Karel Drbohlav

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Commentationes Mathematicae Universitatis Carolinae, Vol. 4 (1963), No. 4, 147--151

Persistent URL: http://dml.cz/dmlcz/104946

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4, 4 (1963)

CONCERNING REPRESENTATIONS OF SMALL CATEGORIES

Karel DRBOHLAV, Praha

The existence of non-concrete categories was proved by J.R. Isbell in [2]. On the other hand the well-known theorem of S. Eilenberg and S. Mac Lane (see e.g. [1] or [3]) states that every small category (the category the objects of which form a set) is concrete. The proof of this fact assigns to every object a the set A consisting exactly of all morphisms α which end in a and it may be used without any change for proving our theorems 1 and 2 .

In what follows we use the following notation. $\mathcal{C}$ is any small category, $\mathcal{C}^{\circ}$ is the set of all objects of $\mathcal{C}$, $H(a, b)$ is the set of all morphisms of $\mathcal{C}$ from the object a into the object b . For $\alpha \in H(a, b)$ and $\beta \in H(b, c)$ the product of α and β , which lies in $H(a, c)$, is written as $\alpha\beta$. In relation to this, for any mapping F from some set A into some set B and for any $a \in A$ the image of a will be denoted by aF , whereas AF means the set of all aF for all $a \in A$. $\mathfrak{m}$ is any infinite cardinality and $\mathcal{U}_{\mathfrak{m}}$ is the category of all sets X with card $X < \mathfrak{m}$ and of all mappings.

<u>Theorem 1</u>. Let card $\mathcal{C}^{\circ} < \mathfrak{m}$ and let card $H(a, b) \leq \mathfrak{m}_1$ for all objects $a, b \in \mathcal{C}^{\circ}$ and for some fixed cardinality $\mathfrak{m}_1 < \mathfrak{m}$. Then $\mathcal{C}$ is isomorphic to some subcategory of $\mathcal{U}_{\mathfrak{m}}$.

<u>Theorem 2</u>. Let card $\mathcal{C}^{\circ} < \mathfrak{m}$ and let card $H(a, b) < \mathfrak{m}$

for all objects $a, b \in \mathcal{C}^o$. If $\mathfrak{m}$ is regular then $\mathcal{C}$ is isomorphic to some subcategory of $\mathcal{U}_{\mathfrak{m}}$.

For the first irregular cardinality $\aleph_\omega$ the following is true.

Theorem 3. There exists a small category $\mathcal{C}$ with the following properties: 1) card $\mathcal{C}^o = \aleph_o$ 2) card $H(a, b) < \aleph_\omega$ for all objects $a, b \in \mathcal{C}^o$ 3) $\mathcal{C}$ is isomorphic to no subcategory of $\mathcal{U}_{\aleph_\omega}$.

Before proving it we formulate our last theorem.

Theorem 4. For any infinite cardinality $\mathfrak{m}$ there exists always a small category $\mathcal{C}$ with the following properties: 1) card $\mathcal{C}^o = \mathfrak{m}$ 2) card $H(a, b) < \aleph_o$ for all objects a, $b \in \mathcal{C}^o$. 3) $\mathcal{C}$ is isomorphic to no subcategory of $\mathcal{U}_{\mathfrak{m}}$.

Proof of the theorem 3. Let $\mathfrak{n}$ be any infinite cardinality and let W be a well-ordered set with card $W = \mathfrak{n}$. Consider a category $\mathcal{C}_{\mathfrak{n}}$ consisting of three objects a, b, c, of identity-morphisms, of some morphisms α_i, β_i, γ_i ($i \in W$) and of their products so that the following is true: 1) $H(a, b)$ is the system $\{\alpha_i\}_{i \in W}$ 2) $H(b, c)$ is the union of disjoint systems $\{\beta_j\}_{j \in W}$ and $\{\gamma_j\}_{j \in W}$ 3) $H(a, c)$ is formed by all products $\alpha_i \beta_j$ and $\alpha_i \gamma_j$ under the assumption that, by definition,

$$\alpha_i \beta_j = \alpha_i \gamma_j \tag{1}$$

holds if and only if $i < j$.

Let us suppose that F is any embedding-functor from $\mathcal{C}_{\mathfrak{n}}$ into the category $\mathcal{U}$ of all sets and of all mappings. Let $A = F(a)$, $B = F(b)$. For every $i \in W$ define B_i by the formula $B_i = \bigcup_{k \leq i} A\, F(\alpha_k)$ so that $B_i \subset B$. It is clear that $B_i \subset B_\ell$ holds for $i < \ell$ ($i, \ell \in W$). We shall prove that

$i < \ell$ implies $B_i \neq B_\ell$. Really, we have $\alpha_\ell \beta_\ell \neq \alpha_\ell \gamma_\ell$ (see(1)) and consequently $F(\alpha_\ell \beta_\ell) \neq F(\alpha_\ell \gamma_\ell)$. Hence there exists an element $x_\ell \in A$ such that $x_\ell F(\alpha_\ell) F(\beta_\ell) \neq$ $\neq x_\ell F(\alpha_\ell) F(\gamma_\ell)$. Putting $y_\ell = x_\ell F(\alpha_\ell)$ we have $y_\ell \in B_\ell$ and

(2) $$y_\ell F(\beta_\ell) \neq y_\ell F(\gamma_\ell)$$

Assume now that $y_\ell \in B_i$ holds for some $i < \ell$. Then it is possible to find $k \leqslant i$ and $x_k \in A$ such that $y_\ell =$ $= x_k F(\alpha_k)$. But $k < \ell$ and thus, by (1), it is $\alpha_k \beta_\ell = \alpha_k \gamma_\ell$. Hence $y_\ell F(\beta_\ell) = y_\ell F(\gamma_\ell)$ in contradiction to (2). Hence $y_\ell \notin B_i$. The mapping $\ell \to y_\ell$ is an injection from W into B, hence $\operatorname{card} B \geqslant \mathfrak{m}$.

