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A REMARK ON LINEAR OPERATORS LEAVING A CONE INVARIANT IN A BANACH SPACE

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In [1] theorems are given about the existence of eigenvalues and eigenvectors of compact linear operators reproducing a cone in a Banach space. In this remark we will call to attention a class of non compact operators for which some of the mentioned theorems are also correct.

Let X be a real Banach space, X' the adjoint space of linear forms and $X_1 = (X \to X)$ the space of linear continuous transformations of space X into itself. The space $\tilde{X}$ will be the complex extension of space X, i.e. the space of pairs $z = (x, y) = x + iy$ with the norm defined as

$$\|z\|_{\tilde{X}} = \sup_{0 \leq \vartheta \leq 2\pi} \|x \cos\vartheta + y \sin\vartheta\|_X .$$

We extend the linear operator $T \in X_1$ from X to $\tilde{X}$ by the prescription:

$$Tz = Tx + i\,Ty, \quad x \in X, \; y \in X.$$

The symbol $\sigma(T)$ means the spectrum of the operator T extended to $\tilde{X}$ and the symbol $\rho(T)$ the resolvent set of this operator.

Let $\mathcal{K}$ be a cone in space X. The operator $T \in X_1$ is called $\mathcal{K}$-positive if $T\mathcal{K} \subset \mathcal{K}$, i.e. for $x \in \mathcal{K}$ also $Tx = y \in \mathcal{K}$. The cone $\mathcal{K}$ is called "volume type" if it has interior points. The operator $T \in X_1$ is called strongly $\mathcal{K}$-positive, if for every vector $x \in \mathcal{K}$, $x \neq \sigma$ a natural $p = p(x)$ exists, such that $T^p x$ lies in the interior of the cone $\mathcal{K}$.

With the help of the cone $\mathcal{K}$ the space X can be partially ordered. We define

$$y \succ x \iff y - x \in \mathcal{K},$$
$$y \succ\succ x \iff y - x \in \mathrm{int}\ \mathcal{K},$$
where $\mathrm{int}\ \mathcal{K}$ is the interior of the cone $\mathcal{K}$.

A cone $\mathcal{K}$ is called "productive" if for every vector $x \in X$ a sequence $\{x_n\}$, $x_n \in \mathcal{K}$ and a numerical sequence $\{c_n\}$ exist such that $x = \lim_{n\to\infty} c_n x_n$. Further we assume that $\mathcal{K}$ is a productive cone.

If $\mathcal{K}$ is a cone in X we define the adjoint cone $\mathcal{K}' \subset X'$ so, that $x' \in \mathcal{K}'$ if $x' \in X'$ and if for $x \in \mathcal{K}$ we have $x'(x) \geqq 0$. The form $x' \in X'$ is called strongly positive, if for $x \in \mathcal{K}$, $x \neq 0$ we have $x'(x) > 0$.

The definitions given above have been adopted from [1].

If $T \in X_1$, then the number $R_T = \sup_{\lambda \in \sigma(T)} |\lambda|$ is called the spectral radius of the operator T. It is well known [4] that $R_T = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$.

The point $\lambda \in \sigma(T)$ is called a Fredholm point of the spectrum of the operator T if it has following properties:

(a) Point λ is an isolated point of the spectrum $\sigma(T)$.

(b) The set $\mathfrak{M}(\lambda)$ of vectors $x \in X$ for which a natural n such that $(T-\lambda I)^n x = \sigma$ exists, forms a finite dimensional lineal.

(c) The space $\tilde{X}$ is a direct sum $\tilde{X} = \mathfrak{M}(\lambda) \oplus \mathfrak{N}(\lambda)$, where $\mathfrak{N}(\lambda)$ is invariant with respect to T and the operator $(T-\lambda I)$ has a continuous inverse operator $(T-\lambda I)^{-1}_{\mathfrak{N}} = R_{\mathfrak{N}}$ on $\mathfrak{N}(\lambda)$.

(d) The equation $(T-\lambda I)x = y$ has a solution in X if and only if $x'(y) = 0$ holds for every functional $x' \in X'$ such that $x'(Tx) = \lambda x'(x)$ for all $x \in X$.

Operator $T \in X_1$ is called Nicolski operator if it can be expressed in the form

$$T = C + D,$$

where $D \in X_1$, C is a compact operator and the inequality

(*) $$R_T > R_D$$

is valid.

