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HOMOLOGICAL FIXED POINT THEOREMS

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A concept generalising the Lefschetz number of a mapping is introduced and examined, leading to a fixed point theorem. It is proved that for a map $f : S^{2n} \to S^{2n}$, f^2 has a fixed point.

The Hopf-Lefschetz theorem [1, ch.XVII, § 1] states that a continuous map f of a triangulable space into itself has a fixed point if a certain numerical characteristic associated with f is nonzero. This characteristic, the Hopf index $J(f)$, may be obtained roughly as follows: f determines an endomorphism on a (sequence of) group; $J(f)$ is then the (sum of) trace of any transformation matrix describing the endomorphism.

The fundamental idea developed in the present paper is that all transformation matrices describing a given endomorphism are similar, so that there are further invariants in addition to the trace. The one considered here is intimately associated with the characteristic polynomial; if non-zero, then some k-th iterate of f has a fixed point, and we may even determine minimal k.

The suggestion is ventured that other invariants of matrix similarity (e.g. the minimal polynomial, the other elementary factors, the characteristic roots) may also prove interesting.

Of the three sections of this paper, the first two are algebraic, and preparatory in character.

1. Single groups

First, the conventions are listed. There is given an integral domain J ; by a group G we shall always mean an abelian group with J as left operators, and with finite rank over J (denoted as rank G). Similarly, a subgroup means a J-invariant subgroup, etc. A homomorphism (i.e., a J-invariant homomorphism) taking a group G into itself will be called a homomorphism of G . A maximal linearly independent subset of a group G will be called a w-base; thus, a base of G is a w-base which generates G . Note that w-bases always exist, but bases need not.

Consider a group G and a homomorphism f of G . With these we may associate - in various ways - two matrices over J ,

$$D = \operatorname{diag} \theta_i , \quad A = (\alpha_{ij}) ,$$

where the $\theta_i, \alpha_{ij} \in J$ are obtained as follows. Take any w-base $x_1, \ldots, x_n$. Since these elements are linearly independent and

$$fx_i, x_1, \ldots, x_n$$

are not, there exist $\theta_i \neq 0$, α_{ij} in J with

$$\theta_i fx_i = \Sigma_j \alpha_{ij} x_j \quad . \tag{1}$$

Thus both D, A are n-square matrices over J (n = rank G), and D is nonsingular.

The next step is to assign a special type of function to each such D, A : for any indeterminate λ over J , set

$$p(D, A; \lambda) = \det (I - \lambda D^{-1}A)$$

(I is the unit n-square matrix over J). We note that p is a nonzero polynomial in λ with coefficients in dJ , the quotient field of J , and with degree $\leq$ rank G . Naturally, the construction is void if rank G = 0 , since matrices of type 0,0 are not defined; in this case we set

$$p(\lambda) = 1$$

Lemma 1. Given G and f , the polynomial $p(D, A; \lambda)$ is independent of D, A .

Thus we may formulate

Definition 1. Define p(f) or $p(f; \lambda)$ as $p(D, A; \lambda)$.

Proof of Lemma 1. We may assume rank G $\neq$ 0 . In matric notation, relation (1) may be written as

$$Df(X) = A X$$

with X a column-vector of the x_i's . Now consider another w-base $X': x_1', \ldots, x_n'$ in G , and

$$D'f(X') = A' X'$$

with D' nonsingular diagonal. Since both X, X' are w-bases, there exist D^*, T with D^* diagonal and both nonsingular, which transform X into X' , i.e.

$$D^* X' = T X \quad .$$

Left-multiply these three relations by adjoints of D, D', D^* respectively, to obtain

$$f(\delta X) = D^a A X, \quad f(\delta X') = D'^a A' X', \quad \delta^* X' = D^{*a} T X \quad .$$

Here $\delta = D^a D = (\det D)^n \neq 0$, etc. Then $f(\delta \delta' \delta^* X')$ may be expressed in two ways, leading to

$$\delta D'^a A' D^{*a} T = \delta' D^{*a} T D^a A \quad .$$

Now continue in the quotient field dJ of J ; here $D^a = \delta D^{-1}$, etc., yielding

$$(D'^{-1} A')(D^{*-1} T) = (D^{*-1} T)(D^{-1} A) \quad .$$

But $U = D^{*-1} T$ is nonsingular, so that

$$D'^{-1}A = U(D^{-1}A)\, U^{-1}$$

and therefore also

$$I - \lambda\, D'^{-1}A' = U(I - \lambda D^{-1} A)\, U^{-1} \quad .$$

Taking determinants, $p(D', A'; \lambda) = p(D, A; \lambda)$, as was to be proved.

