum_{k=1}^{N} A_k f^i_{kk}, \qquad \bar{b} = 0.$$

From the supposition (A) it is easy to follow that the relation

$$\|\bar{F}\bar{u} - \bar{F}\bar{v}\| \leq \lambda \|\bar{u} - \bar{v}\|$$

holds in the set $\bar{\mathcal{L}}$ consisting of all the elements $\bar{u}$ of m_N satisfying the inequalities $p \leq u_i \leq q$, $i = 1, 2, \dots, N$, since

$$\|Fu - Fv\| = \max_i \Big|\sum_{k=1}^{N} A_k [f(t_i, t_k, u_k) - f(t_i, t_k, v_k)]\Big| \leq$$

$$\leq \max_i \Big\{\sum_{k=1}^{N} \frac{1}{N} |f(t_i, t_k, u_k) - f(t_i, t_k, v_k)| \leq$$

$$\leq \lambda \max_k |u_k - v_k| = \lambda \|\bar{u} - \bar{v}\|.$$

Now, we shall choose $X = C\langle 0,1\rangle$, $\bar{X} = m_N$, $X' = C'\langle 0,1\rangle$, where $C'\langle 0, 1\rangle$ is the space of functions continuous on $\langle 0,1\rangle$ and linear on intervals $\langle t_k, t_{k+1}\rangle$, $k = 1, 2, \dots N$. The operators φ_0, φ, φ_0^{-1} are defined in the usual way ([4], Chapter 14) and the relation $\|\varphi\| = \|\varphi_0\| = \|\varphi_0^{-1}\| = 1$ holds.

Let $\bar{x}_1$ be the element defined by formulae

$$\bar{x}_1 = \bar{F}^2\bar{x}_0 + \delta\bar{F}(\bar{F}\bar{x}_0, \bar{x}_0)(\bar{x}_1 - \bar{F}\bar{x}_0), \quad \bar{x}_0 = \varphi x_0.$$

Let there exist a real number $\bar{t}$ $(0 < \bar{t} < 1)$ such that

$$\bar{h} = \frac{\bar{a}\,\bar{t}}{1 - \bar{t}} \|\bar{x}_1 - \bar{x}_0\| < 1,$$

$$\bar{d}_0 + \bar{a}\,(1+\lambda)\,[1+\sigma(\bar{h})]\,\|\bar{x}_1 - \bar{x}_0\| \le \bar{t} < 1 .$$

Then there exists in $\bar{D}$

$$\bar{D} = \{\bar{x} \in m_N,\ \|\bar{x} - \bar{x}_1\| \le \bar{h}\,[1+\sigma(\bar{h}^2)]\,\|\bar{x}_1 - \bar{x}_0\|\} \subset \bar{\mathcal{L}},$$

just one solution $\bar{x}^*$ of the equation (4,17), $\bar{x}^*$ being the limit of the sequence $\{\bar{x}_m\}_0^\infty$ defined by the formula

(4,18) $\quad \bar{x}_{m+1} = \bar{F}^2\bar{x}_m + \delta\bar{F}(\bar{F}\bar{x}_m\,\bar{x}_m)\,(\bar{x}_{m+1} - \bar{F}\bar{x}_m).$

We have the estimates (4,6) - (4,9) .

The supposition 3) is fulfilled with the value $\mu = 1$.

Finally, we shall estimate ε in the estimate 4). We shall define the modulus of the continuity (Lin Chŭn [3]).

