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AN EXTENSION THEOREM FOR SEPARATELY CONTINUOUS FUNCTIONS AND ITS APPLICATION TO FUNCTIONAL ANALYSIS x)

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1. Let S be a family of continuous functions on a topological space T. Consider the value of an $s \in S$ at the point $t \in T$ as a function $f(s, t)$ of two variables on the Cartesian product $S \times T$. If S is taken in the topology of pointwise convergence, the function f will be separately continuous.

2. Every completely regular topological space P possesses a natural extension, a locally convex topological linear space, constructed in the following manner. Denote by $C_\beta(P)$ the Banach space of all bounded continuous functions on P and take the dual space $C_\beta(P)'$ in the weak-star topology $\sigma(C_\beta(P)', C_\beta(P))$. Then P may be considered as a subset of $C_\beta(P)'$.

3. The main problem. Consider a bounded separately continuous function $f(s, t)$ on the product $S \times T$ of two completely regular topological spaces. Now, $S \times T$ is imbedded in the linear space $C_\beta(S)' \times C_\beta(T)'$. Under what conditions may f be extended to a separately continuous bilinear form on $C_\beta(S)' \times C_\beta(T)'$?

4. We say that a function f on $S \times T$ satisfies the

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double limit condition on $S \times T$ if it is impossible to find two (countable!) sequences $s_i \in S$ and $t_j \in T$ such that both $\lim_i \lim_j f(s_i, t_j)$ and $\lim_j \lim_i f(s_i, t_j)$ exist and are different from each other.

5. The main theorem. *Let S and T be two completely regular topological spaces and f a bounded separately continuous function on $S \times T$. There exists a separately continuous bilinear form on $C_\beta(S)' \times C_\beta(T)'$ which extends f if and only if f satisfies the double limit condition on $S \times T$.*

This theorem permits us to obtain statements about the (non metrizable) weak topology of a Banach space from assumptions of a countable character. It contains e.g. the theorems of Krein and Eberlein. A weaker version of this theorem is already contained in [9]. The proof of the main theorem is based on the combinatorial lemma on convex means [8] . The reader is refered to [9] for all information and notation connected with this lemma and its application to problems concerning weak compactness.

6. The following lemma will be used in the proof of the main theorem.

(2,1) *Let X and Y be two completely regular topological spaces and $B(x, y)$ a separately continuous function on $X \times Y$. Suppose that B is bounded on $X \times Y$. Let us define a mapping h of X into $C_\beta(Y)$ and a mapping k of Y into $C_\beta(X)$ by the relation*

$$\langle h(x), y \rangle = \langle x, k(y) \rangle = B(x, y) \quad .$$

Suppose further that B satisfies the double limit condition on $X \times Y$. Let $R \subset X \subset C_\beta(X)'$ and suppose that $r_0 \in C_\beta(X)'$

belongs to the closure of R. Let $\varepsilon > 0$. Then there exists a convex mean $\sum_{r \in R} \lambda(r)\, r$ such that

$$|\langle \sum \lambda(r) r - r_0 , \ k(Y) \rangle | \leq \varepsilon$$

Proof: Let W be the subset of $R \times Y$ where

$$| \langle r - r_0 , \ k(y) \rangle | \geq \frac{\varepsilon}{4}$$

and let M be the subset of $R \times Y$ where

$$| \langle r - r_0 , \ k(y) \rangle | < \frac{\varepsilon}{8}$$

Let $\mathcal{W}$ be the system of all sets $W(y)$ with $y \in Y$. Let β be such that $|B(x, y)| \leq \beta$ on $X \times Y$. Suppose that $M(R, \mathcal{W}, \frac{\varepsilon}{8\beta})$ is empty; it follows from Theorem (3.1) of [9] that there exist two sequences $r_n \in R$ and $y_n \in Y$ such that

$$r_n \in M(y_1) \cap \dots \cap M(y_{n-1}) \cap W(y_n) \cap W(y_{n+1}) \cap \dots$$

so that the double limit condition is violated on $R \times Y$. There exists, accordingly, a $\lambda \in M(R, \mathcal{W}, \frac{\varepsilon}{8\beta})$. We have, for $y \in Y$,

$$|\langle \sum_{r \in R} \lambda(r)\, r - r_0 , \ k(y) \rangle | \leq \sum_{r \in R} \lambda(r) \, | \langle r - r_0 , k(y) \rangle | =$$

$$= \sum_{r \in W(y)} + \sum_{r \in R - W(y)} \leq \frac{\varepsilon}{8\beta} \, 2\beta + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

(2,2) The extension theorem. Let S, T be two completely regular topological spaces and let $B(s, t)$ be a separately continuous function on $S \times T$. Suppose that B is bounded and that it satisfies the double limit condition on $S \times T$. Then B may be extended to a separately continuous bilinear form on $C_\beta(S)' \times C_\beta(T)'$.