<u>Definition 2</u>. Let f be a homomorphism of a group G ; define

(2) $$j(f) = j(f; \lambda) = -\frac{\frac{d}{d\lambda} p(f; \lambda)}{p(f; \lambda)}$$

($d/d\lambda$ denotes algebraic differentiation of polynomials). Then $j(f; \lambda)$ is a rational function of λ with coefficients in dJ (or in J). Since $p(f; 0) = 1$, there is a formal power series expansion,

(3) $$j(f; \lambda) \approx \Sigma_0^\infty t_k \lambda^k$$

where $t_k \in d\,J$ are obtained by the division algorithm from (2); or also by formal differentiation,

$$k!\; t_k = \frac{d^k}{d\lambda^k}\; j(f; \lambda)|_{\lambda=0} \in J \quad ,$$

at least if dJ has characteristic 0 .

The $\approx$ sign in (3) merely denotes a 1-1 linear map of the rational functions in λ over J into infinite sequences of elements from dJ . As trivial examples, for $f = id$, the identity homomorphism, we may take $D = A = I$ to obtain $p(id) = \det(I - \lambda I) = (1 - \lambda)^{\text{rank } G}$,

$$j(id) = \frac{1}{1-\lambda} \text{rank } G \approx \Sigma_0^\infty \text{rank } G \cdot \lambda^k \quad .$$

If we take $f = 0$, then say $D = I$, $A = 0$, $p(0;\lambda) = 1$, $j(0;\lambda) = 0$; this is always the case if rank $G = 0$. If rank $G = 1$, then for any homomorphism f of G,

$$j(f) = \frac{a}{1 - \lambda a} , \qquad a \in dJ$$

The fundamental properties of $j(\,.\,)$ are described in the two theorems to follow. The first develops the algebraic tool needed later; the second is the basis for the topological applications.

<u>Theorem 1.</u> Let f be a homomorphism of a group G, mapping a subgroup H into itself. Then f induces homomorphisms f_H and $f_{G/H}$ of H, G/H respectively, and

$$j(f) = j(f_H) + j(f_{G/H}) .$$

Proof. Define f_H as $f|H$, the partial mapping; by assumption, f_H is a homomorphism of H. Let $h : G \to G/H$ be the natural homomorphism, define $f_{G/H}$ as $h\, f\, h^{-1} \to G/H$ G/H; since $f(H) \subset H$, $f_{G/H}$ is a single-valued homomorphism of G/H.

Now take any w-base $x_1,\dots, x_n$ in H, and a w-base of the form

$$x_1,\dots, x_n, \quad y_1,\dots, y_m$$

in G. Then $h\, y_i$ form a w-base in G/H. In the usual manner, there exist "coefficients" in J with

$$(4) \qquad \begin{aligned} \theta_i\, f\, x_i &= \Sigma_j \alpha_{ij}\, x_j + \Sigma_j \beta_{ij}\, y_j , \\ \theta'_i\, f\, y_i &= \Sigma_j \alpha'_{ij}\, x_j + \Sigma_j \beta'_{ij}\, y_j . \end{aligned}$$

Since f maps H into itself, there must be $\beta_{ij} = 0$, so that the coefficient matrix has the form

$$\begin{pmatrix} A, & 0 \\ A', & B' \end{pmatrix}$$

where matrices A and B' are square . It follows immediately that

(5) $\quad p(f;\lambda) = \det(I - \lambda D^{-1}A) \det(I-\lambda D'^{-1}B')$.

