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A NOTE ON K-POSITIVE OPERATORS

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In this paper we give some generations of the results of the third paragraph of the paper [4].

Let Y be a real Banach space, X the corresponding complex extension defined in evident way. Let Y', X' be the adjoint spaces of Y, X and let $[Y]$, $[X]$ be the spaces of linear bounded operators mapping Y, X into itself. The reader can find the necessary definitions in the paper [4]. Let $K \subset Y$ denote a productive cone and let $K' \subset Y'$ denote the adjoint cone ([2]). By the symbol $\mathfrak{A}_{\infty}(T)$ ([5]p. 292) we denote the set of complex-valued functions which have the following properties: (i) The definition domain $\Delta(f)$ is an open set in the complex plane such that $\Delta(f) \supset \sigma(T)$, where $\sigma(T)$ is the spectrum of the operator T. (ii) The function f is differentiable in $\Delta(f)$ and $f(\lambda)$ is bounded as $|\lambda| \to \infty$.

The operator $T \in [X]$ is called Radon-Nicolski operator (RN-operator), if $T = U + V$, where $V \in [X]$ and $U \in [X]$ is a compact operator such that the inequality $r(T) > r(V)$ holds for the spectral radii $r(T)$, $r(V)$.

Some assertions of the third paragraph of the paper [4] are proved by using the assumption that K is so called volume type cone and that the operator T is strongly K-positive. We can show that these assertions hold also for a class of more general operators.

If $y - x \in K$, where $x, y \in Y$, we write $x \prec y$ or $y \succ x$.

D e f i n i t i o n ([1] p. 261) The K-positive operator ($TK \subset K$) is called u_0-bounded, if there is a vector $u_0 \in K$, $u_0 \neq 0$ (0 denotes the zero vector in Y, X) and if there exist a natural p and positive α, β such that the relations

$$(1) \qquad \alpha u_0 \prec T^p x \prec \beta u_0$$

hold for any $x \in K$, $x \neq 0$.

C o r r e c t i o n [4]. In the proofs of all the theorems of the third paragraph in [4] it is assumed (besides other assumptions) that T is a closed operator for which there exists a function $f \in \mathcal{O}_\infty(T)$ such that $f(T) = U + V$ is a RN-operator and such that

$$(2) \qquad |f(\lambda)| > r(V) \qquad \text{if} \quad |\lambda| = r(T) .$$

The assumption (2) is not referred in [4] and so the corresponding proofs are not correct. We were not succeeded to prove the mentioned theorems without the assumption (2).

L e m m a 1. If $T \in [Y]$ is an u_0-bounded operator, then there exists at most one eigenvector X_0 of the operator T which belongs to the cone K.

P r o o f . Let us assume that $v_1, v_2 \in K$ are two independent eigenvectors of T :

$$(3) \qquad T v_1 = \mu v_1 , \qquad T v_2 = \nu v_2$$

and let

$$(4) \qquad \mu > \nu \geq 0 .$$

From [1] p. 262, lemma 2.2 , it is known that T is also

v_1-bounded and so there exist $p = p(x), \alpha = \alpha(x), \beta = \beta(x)$, such that

$$\alpha v_1 \dashv T^p v_2 \dashv \beta v_1$$

From (3) we deduce $T^p v_2 = \nu^p v_2$ and we obtain the relations

$$(5) \qquad \alpha \mu^n v_1 \dashv \nu^{p+n} v_2 \dashv \beta \mu^n v_1 ,$$

which hold for every natural $n \geqq 1$. It follows from (5) that

$$v_1 \dashv \frac{\nu^p}{\alpha} \left(\frac{\nu}{\mu}\right)^n v_2 .$$

But $(\nu/\mu)^n$ converges to zero by (4) if $n \to \infty$. So $v_1 \dashv 0$ in contrary to $v_1 \neq 0$. Let us assume $\nu = \mu$. But in this case the vectors of the form $v_1 - t v_2$, t real, have the following property ([1] p. 242): There is a t_0 for which

$$v_1 - t v_2 \in K \text{ if } t \leqq t_0 ; \quad v_1 - t v_2 \bar{\in} K \text{ if } t > t_0 .$$

We have already used the fact that the operator T is v_2-bounded, so that

$$T^p (v_1 - t_0 v_2) \vdash \alpha (v_1 - t_0 v_2) v_2 .$$

This relation shows that $v_1 - (t_0 + \alpha/\mu^p) v_2 \in K$ and this is a contradiction to the definition of t_0. The lemma 1 is then proved.