Proof: I. We define first a mapping h of S into $C_\beta(T)$

and a mapping k of T into $C_\beta(S)$ by the relation

(1) $\langle h(s), t \rangle = \langle s, k(t) \rangle = B(s, t)$

If $p \in C_\beta(S)'$ define a function $k'(p)$ on T by the relation

(2) $\langle k'(p), t \rangle = \langle p, k(t) \rangle$

Let us show that $k'(p)$ is continuous on T. Indeed, suppose that $M \subset T$ and $t_o \in T$ belongs to the closure of M and that $|\langle k'(p), m - t_o \rangle| \geqq \varepsilon$ for all $m \in M$ and some $\varepsilon > 0$. Divide the set M into two parts $M^{(+)}$ and $M^{(-)}$ according to the sign of $\langle k'(p), m - t_o \rangle$. Since t_o has to belong to the closure of one of them, we may clearly assume that t_o is in the closure of $M^{(+)}$. According to (2,1) there exists a convex mean $\sum_{m \in M} \lambda(m)\, m$ such that $|\langle h(S), \sum \lambda(m)\, m - t_o \rangle| \leqq \frac{\varepsilon}{2|p|}$ whence

$|\langle S, \sum \lambda(m)\, k(m) - k(t_o) \rangle \leqq \frac{\varepsilon}{2|p|}$. It follows that

$|\langle p, \sum \lambda(m)\, k(m) - k(t_o) \rangle \leqq \frac{\varepsilon}{2}$. This is a contradiction since $\langle k'(p), m - t_o \rangle \geqq \varepsilon$ for each $m \in M^{(+)}$ whence $\langle p, \sum \lambda(m)\, k(m) - k(t_o) \rangle =$

$= \langle k'(p), \sum \lambda(m)\, (m - t_o) \rangle = \sum \lambda(m) \langle k'(p), m - t_o \rangle \geqq \varepsilon$

It follows that k' is a mapping of $C_\beta(S)'$ into $C_\beta(T)$. By (2) and (1), we have

$$\langle k'(s), t \rangle = \langle s, k(t) \rangle = \langle h(s), t \rangle$$

so that K' is an extension of h.

II. In the same manner we obtain a mapping h' of $C_\beta(T)'$ into $C_\beta(S)$ defined by

(3) $\langle s, h'(q) \rangle = \langle h(s), q \rangle$

III. Now let $p \in C_\beta(S)'$, $q \in C_\beta(T)'$. Since $h'(q) \in C_\beta(S)$, the expression $\langle p, h'(q) \rangle$ has a meaning; similarly, $\langle k'(p), q \rangle$ also may be defined. If we show that

(4) $\langle k'(p), q \rangle = \langle p, h'(q) \rangle$

it will be sufficient to put $B^*(p, q) = \langle k'(p), q \rangle$ to have the desired extension. Indeed, $B^*(s, t) = \langle k'(s), t \rangle = \langle s, k(t) \rangle = B(s, t)$ by (2) and (1) . If p is fixed, we have $k'(p) \in C_\beta(T)$ so that $k'(p)$ is continuous on $C_\beta(T)'$. If q is fixed, we have $h'(q) \in C_\beta(S)$ so that $h'(q)$ is continuous on $C_\beta(S)'$.