L e m m a [3]. Let T be a Nicolski operator. If $\lambda \in \sigma(T)$ and $|\lambda| > R_D$, then λ is a Fredholm point of the spectrum of the operator T.

P r o o f . Let $R(\lambda, T) = (\lambda I - A)^{-1}$ be the resolvent of the operator $A \in X_1$. For $\lambda \in \rho(T), |\lambda| > R_D$ we have clearly the equality

$$R(\lambda, T) - R(\lambda, D) = R(\lambda, T) C R(\lambda, D)$$

i.e.

$$R(\lambda, T) = \lambda R(\lambda, D)\left[\lambda I - C R(1, \tfrac{1}{\lambda} D)\right]^{-1}.$$

It can be seen from this expression that the resolvent $R(\lambda, T)$ for $|\lambda| > R_D$ is a product of bounded linear operator $R(\lambda, D)$ and of the resolvent of the compact operator $C R(1, \frac{1}{\lambda} D)$. The assertion of the lemma follows from the properties of the resolvent of a compact operator.

T h e o r e m 1. Let operator T be a $\mathcal{K}$-positive Nicolski operator. A positive eigenvalue μ_0, for which

$$|\lambda| \leqq \mu_0, \quad \lambda \in \sigma(T)$$

then lies in the spectrum of operator T. To this eigenvalue corresponds at least one eigenvector $x_0 \in \mathcal{K}$, $\|x_0\| = 1$ of the operator T: $T x_0 = \mu_0 x_0$ and at least one eigenfunctional $x_0' \in \mathcal{K}'$ $\|x_0'\| = 1$ of the adjoint operator T': $T' x_0' = \mu_0 x_0'$.

T h e o r e m 2. Assumptions:

1. $\mathcal{K}$-positive operator T can be expressed in the form
$$T = C + D,$$
where $D \in X_1$ and $C \neq \Theta$ is compact operator (Θ denote zero-operator).
2. There exists such a natural number n and vector

$u \in \mathcal{K}$ that

$$\|u\|_X = 1, \quad d = \inf_{x \in \mathcal{K}} \|x + u\|_X = 1$$

and such a positive constant c that

$$T^p u \succ c u,$$

where $\sqrt[p]{c} > R_D$.

Assertion: A positive eigenvalue μ_0 exists in the spectrum of operator T and the inequalities

$$\mu_0 \geqq \sqrt[p]{c}, \quad \mu_0 \geqq |\lambda|, \; \lambda \in \sigma(T)$$

hold.

Thus the operator T is $\mathcal{K}$-positive Nicolski operator so that the assertions of theorem 1 are valid.

T h e o r e m 3. Assumptions:

1. $\mathcal{K}$ is a volume type cone.
2. T is a strongly $\mathcal{K}$-positive Nicolski operator.

Assertions:

(a) Operator T has one and only one eigenvector x_0 in $\mathcal{K}$ and for this eigenvector we have

$$T x_0 = \mu_0 x_0, \quad \|x_0\| = 1, \quad x_0 \succ 0.$$

(b) The adjoint operator T' has one and only one eigenfunctional x_0' in $\mathcal{K}'$ and this will be a strongly positive functional: $T' x_0' = \mu_0 x_0'$, $\|x_0'\|_{X'} = 1$, $x'(x) > 0$ for $x \in \mathcal{K}$, $x \neq \sigma$.

(c) The eigenvalue μ_0 corresponding to the eigenvectors x_0, x_0' is a positive simple dominant eigenvalue of the operators T, T':

$$\mu_0 > |\lambda|, \quad \lambda \in \sigma(T).$$

Theorems 1 - 3 can be proved analogically as the corresponding theorems for compact operators in [1], since the assertions of these theorems follow from the properties of points of the spectrum of the operator T lying outside the circle $|\lambda| \leqq R_D$. These properties, according to the lemma, are the same for compact operators and for Nicolski operators.

The assumption (*) in theorems 1 and 3 cannot be omitted. This can be demonstrated on the following example.