Let δ, η be sufficiently small non-negative numbers.

$$\omega_x(\delta) = \sup |x(s') - x(s'')| \qquad (|s'-s''| \le \delta);$$

$$\omega_f^{(s)}(\delta) = \sup |f(s',t,u) - f(s'',t,u)| \quad (|s'-s''| \le \delta);$$

$$\omega_f^{(t)}(\delta) = \sup |f(s,t',u) - f(s,t'',u)| \quad (|t'-t''| \le \delta);$$

$$\omega_f(\delta,\eta) = \sup |f(s,t',u') - f(s,t'',u'')| \qquad (|t'-t''| \le \delta,\ |u'-u''| \le \eta),$$

where $u \in C\langle 0,1\rangle$, $0 \le s, t \le 1$. Then

$$\varepsilon = \|F\varphi_0^{-1}\bar{x}^* - \varphi_0^{-1}\bar{F}\bar{x}^*\| = \max_s \Big|\int_0^1 f(s,t,\bar{x}^*(t))\,dt - \sum_{k=1}^N A_k \cdot f(s,t_k,\bar{x}^*(t_k))\Big| = \max_s \Big|\sum_{k=1}^N \int_{t_k}^{t_{k+1}} |f(s,t,\bar{x}^*(t)) - \sum_{k=1}^N \int_{t_k}^{t_{k+1}} f(s,t_k,\bar{x}^*(t_k))\,dt\Big| \le \max_s \sum_{k=1}^N \int_{t_k}^{t_{k+1}} |f(s,t,\bar{x}^*(t)) - f(s,t_k,\bar{x}^*(t_k))|\,dt \le \omega_f\Big(\frac{1}{N},\ \omega_{\bar{x}^*}\Big(\frac{1}{N}\Big)\Big) .$$

But $\omega_{\bar{x}^*}\left(\frac{1}{N}\right) = \sup|\bar{x}^*(s') - \bar{x}^*(s'')| = \sup\Big|\sum_{k=1}^N A_k f(s',t_k,x_k^*) - \sum_{k=1}^N A_k f(s'',t_k,x_k^*)\Big| \le \sup \sum_{k=1}^N |A_k|\,|f(s',t_k,x_k^*) - f(s'',t_k,x_k^*)| \le \omega_f^{(s)}\left(\frac{1}{N}\right)$.

By means of the easily provable inequality

$$\varepsilon \leq \omega_f\left(\frac{1}{N},\ \omega_f^{(s)}\left(\frac{1}{N}\right)\right) \leq \omega_f^{(t)}\left(\frac{1}{N}\right) + \lambda\,\omega_f^{(s)}\left(\frac{1}{N}\right),$$

we get

$$\|x^* - \varphi_0^{-1}\bar{x}_m\| \leq \frac{1}{1-\lambda}\left\{\omega_f^{(t)}\left(\frac{1}{N}\right) + \lambda\,\omega_f^{(s)}\left(\frac{1}{N}\right)\right\} + \bar{h}^{2^{m-1}}\left[1+\sigma(\bar{h}^{2^m})\right]\|\bar{x}_m - \bar{x}_{m-1}\|,$$

$$\|x^* - \varphi_0^{-1}\bar{x}_m\| \leq \frac{1}{1-\lambda}\left\{\omega_f^{(t)}\left(\frac{1}{N}\right) + \lambda\,\omega_f^{(s)}\left(\frac{1}{N}\right)\right\} + \bar{h}^{2^{m-1}}\left[1+\sigma(\bar{h}^{2^m})\right]\|\bar{x}_1 - \bar{x}_0\|,$$

$$\|x^* - \varphi_0^{-1}\bar{x}_m\| \leq \frac{1}{1-\lambda}\left\{\omega_f^{(t)}\left(\frac{1}{N}\right) + \lambda\,\omega_f^{(s)}\left(\frac{1}{N}\right) + \bar{a}\bar{t}\,\|\bar{x}_m - \bar{x}_{m-1}\|^2\right\},$$

$$\|x^* - \varphi_0^{-1}\bar{x}_m\| \leq \frac{1}{1-\lambda}\left\{\omega_f^{(t)}\left(\frac{1}{N}\right) + \lambda\,\omega_f^{(s)}\left(\frac{1}{N}\right) + \bar{a}\bar{t}\,\bar{h}^{2^m-2}\|\bar{x}_1 - \bar{x}_0\|^2\right\}.$$

Evidently,

$$\lim_{N,\,m\to\infty} \|x^* - \varphi_0^{-1}\bar{x}_m^{(N)}\| = 0 .$$

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