IV. To prove (4), suppose that $|p| \leqq 1$, $|q| \leqq 1$ and let $\varepsilon > 0$ be given. Let V be the set of all linear combinations $\sum \omega_i s_i$ with $\sum |\omega_i| \leqq 1$ so that V is dense in the unit ball of $C_\beta(S)'$. Let R be the set of those $v \in V$ for which

(5) $|\langle v - p, h'(q) \rangle \leqq \varepsilon$

so that p belongs to the closure of R . Let us show now that it is sufficient to find a $v \in R$ such that

(6) $|\langle v - p, k(T) \rangle| \leqq \varepsilon$

Indeed, we have by (2) and (6)

$$|\langle k'(p), t \rangle - \langle k'(v), t \rangle| = |\langle p - v, k(t) \rangle| \leqq \varepsilon$$

for all $t \in T$ whence

(7) $|\langle k'(p), q \rangle - \langle k'(v), q \rangle| \leqq \varepsilon$

Since $v \in V$ and k' is an extension of h , we have further

$$\langle k'(v), q \rangle = \langle h(v), q \rangle = \langle v, h'(q) \rangle$$

which, together with (7), yields

(8) $|\langle k'(p), q \rangle - \langle v, h'(q) \rangle| \leqq \varepsilon$

On the other hand, $v \in R$ so that, by (5)

$$|\langle v, h'(q)\rangle - \langle p, h'(q)\rangle| \leq \varepsilon$$

and this, combined with (8) gives

$$|\langle k'(p), q\rangle - \langle p, h'(q)\rangle| \leq 2\,\varepsilon$$

V. The proof will be complete if we show that there exists a $v \in R$ such that

$$|\langle v - p, \; k(T)\rangle| \leq \varepsilon$$

Since p belongs to the closure of R, it follows from (2,1) that there exists a convex mean $\sum_{r \in R} \lambda(r) r$ with $|\langle \sum \lambda(r)\, r - p, \; k(T)\rangle| \leq \varepsilon$ or there exist two sequences r_i, t_j with

(9) $\quad r_n \in M(t_1) \cap \ldots M(t_{n-1}) \cap W(t_n) \cap W(t_{n+1}) \cap \ldots$

where M and W are the subsets of $R \times T$ where $|\langle r - p, k(t)\rangle|$ is respectively $< \frac{1}{2}\varepsilon$ and $\geq \varepsilon$.

If we show that (9) is impossible it will be sufficient to take $v = \sum \lambda(r)\, r$.

Now let t_j^* be a subsequence of t_j and $t_o \in C_\beta(T)'$ an accumulation point of the sequence t_j^* such that

(10) $\quad \lim \langle h(r_i), t_j^*\rangle = \langle h(r_i), t_o\rangle \quad$ for each i

and

(11) $\quad |\langle k'(p), t_j^* - t_o\rangle| \leq \frac{1}{8}\,\varepsilon \qquad$ for each j

By (2,1) there exists a convex mean $\sum \lambda_j t_j^*$ such that

(12) $\quad |\langle h(S), \sum \lambda_j t_j^* - t_o\rangle| \leq \frac{1}{8}\,\varepsilon$

Let i be given. It follows from (9) that, for large j,

$$|\langle r_i - p,\ k(t_j)\rangle| \geqq \varepsilon$$

or, which is the same

$$|\langle h(r_i),\ t_j\rangle - \langle k'(p),\ t_j\rangle| \geqq \varepsilon\ ;$$

this, together with (10) and (11), yields

$$(13)\quad |\langle h(r_i),\ t_o\rangle - \langle k'(p),\ t_o\rangle| \geqq \frac{7}{8}\,\varepsilon$$

Now let i be greater than any of the indices of the t_s which occur in the expression $\Sigma\,\lambda_j\,t_j^*$. It follows from (9) that, for $i > s$,

$$|\langle r_i - p,\ k(t_s)\rangle| < \frac{1}{2}\,\varepsilon$$

or

$$|\langle h(r_i),\ t_s\rangle - \langle k'(p),\ t_s\rangle| \leqq \frac{1}{2}\,\varepsilon\ ;$$

together with (11), we have

$$|\langle h(r_i),\ t_s\rangle - \langle k'(p),\ t_o\rangle| \leqq \frac{5}{8}\,\varepsilon$$

so that it follows from (12)

$$|\langle h(r_i),\ t_o\rangle - \langle k'(p),\ t_o\rangle| \leqq \frac{6}{8}\,\varepsilon$$

which is a contradiction with (13). The proof is complete.

To conclude, let us point out some questions arising in connection with the present remark. A systematic study of the convex extension of a given topological space seems to be indicated. Also, it would be interesting to obtain more information about the inductive topology of the cartesian product $S \times T$, i.e. the topology which yields as continuous functions, exactly the system of all separately continuous functions.

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