Let $X = C(\langle 0,1\rangle)$ be a space of continuous functions on $\langle 0,1\rangle$ with the usual norm $\|x\|_X = \max_{t\in\langle 0,1\rangle} |x(t)|, x\in X$. Let $\mathcal{K} \subset C(\langle 0,1\rangle)$ be a cone of nonnegative functions in $C(\langle 0,1\rangle)$. It is known that $\mathcal{K}$ is a volume type cone [2]. Further let $C = C(s,t)$ be continuous on $\langle 0,1\rangle \times \langle 0,1\rangle$,

$$Cx = y: \quad y(s) = \int_0^s C(s,t)x(t)dt,$$

$$C(s,t) > 0 \quad \text{for} \quad s,t \in \langle 0,1\rangle, \; D = I, T = C + D.$$

Evidently we have $1 = R_D = R_{C+D}$. If such a function $x_0 \in \mathcal{K}$, $x_0(t) \geqq 0$ existed that

$$\int_0^s C(s,t)x_0(t)\,dt + x_0(s) = \lambda x_0(s),$$

then the operator C would have an eigenvalue and we know that this is not so.

Using theorems 1 – 3 it is easy to prove the following theorems about the dependence of eigenvalues of Nicolski operators on a parameter.

Let $G = \langle \beta_0, \beta_1\rangle$ be an interval of real numbers. The operator-function $T = T(\beta)$ (for short just "operator") is called continuous in the point $\beta_0 \in G$ if for every $\varepsilon > 0$ a $\delta > 0$ exists, such that for $|\beta - \beta_0| < \delta$ we have

$$\| T(\beta) - T(\beta_0)\|_{X_1} < \varepsilon .$$

If $T = T(\beta)$ is continuous in every point $\beta \in G$ we say that it is continuous with respect to β in G.

T h e o r e m 4. Assumptions:

1. For every $\beta \in G$ is $T(\beta) = C(\beta) + D(\beta)$ a Nicolski operator.
2. The operator $T = T(\beta)$ is continuous with respect to β in G
3. The value $\mu_0 = \mu(\beta_0)$, $|\mu_0| > R_{D(\beta_0)}$ is an eigenvalue of multiplicity ν_0 of the operator $T(\beta_0)$.

Assertion: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|\beta - \beta_0| < \delta$ then p $(p \geqq 1)$ values

$\mu_1(\beta), \dots, \mu_p(\beta)$ exist, which are eigenvalues of the multiplicites $q_1(\beta), \dots, q_p(\beta)$ of the operator $T(\beta)$ and we have

$$|\mu_j(\beta) - \mu_0(\beta_0)| < \varepsilon, \; j = 1, \dots, p; \; \nu_0 = \sum_{k=1}^{p} q_k(\beta).$$

C o r o l a r y . If $\mu_0(\beta_0)$ is a simple eigenvalue of the operator $T(\beta_0)$ then under the conditions of theorem 4 the eigenvalue $\mu_0 = \mu_0(\beta)$ is a continuous function of the variable $\beta \in G$.

R e m a r k. According to the theorem 3 a simple positive eigenvalue μ_0 corresponds to a strongly $\mathcal{K}$-positive Nicolski operator T. Thus if the operator $T(\beta)$ is strongly $\mathcal{K}$-positive for every $\beta \in G$ and continuous with respect to β in G then $\mu_0 = \mu_0(\beta)$ is a positive continuous function in G.

We shall show that under certain assumptions μ_0 is also a purely monotonous function.

T h e o r e m 5. Assumptions:

1. $\mathcal{K}$ is a volume type cone.
2. For every vector $u \succ\succ \sigma$ we have
$$\inf_{x \in \mathcal{K}} \|x + u\|_X \geqq \|u\|_X .$$
3. $T(\beta)$ is a strongly $\mathcal{K}$-positive Nicolski operator for every $\beta \in G$.
4. The operator-function $T = T(\beta)$ is continuous with respect to β in G.
5. For the spectral radii $R_{D(\beta)}$ and $R_{T(\beta)}$ we have
$$R_{T(\beta)} \geqq R > R_{D(\beta)} \quad \text{for } \beta \in G$$
with R independent on β.
6. For every vector $x \succ\succ \sigma$ and for $\beta' < \beta''$ we have
$$[T(\beta') - T(\beta'')]\, x \succ \alpha(\beta', \beta'')\, x,$$
where $\alpha(\beta', \beta'') > 0$.

Assertion: The inequality
$$\mu_0(\beta') > \mu_0(\beta''), \; \beta' < \beta''$$

holds for the dominant eigenvalues $\mu_0(\beta'), \mu_0(\beta'')$ of the operators $T(\beta'), T(\beta'')$.

The given theorems 1 - 5 are applied in [5] and [6] to prove the existence of so called critical parameters of certain systems in which certain types of nuclear reactions take place. These papers will be published in "Aplikace matematiky".

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