D e f i n i t i o n . An u_0-bounded operator T is called strongly u_0-bounded, if there exists numbers $p = p(x)$ (natural), $\gamma = \gamma(x)$, (real) such that the relation

$$(6) \qquad \gamma T^p x \dashv u_0$$

holds for every vector $x \in Y$.

L e m m a 2. Let T be an u_0-bounded operator and let x_0 be a K-positive eigenvector $(x_0 \in K)$ of the operator

T corresponding to the eigenvalue μ_0 . Then the power T^k is also x_0-bounded for all natural k .

Proof is evident.

T h e o r e m 1. Assumptions:

1. The operator T is strongly u_0-bounded.
2. There is a function $f \in \mathcal{O}_{\infty}(T)$ such that $f(T) = U + V$ is RN-operator and such that the inequality (2) holds.

Then there exists one and only one eigenvector $x_0 \in K$ of the operator T . The eigenvalue μ_0 corresponding to this eigenvector x_0 is positive, simple and dominant, i.e. the inequalities

(7) $$|\lambda| < \mu_0$$

hold for $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$.

To the eigenvalue μ_0 corresponds an eigenvector $x_0' \in K'$ of the adjoint operator T' and this vector is a strongly positive form, i.e.

$$x_0'(X) > 0 \text{ if } x \in K , x \neq 0 .$$

P r o o f . From the paper [4], theorem 3.2 , it follows the existence of an eigenvalue $\mu_0 > 0$ of the operators T, T' such that

$$|\lambda| \leqq \mu_0 , \quad |\bar{\lambda}| \leqq \mu_0$$

for $\lambda \in \sigma(T)$, $\bar{\lambda} \in \sigma(T')$ and the existence of eigenvectors $x_0 \in K$, $x_0' \in K'$ of the operators T, T' corresponding to the value μ_0 .

From the lemma 2.2 of [1] p.262 and from the x_0-boundedness of T it follows that

$$\alpha x_0'(x_0) \leqq x_0'(T^p x) = \mu_0^p x_0'(x) \leqq \beta x_0'(x_0)$$

and from these relations we deduce the following result

$$x_0'(x_0) = 0 \Longleftrightarrow x_0'(x) = 0 \quad \text{for all} \quad x \in K .$$

Thus $x_0'(x_0)$ must be positive and therefore $x_0'(x) > 0$ if $x \in K$, $x \neq 0$.

We shall prove that the eigenvalue μ_0 is simple. Let $x_0 \in K$, $v \in Y$ be two independent eigenvectors of the operator T corresponding to the eigenvalue μ_0. Then we have by the assumption 1 that $x_0 - \gamma' v \in K$, so that $x_0 - \gamma' v = z$, $\gamma' \leqq \alpha(x_0)\gamma(\mu_0)^{p(v)-p(x_0)}$ is an eigenvector of the operator T corresponding to the value μ_0. Thus by lemma 1 $z = \eta x_0$, where η is a real constant and therefore $v = \eta x_0$.

If there is a vector $y \in Y$ such that

$$(T - \mu_0 I)^{r-1} y \neq 0 , \quad (T - \mu_0 I)^r y = 0$$

for some $r > 1$, then the vector $z = (T - \mu_0 I)^{r-1} y$ is an eigenvector of the operator T corresponding to the eigenvalue μ_0. Thus $z = \eta x_0$, where $x_0 = \mu_0^{-1} T x_0$, $x_0 \in K$, $x_0 \neq 0$. The above considerations give $x_0'(x_0) > 0$ where $x_0' = \mu_0^{-1} T' x_0'$, $x_0' \in K'$, $x_0' \neq 0$. Thus

$$0 < \left| \frac{1}{\eta} x_0'(x_0) \right| = |x_0'(z)| = |[(T' - \mu_0 I')^{r-1} x_0'](y)| = 0$$

and this contradiction proves the simplicity of the eigenvalue μ_0 in regard to T.

Similarly we shall prove the simplicity of the eigenvalue μ_0 with regard to T'. We prove that every eigenvector v' of the operator T' corresponding to μ_0 has the form $\eta x_0'$, where $x_0' = \mu_0^{-1} T' x_0'$, $x_0' \in K$, $x_0' \neq 0$ for some suitable real η. Let us assume that x_0' and v' are linearly independent.