Obviously $\det(I-\lambda D^{-1}A) = p(f_H;\lambda)$; consider the second factor. In G/H we have the w-base $hy_1,\ldots,hy_m$; also $f_{G/H}\, h = h\, f$; from (4), then $\theta_i f_{G/H} hy_i = \theta_i h\, f\, y_i =$ $= \sum \alpha'_{ij} h\, x_i + \sum \beta'_{ij}\, h\, y_i = \sum \beta'_{ij}\, h\, y_i$.

Thus $p(f_{G/H};\lambda) = \det(I - \lambda D'^{-1} B')$, and (5) reduces to

$$p(f) = p(f_H)\, p(f_{G/H}) ,$$

yielding the required result immediately.

The reason for concentrating on $j(\,.\,)$ rather than $p(\,.\,)$ may now be apparent: sums are easier to work with than products - e.g. the proof of corollary 2 to theorem 2 would become unnecessarily unwieldy. On the other hand, some information may be lost in the transition from $p(\,.\,)$ to $j(\,.\,)$: thus if J has characteristic 2 and $p(f;\lambda) =$ $= 1 + \lambda^2$, then $j(f;\lambda) = 0$.

The following consequence is immediate.

Corollary. Let

$$G_1 \supset G_2 \supset \ldots \supset G_n \supset G_{n+1} = 0$$

be groups, and f a homomorphism of G_1 with $f(G_k) \subset G_k$. Denote by f_k the homomorphism of G_k/G_{k+1} induced by f . Then $j(f) = \sum_1^n j(f_k)$.

Lemma 2. Let G be a group and H the J-periodic part of G, consisting of all $x \in G$ with $\theta x = 0$ for some $\theta \neq 0$ in J. Then any homomorphism f of G maps H into itself, and $j(f) = j(f_{G/H})$.

(The proof is trivial: rank $H = 0$, so that in theorem 1 $j(f_H) = 0$.)

Thus $j(\ .\ )$ does not account for the behavior of f on the J-periodic part of G; in particular, if f maps G into H, then $j(f) = j(f_{G/H}) = j(0) = 0$. There is also a converse result:

Lemma 3. Assume J has characteristic 0. If f is a homomorphism of a group G and $j(f) = 0$, then f maps G into the J-periodic part of G. In particular, if G is J-free, then $j(f) = 0$ iff $f = 0$.

Proof. $j(f) = 0$ implies $p(f;\lambda)$ has degree 0, so that

$$p(f;\ \lambda) = p(f;\ 0) = 1$$

Thus rank $f(G) = 0$, completing the proof.

Theorem 2. Given a homomorphism f of a group G. Take D, A as in definition 1; then trace $(D^{-1}A)$ does not depend on D, A, and will be denoted by $tr(f)$. Furthermore,

$$j(f;\ \lambda) \approx \Sigma_0^{\infty} tr(f^{k+1})\ \lambda^k\ .$$

Proof. Consider f fixed, so that $p(f;\lambda)$ is a polynomial in $dJ[\lambda]$. Let $F = dJ[\lambda_1, \ldots, \lambda_n]$ be the root field of $\det(\lambda I - D^{-1}A) = 0$ (the characteristic equation of $D^{-1}A$). Then $p(f;\lambda)$ decomposes in F,

$$p(f;\lambda) = p_0\ \Pi_j(1 - \lambda\,\lambda_j)$$

with $0 \neq p_0 \in dJ$ (we may even omit all $\lambda_j = 0$). Hence

$$j(f;\lambda) = \Sigma_j \frac{\lambda_j}{1-\lambda\lambda_j} \approx \Sigma_{k=0}^{\infty}(\Sigma_{j=1}^{n} \lambda^{k+1})\lambda^k$$

and obviously $\Sigma_j \lambda_j^{k+1}$ = trace $(D^{-1}A)^{k+1} = tr(f^{k+1})$;

this completes the proof.

<u>Corollary 1</u>. $j(f;\ 0) = tr(f)$;

$$j(f^2;\ \lambda^2) = \frac{1}{2\lambda}(j(f;\ \lambda) - j(f;\ -\lambda)) \ ;$$

if $f : G \approx G$ is an isomorphism,

$$\lambda j(f;\lambda) + \frac{1}{\lambda}\ j(f^{-1};\ \frac{1}{\lambda}) = \text{rank } G \ .$$

There are direct consequences. The next corollary will be needed later.

