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A NOTE ON EPIMORPHISMS IN ALGEBRAIC CATEGORIES

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In this note the notion of a category will be used in the sense of [1]. Thus, the product of $\alpha : a \to b$ and $\beta : b \to c$ will be denoted by $\alpha\beta : a \to c$. If $\alpha\beta = \alpha\gamma$ always implies $\beta = \gamma$ as far as $\alpha\beta$ and $\alpha\gamma$ are defined, the morphism α is called an epimorphism.

Most categories we meet in practice satisfy the following conditions:

1) their objects are some ordered couples of sets,

2) every morphism $\alpha : (A, A') \to (B, B')$ is a mapping from A into B,

3) if $\alpha : (A, A') \to (B, B')$ and $\beta : (B, B') \to (C, C')$ then we get $\alpha\beta$ by composing the mappings α and β.

For example every group G can be taken for a couple (A, A') where A is the underlying set of G and A' a suitable subset of $A \times A \times A$ determining the structure of G. In this way the category of groups is a category of the type mentioned above. But it is quite clear that the same is true e.g. for any algebraical category, for any category of topological spaces etc.

Now let $\mathcal{C}$ be a category satisfying our three conditions 1), 2) and 3), and let $\alpha : (A, A') \to (B, B')$. Then, if α maps A onto B, it is an epimorphism, as it can be proved in an easy way.

For our category $\mathcal{C}$ we can try to formulate a "converse theorem":

4) If $\alpha : (A, A') \longrightarrow (B, B')$ is an epimorphism, then it maps A onto B.

This converse theorem, however, is not true in general. But it is true e.g. in the category of all groups, as it is mentioned in [1], § 6, No 5. But there is a brief note in this paper, too, that it holds in a corresponding way in other algebraic categories, what may make the reader believe, that it is true e.g. in the category of all semi-groups or in the category of all rings, etc. This is of course not the case as we shall show by the following simple examples.

Let $\mathcal{S}$ be the category of all semigroups. Consider two real intervals $A = (0, 1\rangle$ and $B = (0, \infty)$ and let (A, A') and (B, B') denote the corresponding multiplicative semigroups. Let α be the identity mapping of A into B. Then $\alpha : (A, A') \longrightarrow (B, B')$ is an epimorphism in $\mathcal{S}$ though it does not map A onto B.

Really, let us have any semigroup (C, C'), let $\beta : (B, B') \rightarrow (C, C')$ and $\gamma : (B, B') \longrightarrow (C, C')$ be any two morphisms in $\mathcal{S}$ and let $\alpha\beta = \alpha\gamma$. If $\beta \neq \gamma$, then $x^\beta \neq x^\gamma$ for some $x \in B$, $x > 1$. Because β and γ are homomorphisms, we have

$$[x^\beta (\tfrac{1}{x})^\beta]\, x^\gamma = 1^\beta \,.\, x^\gamma = 1^{\alpha\beta} \,.\, x^\gamma = 1^{\alpha\gamma} \,.\, x^\gamma = 1^\gamma \,.\, x^\gamma = x^\gamma$$

and

$$x^\beta [(\tfrac{1}{x})^\beta x^\gamma] = x^\beta [(\tfrac{1}{x})^{\alpha\beta} x^\gamma] = x^\beta [(\tfrac{1}{x})^{\alpha\gamma} x^\gamma] = x^\beta [(\tfrac{1}{x})^\gamma x^\gamma] =$$
$$= x^\beta \,.\, 1^\gamma = x^\beta \,.\, 1^\beta = x^\beta \,.$$

But (C, C') is a semigroup, hence $x^{\beta} = x^{\gamma}$. This contradiction gives $\beta = \gamma$ and so α is an epimorphism. Hence the "converse theorem" 4) does not hold in $\mathcal{S}$.

Now let $\mathcal{R}$ be the category of all rings, Z the set of all rational integers, F the set of all rational numbers, α the identity mapping of Z into F and (Z, Z') and (F, F') the rings corresponding to Z and to F .

We shall show again that α is an epimorphism in $\mathcal{R}$ though it does not map Z onto F .

Let us have any ring (C, C') let $\beta : (F, F') \to (C, C')$ and $\gamma : (F, F') \to (C, C')$ be any two morphisms in $\mathcal{R}$ and let $\alpha\beta = \alpha\gamma$. Then, again, if $\beta \neq \gamma$, it is $(\frac{r}{s})^{\beta} \neq (\frac{r}{s})^{\gamma}$ for some rational integers r and s where $s \neq 0$ and where r may be supposed to be positive. Because β and γ are homomorphisms, we have $(\frac{1}{s})^{\beta} \neq (\frac{1}{s})^{\gamma}$. Now again

$$[(\tfrac{1}{s})^{\beta}\, s^{\beta}]\,(\tfrac{1}{s})^{\gamma} = 1^{\beta} \,.\, (\tfrac{1}{s})^{\gamma} = 1^{\gamma} \,.\, (\tfrac{1}{s})^{\gamma} = (\tfrac{1}{s})^{\gamma} \quad \text{and}$$

$$(\tfrac{1}{s})^{\beta}\,[s^{\beta}\,(\tfrac{1}{s})^{\gamma}] = (\tfrac{1}{s})^{\beta}\,[s^{\gamma}\,(\tfrac{1}{s})^{\gamma}] = (\tfrac{1}{s})^{\beta} \,.\, 1^{\gamma} = (\tfrac{1}{s})^{\beta}.$$