On the unit sphere $S_1 = \{x \mid x \in X , \|x\| = 1$ we have

(8) $z'_t(x) = x'_o(x) - t\, v'(x) \geqq 0$ for all real $t \leqq t_o$, where $|t_o| < \infty$.

From the inequalities

$$\alpha\, z'_t(x_o) \leqq \mu_o^p\, z'_t(x) \leqq \beta\, z'_t(x_o)$$

it follows that either $z'_t(x) = 0$ for all $x \in K$, or $z'_t(x) > 0$ for $x \in K$, $\|x\| = 1$ and thus $z'_t(x) > 0$ for $x \in K$, $x \neq 0$. But the first possibility is in contradiction to the assumption of linear independence of x'_o, v'. Thus $z'_t(x) > 0$ for $x \in K$, $x \neq 0$. Let us assume that t_o is such that $z'_t \in K$ for $t \leqq t_o$ and

(9) $\quad z'_t \bar{\in} K'$ for $t > t_o$.

Let $\|x'_o\|$, $\|v'\|$ be the norms of the forms x'_o, v'. Then we have by (8)

(10) $$z'_o(x) - \frac{\alpha}{\mu_o^p}\, z'_o(x_o)\, \frac{v'(x)}{\|v'\|} \geqq 0 \text{ for } x \in K,$$

$x \neq 0$,

where $z'_o = z'_{t_o}$. The relation (10) implies

$$x'_o(x) - t_o\, v'(x) - \frac{\alpha}{\mu_o^p}\, \frac{z'_o(x_o)}{\|v'\|}\, v'(x) \geqq 0,$$

or $x'_o - \{[t_o + (\alpha/\mu_o^p)]\, (z'_o(x_o)/\|v'\|\}\, v' \in K$ which is impossible for (9). The linear independence of x'_o, v' is false. Thus $v' = \eta' x'_o$ for some real η'.

Let y' be a form lying in Y' such that $v' = (T' - \mu_o I')^{r-1} y' \neq 0$, $(T' - \mu_o I')^r y' = 0$ for some $r > 1$ (I' - denotes the identity-operator mapping Y' onto itself). It is evident that v' is an eigenvector of the operator T' corresponding to the eigenvalue μ_o. We then have

$$0 < |\eta' x'_o(x_o)| = |v'(x_o)| = |[(T' - \mu_o I')^{r-1} y']\, (x_o)| = 0.$$

The simplicity of the value μ_o with regard to T' is also proved.

To prove the strong inequality (7) let us assume the contrary, i.e. let $|\nu| \in \sigma(T)$ be an eigenvalue such that $|\nu| = \mu$. Let us put $\nu = \mu_o e^{i\varphi}$ and let us denote $v = v_1 + i v_2$, $v_1, v_2 \in Y$ the corresponding eigenvector: $Tv = \nu v$.

We shall investigate two cases:

Case A. There is a positive integer q such that $\nu^q = \mu_o^q$. Then $T^q v = \mu_o^q v$ and therefore the eigenvectors x_o, v lie in the eigenmanifold of the operator T^q corresponding to μ_o^q. From this it follows that either v is a real vector, or both the vectors v_1, v_2 are also eigenvectors of T. From the strong x_o-boundedness of T we obtain (u denotes one of the vectors v, v_1, v_2) that

(11) $\gamma T^p u \to x_o$ for the real u.

When the vector u is a real eigenvector of the operator T, we deduce from (11) that

$$0 \to x_o - \gamma T^p u = x_o - \gamma \nu^p u = z_o .$$

Thus z_o is K-positive eigenvector of the operator T^q corresponding to the eigenvalue μ_o^q. By the lemma 2 T^q is x_o-bounded and thus z_o is a real multiple of x_o. Thus $\nu = \mu_o$ in the case A.