<u>Corollary 2</u>. The following assertions are equivalent:

1^o there is an m such that $j(f^m) = j(f^{m+k})$ for all $k \geq 0$ and

2^o $\quad j(f;\lambda) = \dfrac{c}{1-\lambda}$, $\quad c \in J$.

Proof. 2^o implies $tr(f^k) = c$ for all $k \geq 1$, and in particular,

$$j(f^m;\ \lambda) = \frac{c}{1-\lambda} \ ,$$

independent of m ; a fortiori, 1^o .

Assume 1^o . Then in particular $tr(f^m) = tr(f^{m+k})$; with the notation used in the proof of theorem 2 ,

$$\Sigma_j \lambda_j^m = \Sigma_j \lambda_j^{m+k}$$

where we may assume all $\lambda_j \neq 0$. Now collect all equal λ'_j s , so that $\Sigma_t m_t \lambda_t^m = \Sigma_t m_t \lambda_t^{m+k}$ with distinct

λ_r and positive integers m_r. Also, omit all λ_j with $\lambda_j = 1$. Then we obtain

$$\Sigma_r \, m_r \, \lambda_r^m \, (\lambda_r^k - 1) = 0 \quad .$$

Choose $k = 1, 2, \ldots,$ (number of λ_r' s). It is easily shown that

$$\det(\lambda_r^k - 1) = \prod_r (\lambda_r - 1) \, . \, \prod_{r<s} (\lambda_r - \lambda_s) \neq 0 \; ;$$

therefore all $m_r \lambda_r^m = 0$, i.e. all $m_r \cdot 1 = 0$. Thus the characteristic of J divides all m_r, i.e. all m_t except that corresponding to $\lambda_{t'} = 1$. Thus, finally,

$$j(f;\lambda) \approx \Sigma_{k=0}^{\infty} (\Sigma_j \lambda_j^{k+1}) \lambda^k = \Sigma_{k=0}^{\infty} (\Sigma_t \, m_t \lambda_t^{k+1}) \lambda^k =$$

$$= \Sigma_{k=0}^{\infty} m_{t'} \, \lambda^k \approx \frac{m_{t'}}{1 - \lambda}$$

as was to be proved.

2. Group sequences

A sequence of groups $\{G_q\}_{-\infty}^{\infty}$ shall mean a mapping $q \to G_q$ of the integers into a class of groups, such that $G_q = 0$ except for a finite set of q' s, i.e. essentially a finite sequence. (The conventions of section 1 are preserved; in particular, all G_q have the same integrity domain as left operators.) A lower sequence consists of a sequence of groups $\{G_q\}$ and a sequence of homomorphisms $\{\partial_q\}$ such that

$$\partial_q : G_q \to G_{q-1} \; , \qquad \partial_{q-1} \, \partial_q = 0 \; .$$

An exact sequence is a lower sequence with

$$\text{image } \partial_q = \text{kernel } \partial_{q-1} \quad .$$

Finally, a homomorphism $f : G \to G'$ of lower sequences

$$G = \{G_q, \partial_q\}, \qquad G' = \{G'_q, \partial'_q\}$$

is a sequence of homomorphisms $f = \{f_q\}$ with

$$f_q : G_q \to G'_q, \qquad \partial'_q f_q = f_{q-1} \partial_q .$$

(As a curious example, $\{\partial_q\} : \{G_q, \partial_q\} \to \{G_{q-1}, \partial_{q-1}\}$.) In the case that $G = G'$, f will again be called a homomorphism of G .

The Euler characteristic χ of a sequence of groups $G = \{G_q\}$ is defined as

$$\chi(G) = \sum_{-\infty}^{\infty} (-1)^q \operatorname{rank} G_q .$$

Definition 3 . Let $f = \{f_q\}$ be a homomorphism of a sequence of groups. The Lefschetz number of f is defined as the following element of dJ :

$$J(f) = \sum_{-\infty}^{\infty} (-1)^q \operatorname{tr}(f_q) .$$

We define the generalised Lefschetz invariant of f

(6) $$\operatorname{gli}(f) = \operatorname{gli}(f;\lambda) = \sum_{-\infty}^{\infty} (-1)^q j(f_q;\lambda) ,$$

a rational function in λ over dJ (or J) .