This result gives us the possibility of constructing a small category $\mathcal{C}$ which satisfies conditions of our theorem 3. Consider categories $\mathcal{C}_\mathfrak{m}$ for all infinite cardinalities $\mathfrak{m} < \aleph_\omega$. Let the objects of $\mathcal{C}_\mathfrak{m}$ be denoted by $a_\mathfrak{m}$, $b_\mathfrak{m}$, $c_\mathfrak{m}$. Now, we identify all objects $b_\mathfrak{m}$ by putting $b_\mathfrak{m} = b$ and by considering sets $H(a_{\mathfrak{m}_1}, c_{\mathfrak{m}_2})$ for $\mathfrak{m}_1 \neq \mathfrak{m}_2$ as being formed by all formal products $\xi \eta$ with $\xi \in H(a_{\mathfrak{m}_1}, b)$ and $\eta \in H(b, c_{\mathfrak{m}_2})$. In this way we get a new category $\mathcal{C}$ which satisfies all conditions of theorem 3. Especially, for any embedding-functor F from $\mathcal{C}$ into $\mathcal{U}$ we have $\operatorname{card} F(b) \geqslant \mathfrak{m}$ for any $\mathfrak{m} < \aleph_\omega$ hence $\operatorname{card} F(b) \geqslant \aleph_\omega$.

Remark. A slight modification of this proof gives us an example of a category $\mathcal{C}$ which, like that of Isbell [2], is not concrete. We have only to force $\operatorname{card} F(b) \geqslant \mathfrak{m}$ for any cardinality $\mathfrak{m}$ what may be done by taking categories $\mathcal{C}_\mathfrak{m}$ for all cardinalities $\mathfrak{m}$ and by identifying their "middle" objects $b_{\mathfrak{m}}$ in a way similar to that described above.

Proof of theorem 4. Let $\mathfrak{m}$ be any infinite cardinality and let W be a well-ordered set with card $W = \mathfrak{m}$. Let the objects of $\mathcal{C}$ be any symbols a_i $(i \in W)$, b, c_j $(j \in W)$ so that card $\mathcal{C}^o = \mathfrak{m}$. Assume that each $H(a_i, b)$ consists of exactly two morphisms α_i and β_i whereas each $H(b, c_j)$ contains exactly one morphism γ_j. The sets $H(a_i, c_j)$ consist of products $\alpha_i \gamma_j$ and $\beta_i \gamma_j$ and we put, by definition,

(3) $\qquad \alpha_i \gamma_j = \beta_i \gamma_j$

if and only if $i < j$.

No other morphisms are in $\mathcal{C}$ besides identity-morphisms, of course.

Let F be any embedding-functor from $\mathcal{C}$ into the category $\mathcal{U}$ of all sets. We define to every $i \in W$ a binary relation S_i on $F(b)$ by putting $y\, S_i\, z$ if and only if there exist some $k \leq i$ and some $x_k \in F(a_k)$ such that $y = x_k F(\alpha_k)$ and $z = x_k F(\beta_k)$. It is clear that $S_i \subset S_\ell$ holds for $i < \ell$ $(i, \ell \in W)$. We shall prove that $i < \ell$ implies $S_i \neq S_\ell$. By (3) we have $\alpha_\ell \gamma_\ell \neq \beta_\ell \gamma_\ell$ and consequently $F(\alpha_\ell \gamma_\ell) \neq F(\beta_\ell \gamma_\ell)$. Hence there exists an element $x_\ell \in F(a_\ell)$ such that

(4) $\qquad x_\ell F(\alpha_\ell) F(\gamma_\ell) \neq x_\ell F(\beta_\ell) F(\gamma_\ell)$

Putting $y = x_\ell F(\alpha_\ell)$ and $z = x_\ell F(\beta_\ell)$ we have $y\, S_\ell\, z$. Assume that $y\, S_i\, z$ is true for some $i < \ell$. Then it is $y = x_k F(\alpha_k)$ and $z = x_k F(\beta_k)$ for some $k \leq i$ and for some $x_k \in F(a_k)$. But $k < \ell$ implies $\alpha_k \gamma_\ell = \beta_k \gamma_\ell$ and $x_k F(\alpha_k) F(\gamma_\ell) = x_k F(\beta_k) F(\gamma_\ell)$. Hence $y F(\gamma_\ell) = z F(\gamma_\ell)$ in contradiction to (4). It follows $\mathfrak{m} = \text{card } W \leq \text{card } (F(b) \times F(b)) = \text{card } F(b)$. The category $\mathcal{C}$ satisfies all conditions of theorem 4 .

References

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