It follows again $\beta = \gamma$ and so α is an epimorphism. Hence the "converse theorem" 4) does not hold in $\mathcal{R}$.

On the other hand the "converse theorem" 4) is true in the category of all <u>universal algebras</u> of any fixed type. For to prove it, it is not necessary to use the notion of a free product of two isotypic universal algebras with isomorphic subalgebras, as it is done e.g. in the case of groups, but we may proceed in a very much simpler way.

First of all, we can express the notion of an universal algebra in a form more convenient for our purpose: Let Λ be a set and ν a mapping of Λ into the set of all positive integers. Let (A, A') be an ordered couple of sets such that

5) every element of A' is a finite sequence of the form

$$(*) \qquad (y;\ x_1,\ x_2,\ \ldots,\ x_{\nu(\lambda)};\ \lambda)$$

where $\lambda \in \Lambda$ and $y, x_1, x_2, \ldots, x_{\nu(\lambda)}$ belong to A,

6) for every $\lambda \in \Lambda$ and for any $x_1, x_2, \ldots, x_{\nu(\lambda)}$ belonging to A there exists exactly one $y \in A$ such that the sequence $(*)$ belongs to A'. Then the ordered couple (A, A') is called a <u>universal algebra</u> of type $[\Lambda, \nu]$. If, in the previous definition, in the condition 6), we use the phrase "at most one" instead of "exactly", we get the definition of a <u>partial universal algebra</u> of type $[\Lambda, \nu]$. Every partial universal algebra (A, A'') can be embedded into a universal algebra (A, A') with the same underlying set A and with $A'' \subset A'$.

A <u>homomorphism</u> φ of a universal algebra (A, A') into a universal algebra (B, B') with the same type $[\Lambda, \nu]$ is any mapping from A into B such that if $(*)$ belongs to A' then $(y^{\varphi};\ x_1^{\varphi},\ x_2^{\varphi},\ \ldots,\ x_{\nu(\lambda)}^{\varphi};\ \lambda) \in B'$. If φ is, at the same time, a one-to-one mapping from A onto B, then it is called an <u>isomorphism</u>. A universal algebra (A_1, A_1') is called a <u>subalgebra</u> of (A, A'), if $A_1 \subset A$ and $A'_1 \subset A'$.

All universal algebras of a given type $[\Lambda, \nu]$ form a <u>category</u> $\mathcal{U}_{\Lambda,\nu}$ with respect to homomorphisms. $\mathcal{U}_{\Lambda,\nu}$ satisfies the conditions 1), 2) and 3). As already mentioned, in $\mathcal{U}_{\Lambda,\nu}$ the converse theorem 4) is true. We shall prove it now.

Let α be a homomorphism from (A, A') into (B, B'), both algebras belonging to $\mathcal{U}_{\Lambda,\nu}$. Let $A^{\alpha} \neq B$ and we shall show that α is not an epimorphism. Really, α maps (A, A') onto a subalgebra (B_1, B_1') of (B, B'), where $B_1 = A^{\alpha} \neq B$. Now, it is always possible to find in $\mathcal{U}_{\Lambda,\nu}$

an algebra (B_2, B_2') and an isomorphism ι from (B, B') onto (B_2, B_2') such that $B \cap B_2 = B_1$, $B' \cap B_2' = B_1'$ and $x^{\iota} = x$ for every $x \in B_1$. The couple (C, C'') , where $C = B \cup B_2$ and $C'' = B' \cup B_2'$, is a partial universal algebra of type $[\Lambda, \nu]$. This can be of course embedded into a universal algebra $(C, C') \in \mathcal{U}_{\Lambda,\nu}$. Now, (B, B') and (B_2, B_2') are subalgebras of (C, C'), and we shall denote by μ and μ_2 the identity mappings of them into (C, C') . Now, we have $\alpha\mu = \alpha\iota\mu_2$ but at the same time $\mu \neq \iota\mu_2$. Thus α is no epimorphism.

R e f e r e n c e s

[1] A.G.KUROŠ, A.H. LIVŠIC, E.G. ŠUL´GEIFER: Osnovy teorii kategorij, Uspehi Mat.Nauk, XV, 6(96),1960, 3 - 52 .