As an example,

$$j(\mathrm{id}) = \frac{1}{1-\lambda} \chi(G), \qquad j(0) = 0 .$$

Further results may be obtained from those of the preceding section by assembling them as prescribed in (6). Thus, from theorem 2 there follows immediately

Lemma 4. $\operatorname{gli}(f;\lambda) \approx \sum_{0}^{\infty} J(f^{k+1}) \lambda^k$.

From theorem 1 we have

$$\operatorname{gli}(f) = \operatorname{gli}(f_H) + \operatorname{gli}(f_{G/H})$$

for homomorphisms $f = \{f_q\}$ on $G = \{G_q\}$ such that f_q map subgroups $H_q \subset G_q$ onto themselves. In particular (cf. lemma 2)

$$\text{(7)} \qquad gli(f) = gli(f_{G/H})$$

where $H = \{H_q\}$ consists of the J-periodic parts of G_q .

<u>Theorem 3</u>. Let f be a homomorphism of a lower sequence $G = \{G_q, \partial_q\}$; consider the sequence of groups $G^\wedge = \{$kernel $\partial_q/$ /image $\partial_{q+1}\}$ and the homomorphism $f^\wedge$ of $G^\wedge$ induced by f . Then $gli(f) = gli(f^\wedge)$.

Proof. Define

$$B_q = \text{image } \partial_{q+1}, \quad Z_q = \text{kernel } \partial_q, \quad G_q^\wedge = Z_q/B_q .$$

Since G is lower, $B_q \subset Z_q$ and $G_q^\wedge$ is defined. Set $g_q = f_q \mid B_q$, let f_q' be induced by f_q on G_q/B_q , set $f_q^\wedge = f_q' \mid Z_q$, let f_q'' be induced by f_q' on

$$(G_q/B_q)/Z_q = G_q/Z_q .$$

From the commutativity relation it follows that this is possible. Then theorem 1 applied twice yields

$$j(f_q) = j(g_q) + j(f_q') = j(g_q) + j(f_q^\wedge) + j(f_q'') .$$

Since ∂_q maps G_q/Z_q isomorphically onto B_{q-1} , we have $j(f_q'') = j(g_{q-1})$, and thus

$$j(f_q) = j(f_q^\wedge) + (j(g_q) + j(g_{q-1})) .$$

Therefore

$$gli(f) = gli(f^\wedge) + \textstyle\sum_{-\infty}^{\infty}(-1)^q\,(j(g_q) + j(g_{q-1})) = gli(f^\wedge)$$

since, for large $|q|$, $G_q = 0$ and thus $g_q = 0$. This completes the proof.

For exact sequences kernel ∂_q = image ∂_{q+1} , $f^{\wedge} = 0$, and therefore

__Theorem 4__. For any homomorphism f of an exact sequence of groups, $gli(f) = 0$. There is a weak converse to this theorem, applying to free groups.

__Lemma 5__. If G is a lower sequence of free groups (of finite rank), and if $gli(f) = 0$ for every homomorphism f of G , then G is exact.

Proof. Assume the free lower sequence $G = \{G_q, \partial_q\}$ is not exact, so that there is a q and a generator x_1 of $G_q = [x_1, \ldots, x_n]$ with

$$\partial_q x_1 = 0 , \qquad x_1 \notin \text{image } \partial_{q+1} .$$

Now define $f_q : G_q \rightarrow G_q$ by

$$f_q x_1 = x_1 , \qquad f_q x_i = 0 \quad \text{for} \quad i > 1 ;$$

and $f_j : G_j \rightarrow G_j$ by $f_j = 0$ for $j \neq q$. It is easily seen that $f = \{f_q\}$ is a homomorphism of the lower sequence G , and

$$p(f_q) = 1 - \lambda \quad , \quad j(f_q) = \frac{I}{1 - \lambda} \quad ,$$

$$gli(f) = \frac{(-1)^q}{1 - \lambda} \neq 0 \quad .$$

Thus for lower sequences G , the generalised Lefschetz invariant gli may be considered a measure of the departure of G from exactness.