Case B. There does not exist a natural q such that $\nu^q = \mu_o^q$. Let us investigate the operator $W = T + \varepsilon T^2$, where $\varepsilon > 0$. Then $\tau = \nu + \varepsilon \nu^2$ is an eigenvalue of the operator W. Evidently it is $\nu + \varepsilon \nu^2 = \mu_o \{\cos\varphi + \varepsilon \mu_o \cos 2\varphi + i \sin\varphi + i\varepsilon \sin 2\varphi\}$ from which

$$|\tau| = |\nu + \varepsilon \nu^2| = \mu_o (1 + \varepsilon^2 \mu_o^2 + 2\varepsilon \mu_o \cos\varphi)^{\frac{1}{2}} .$$

Let $\varepsilon > 0$ be such that $\nu + \varepsilon \nu^2 = |\tau| \exp\{i\psi\}$, where $\psi = 2\pi/k$ for some natural k. Then it will be $\tau^k = (\mu_0 + \varepsilon \mu_0^2)^k$ and thus by the case A $\nu = \eta x_0$ for some real constant η.

The assumption that there is an eigenvalue ν for which $|\nu| = \mu_0$ is false and thus the inequality (7) holds. The theorem 1 is proved.

R e m a r k . The Theorem 1 generalizes the theorem 3.4 of the paper [4], since every strongly K-positive operator T is also strongly u_0-bounded, where u_0 is an arbitrary vector of the interior of the volume-type cone K ([1] p.267).

Also the assertion 3.5 of the paper [4] can be generalized.

Let $T \in [Y]$ and let

$$(12) \qquad R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_0)^k T_k + \sum_{k=1}^{\infty} (\lambda - \mu_0)^{-k} B_k$$

be the Laurent expansion of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ in the neighborhood of the isolated singularity $\mu_0 \in \sigma(T)$. It is known ([5]p.305) that $T_k \in [X]$, $k = 0, 1, \ldots$ and

$$B_1 = \frac{1}{2\pi i} \int_\Gamma R(\lambda, T) \quad , \quad B_{k+1} = (T - \mu_0 I) B_k \quad , \; k = 1, 2, \ldots,$$

where Γ is the boundary of a circle C having the property $\bar{C} \cap \sigma(T) = \{\mu_0\}$ ($\bar{C}$ - denotes the closure of C).

From the Theorem 1 it follows that B_k, $k = 2, 3, \ldots$ in the expansion of the resolvent (12) of a strongly u_0-bounded operator for which $f(T) = U + V$ is a RN-operator, where $f \in \mathfrak{A}_\infty(T)$ and f fulfils the inequalities (2), are zero-operators. Moreover it holds the following

T h e o r e m 2 . Let us assume that

1. T is a strongly u_0-bounded operator.

2. There is a function $f \in \mathcal{O}_{\infty}(T)$ such that $f(T) = U + V$ is a RN-operator.

3. For the function f the inequalities (2) are fulfilled.

Then the operator B_1 in the expansion (12) of the resolvent $R(\lambda, T)$ is also strongly u_0-bounded.

Proof. Let μ_0 be the eigenvalue for which $|\lambda| < \mu_0$ if $\lambda \in \sigma(T)$. It is known ([5] p. 306) that B_1 is a projector. Thus $B_1 = B_1^k$ for arbitrary $k \geq 1$.

From the relations

$$\alpha(x)\, u_0 \prec T^{p(x)}\, x \prec \beta(x)\, u_0 , \quad x \in K , \quad x \neq 0 ,$$

it follows ($p_1 = p(B_1 x)$, $\alpha(x_2) > 0$, $\beta(x_2) > 0$, since $\mu_0^{-n} T^n \to B_1$ ([3]), $x_2 = B_1 x$)

$$\alpha(B_1 x)\, u_0 \prec T^{p_1}\, B_1 x \prec \beta(B_1 x)\, u_0 .$$

But $\mu_0^{-p}\, T^p B_1 = \mu_0^{-p}\, B_1 T^p = B_1$ for arbitrary natural p.

Therefore

$$\frac{\alpha(B_1 x)}{\mu_0^p}\, u_0 \prec B_1 x \prec \frac{\beta(B_1 x)}{\mu_0^p}\, u_0 ,$$

which proves the u_0-boundedness of the operator B_1.

The strongly u_0-boundedness of the operator B_1 it follows from this same argument and from the relations

$$u_0 \succ \gamma(B_1 x)\, \mu_0^{-p}\, T^p B_1 x = \gamma_1 B_1 x .$$

We have just proved that the vector $y = B_1 x$, where $x \in K$, $x \neq 0$, is an eigenvector of the operator T corresponding to the eigenvalue μ_0.

This property is very important to the construction of the eigenelements μ_0, x_0 of the operator T by the Kellogg's iterative method (see [3]).

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