3. __Homology__

The convention in this section is that the __spaces__ X , and the __continuous maps__ f __of__ X , $f : X \rightarrow X$, belong to an admissible category for a homology theory [cf. 2, ch.I], with the further restriction that the homology groups of a space are to form a sequence of groups in the sense of section 2 . In particular, triangulable spaces and their continuous maps satisfy these conditions.

If f is a continuous map of a space X , we denote by $f_* = \{f_{*q}\}$ the associated homomorphism of the sequence of homology groups of X . Then $j(f_{*q})$ is defined, and will be denoted by $j_q(f)$; similarly $J(f_*)$ and $gli(f_*)$ are defined, and will be denoted by $J(f)$ and $gli(f)$. This would be ambiguous if X were, on its own, a group sequence, and we would speak of both say $J(f)$ and $J(f_*)$; however, this case will not occur here.

Then the Euler characteristic $\chi(X)$ and the Lefschetz number $J(f)$ assume their classical meaning [cf. 1, ch.XVII , § 1.3] . Theorem 4 has several applications. As an illustration, consider a proper triad [2, ch.1] of spaces $(A \cup B; A, B)$ and its Mayer-Vietoris sequence

$$\ldots \to H_q(A \cap B) \to H_q(A) + H_q(B) \to H_q(A \cup B) \to$$
$$\to H_{q-1}(A \cap B) \to \ldots$$

Let f be a continuous map of $A \cup B$, taking A, B into themselves. Since the Mayer-Vietoris sequence is exact, $gli(f) = 0$. Assembling terms,

$$0 = gli(f) = gli(f_{A \cup B}) - (gli(f_A) + gli(f_B)) + gli(f_{A \cap B})$$

on applying theorem 1 to be the direct sum terms. Hence

$$gli(f_{A \cup B}) + gli(f_{A \cap B}) = gli(f_A) + gli(f_B) ,$$

the generalised Mayer-Vietoris formula: it reduces to the classical one on taking f = identity and multiplying through by $1 - \lambda$.

Similar arguments may be carried out for other exact sequences of homology groups. E.g.

<u>Lemma 6</u>. $gli(f_{A,C}) = gli(f_{A,B}) + gli(f_{B,C})$

for a triple $A \supset B \supset C$ of spaces and a continuous map f of A , taking B, C into themselves.

Homotopic maps f_1, f_2 of a space X have coinciding $f_{1*} = f_{2*}$, so that also $j_q(f_1) = j_q(f_2)$ and $gli(f_1) = gli(f_2)$. A related result is

<u>Lemma 7</u>. To every triangulable metric space X there is an $\varepsilon > 0$ such that if two continuous maps f_1, f_2 of X are ε-near, then $j_q(f_1) = j_q(f_2)$ for all q .

Proof. Take a triangulation of X , and let $\{U_i\}$ be the covering of X by open stars of vertices; let 2ε be the Lebesgue number of this covering.

Now take ε-near continuous maps f_1, f_2 of X . Then, in $X \times X$, each point of

$$\{[f_1 x, \ f_2 x] : x \in X\}$$

is ε-near the diagonal, so that $\{U_i \times U_i\}$ cover this set. Then $\{f_1^{-1}(U_i) \cap f_2^{-1}(U_i)\}$ cover X . It only remains to proceed as in the classical simplicial approximation theorem [2, ch.II, § 7] to obtain a common simplicial approximation g to both f_1, f_2 , whereupon $j_q(f_1) = j_q(g) = j_q(f_2)$.

Lemma 7 may also be formulated thus: consider the set of rational functions over J in the discrete topology, and the set of continuous mappings of a triangulable space with the uniform topology; then j_q is uniformly continuous.

Our main interest is in the Lefschetz-Hopf homological fixed-point theory, and specifially, with these four statements [cf.1; ch.XVII, § 1] :

1^o The Lefschetz number $J(f)$ may be computed within the chain complex (Hopf formula),

2^o It may also be computed within the weak homology groups (Betti groups modulo their periodic parts),

3^o It may also be computed within the homology groups with integers-mod 2 as coefficient group,

4^o If $J(f) \neq 0$ then f has a fixed point.

Assertion 1^o is reduced to a group-theoretic proposition, and generalised to the gli invariant in lemma 2 and formula (7). Similarly for 2^o, in theorem 3 ; in fact, this holds also for the j_q - invariants. Assertion 3^o is not group-theoretic, and will be noticed in lemma 8 .

Concerning 4^o, we may apply this result itself to obtain the following generalisation of the Hopf-Lefschetz theorem: Let f be a continuous map of a triangulable space into itself. If $gli(f) \neq 0$, then some iterate of f has a fixed point. More precisely, if the k-th coefficient $J(f^k)$ of the formal series of $gli(f ; \lambda)$ is nonzero, then f^k has a fixed point.

This generalisation is rather trivial (nevertheless, see the corollary below). A more interesting result may be obtained in conjunction with lemma 6 (with $C = \emptyset$; a formulation for triples is also possible):

Theorem 5. Let (X, Y) be a triangulable pair of spaces, and f a continuous map of X taking Y into itself. If

$$gli(f) \neq gli(f|Y)$$

then some iterate of f has a fixed point in $\overline{X - Y}$. More precisely, if $\frac{d^k}{d\lambda^k}[\,\mathrm{gli}(f;\lambda) - \mathrm{gli}(f/Y;\lambda)]_{\lambda=0} \neq 0$, then f^{k+1} has a fixed point in $\overline{X - Y}$.

<u>Corollary</u>. For every continuous map f of an even-dimensional sphere into itself, either f or f^2 has a fixed point.

Proof. The statement is manifestly true for S^0; therefore consider S^{2n} with $n > 0$. Take the integers C as coefficient group. It is well known that $H_0(S^{2n}) = C = H_{2n}(S^{2n})$, the remaining groups being trivial. Also, it is known that

$$\mathrm{tr}(f_{*0}) = 1 \;, \quad \mathrm{tr}(f_{*2n}) = d$$

where the integer d is called the degree of f [1, XVII, § 1.43]. Since the corresponding groups have rank 1, we must have

$$j_0(f) = \frac{1}{1-\lambda} \quad , \quad j_{2n}(f) = \frac{d}{1-\lambda d}$$

and therefore

$$\mathrm{gli}(f) = \frac{1}{1-\lambda} + \frac{1}{1-\lambda d} \approx \Sigma_0^{\infty}(1 + d^{k+1})\lambda^k \quad .$$

Thus <u>either</u> $1 + d \neq 0$, the first coefficient $J(f)$ is nonzero, f has fixed point; <u>or</u> $d = -1$, whereupon the second coefficient $j(f^2) = 2$, and f^2 has a fixed point.

<u>Conjecture</u>. Let f be a continuous map of a product of n simplexes and m even-dimensional spheres. Then one of f, f^2, f^4, ..., f^{2^m} has a fixed point.

For odd-dimensional spheres, the situation is also odd: it is possible that no iterate of a map f has any fixed points (e.g. in S^1 , $f(z) = e^{2\pi i \alpha} z$ with real irrational). However, then f must map onto, since otherwise it would be inessential [2, ch.XI, § 2], i.e. homotopic to a constant map c , which then has $gli(c) = j_0(c) \neq 0$: more generally, all retractions have some $j_q \neq 0$. A further generalisation of this is the following

<u>Theorem 6</u>. Let f be a continuous map of a space X , and let $f^n \to f^\infty$ uniformly with $n \to \infty$. Then $Y = f^\infty(X)$ is the set of fixed points of f, and

$$j_q(f^m) = \frac{\text{rank } H_q(Y)}{1 - \lambda}$$

for all q and $1 \leq m \leq \infty$.

Proof. First take the <u>special case</u> that f is a retraction: then $f^n = f$ for all $n \geq 1$, $f^\infty = f$, $Y = f(X)$ is indeed the set of fixed points of f . Let $i : Y \subset X$ be the inclusion map, and $g : Y \to Y$ the map induced by f ; thus $f = ig$, and $gi = id_Y$, the identity map of Y . Furthermore [2, ch.I, exercise C2] ,

$$H_q(X) = \text{image } i_{*q} + \text{kernel } g_{*q} \quad .$$

From theorem 1, then, $j_q(f) = j_q(f_1) + j_q(f_2)$ where f_1, f_2 are induced by the direct summands.

For $x \in$ image i_{*q} we have $f_1 x = f_{*q} i_{*q} y =$

$= i_{*q} g_{*q} i_{*q} y = i_{*q} y = x$, i.e. f_1 is the identity map of image i_q . Since i_{*q} is 1 - 1 ,

$$j_q(f_1) = j_q(id_Y) = \frac{\text{rank } H_q(Y)}{1 - \lambda} \quad .$$

As for the second term, take $x \in$ kernel g_{*q} ; then $f_2 x =$

$= f_{*q}\, x = i_{*q}\, g_{*q}\, x = 0$, so that $f_2 = 0$ and $j_q(f_2) = 0$. Thus finally

$$j_q(f) = j_q(f_1) + j_q(f_2) = j_q(f_1) \ ,$$

proving the special case of our theorem.

Now return to the <u>general case</u> described in the assumptions of the theorem. It is simple to show that Y is the set of fixed points of f . Obviously f^{∞} is a retraction of X to Y , so that the special case applies,

$$j_q(f^{\infty}) = \frac{\text{rank } H_q(Y)}{1 - \lambda} \quad .$$

Since $f^n \to f^{\infty}$ uniformly, $j_q(f^n) = j_q(f^{\infty})$ for all sufficiently large n ; now merely apply corollary 2 to theorem 2 . This concludes the proof of theorem 6 .

<u>Problem</u>. Prove that $j_q(f) = \dfrac{\text{rank } H_q(Y)}{1 - \lambda}$ whenever f is a continuous map of a space X , and Y is the set of fixed points of f .(That is, without assuming that f^n converges uniformly.)

As an elementary illustration to theorem 6 , consider a contraction map f of the unit ball in euclidean n-space. By the Banach theorem, f^n converges uniformly to a constant map, whose value is the unique fixed point of f . Hence

$$j_q(f) = 0 \quad \text{for} \quad q \neq 0 \ ,$$

$$\text{gli}(f) = j_0(f) = \frac{1}{1 - \lambda}$$

Finally, we shall consider the dependence of the gli characteristic on the coefficient group of the homology theory. The argument depends, essentially, on these two assertions: the invariance of homology theory theorem [2, ch.III, § 10], and our theorem 3 applied to show that gli may be computed within, say, the ordered chain complex.

Thus, consider two homology theories $\mathcal{H}$, $\overline{\mathcal{H}}$ on triangulable spaces; $\mathcal{H}$ is to have as coefficient groups the integers; the coefficient group of $\overline{\mathcal{H}}$ is G, an abelian group with an integrity domain J as left operators. Let e be the unit element of J. Let j_q, $\overline{j}_q$ and gli, $\overline{\text{gli}}$ be the corresponding characteristics of continuous maps. Then

<u>Lemma 8</u>. $\overline{\text{gli}}(f) = \text{gli}(f)\, e$

(Note that $\text{gli}(f) \neq 0 = \overline{\text{gli}}(f)$ is not excluded.)

Proof. By invariance of homology theory, $\mathcal{H}$ may be obtained from the ordered chain complex $O = \{C_q(K), \partial_q\}$ corresponding to a simplicial complex K, and $\overline{\mathcal{H}}$ may be obtained similarly from $\overline{O} = \{C_q(K) \otimes G, \partial_q\}$. Take a simplicial map f of K, and the homomorphism f_* of O induced by f. To define $j(f_{*q})$, matrices D, A over C were employed. But then De, Ae may be used to define $\overline{j}(\overline{f}_{*q})$ for the homomorphism $\overline{f}_*$ of $\overline{O}$ induced by f, and thus $\overline{j}(\overline{f}_{*q}) = j(f_{*q})\, e$.

R e f e r e n c e s

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