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NOTE ON THE PAPER OF I.FENYŐ: ON THE SOLUTION OF NON-LINEAR EQUATIONS IN BANACH SPACE

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It is shown that in the well known article of I.Fenyő [1] the proofs of both theorems are erroneous. Theorem 3 is a correction of the theorem 2 ([1]).

Let the equation

$$f(x) = y \tag{1}$$

be given, where a non-linear operator $f(x)$ maps Banach space X in Banach space Y. We denote by $F(x)$ the expression $f(x)-y$. I.Fenyő proves [1] the following theorems:

THEOREM 1. Let $f(x)$ be a non-linear operator which maps B-space X in B-space Y and has in r-neighbourhood of the element $x_0 \in X$ the following properties:

1) In the point x_0 it has the Frechet's derivation, there exists $f'(x_0)^{-1}$ and $\|f'(x_0)^{-1}\| \leqq B$

2) $\|f'(x)-f'(x_0)\| \leqq C\|x-x_0\|$ for $\|x-x_0\| \leqq r < \frac{1}{3BC}$.

Then the equation (1) has a unique solution x^* in r-neighbourhood of x_0 for every $y \in Y$ which yields the inequality

$$\|y-f(x_0)\| \leqq \eta = \frac{2-3BCr}{11B} r . \tag{2}$$

The iterative process defined by the equality

$$x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n = 1, 2, \ldots$$

is convergent in the norm of the space X to the solution x^* of (1).

THEOREM 2. Let $f(x)$ be a non-linear operator which maps B-space X in B-space Y and has in r-neighbourhood of the element $x_0 \in X$ a continuous Frechet's derivation. If there the operator $f'(x_0)^{-1}$ exists and the inequality

$$\|f'(x_0)^{-1} [y-f(x_0)]\| \leqq r(1-q)$$

holds for every x for which $\|x-x_0\| \leqq r$, then the equation (1) has a unique solution x^* in r-neighbourhood of the element $x_0 \in X$ for every $y \in Y$ which yields the inequality $\|f'(x_0)^{-1}[y-f(x_0)]\| \leqq r(1-q)$.

The modified Newton-Kantorowitch iterative process defined by the equality $\xi_n = \xi_{n-1} - [F'(x_0)]^{-1} F(\xi_{n-1})$, where $F(x) = f(x) - y$, is convergent in the norm of the space X to the solution x^* of (1).

The proof of theorem 1 is establisned on the validity of the inequalities $\|F(x)\| \leqq \eta$, $\|F'(x)\| \leqq \frac{\eta}{r}$ yielding in $\|x-x_0\| \leqq r$. We show that these estimations are not held. Above all the estimation $\|F(x)\| \leqq \eta$ does not follow from the assumptions of theorem 1 for every x from $\|x-x_0\| \leqq r$.

Further the inequality $\|F'(x)\| \leqq \frac{\eta}{r}$ does not hold for every x for which is $\|x-x_0\| \leqq r$ valid.

PROOF. There exists the operator

$$T(x) = \left(J - f'(x_0)^{-1}[f(x_0) - f'(x)]\right)^{-1}$$

and

$$\|T(x)\| \leqq \frac{1}{1-BCr} < \frac{3}{2} \quad \text{for every } x \text{ from}$$

$\|x-x_0\| \leqq r$.

It follows from the inequalities

$$\|f'(x_0)^{-1}[f'(x_0)-f'(x)]\| \leqq \|f'(x_0)^{-1}\| \ \|f'(x_0)-f'(x)\| \leqq$$

$$\leqq BC\|x-x_0\| \leqq BCr < \frac{1}{3}$$

and Banach theorem. Further we obtain

$$F'(x_0)T^{-1}(x) = F'(x_0)\left(J - f'(x_0)^{-1}[f'(x_0)-f'(x)]\right) =$$

$$= F'(x_0)f'(x_0)^{-1} f'(x) = F'(x)$$

$$F'(x)^{-1} = \left\{F'(x_0)T'(x)\right\}^{-1} = T(x) \ F'(x_0)^{-1} .$$

Therefore the operator $F'(x)^{-1}$ exists in r-neighbourhood of the element x_0 and $\|F'(x)^{-1}\| < \frac{3}{2}B$.

; Let us now assume that $\|F'(x)\| \leqq \frac{\eta}{r}$ for every x from $\|x - x_0\| \leqq r$. Then

$$1 = \|F'(x)^{-1} F'(x)\| \leqq \|F'(x)^{-1}\| \; \|F'(x)\| < \frac{3}{2} B \frac{\eta}{r} =$$

$$= \frac{3}{2} B \; \frac{2 - 3BCr}{11B} < \frac{3}{11} < 1 .$$

Therefore the above mentioned estimation does not hold. Further the application of theorem 1 is not advantageous, because in practical problems it is yielding when $\|x^* - x_0\| < \varepsilon$, where ε is rather small such that x_0 can be considered as a good approximate solution of the equation (1).

The proof of theorem 2 (see [1]) is erroneous because:
a) the equalities (25), (26) (see [1]) do not hold because Lagrange formula has no validity for operators in general (see [2] , § 3)
b) from the condition of theorem 2 does not follow the unicity of the solution x^* of (1) in r-neighbourhood of $x_0 \in X$.

THEOREM 3. Let $f(x)$ be a non-linear operator which maps B-space X in B-space Y and has the following properties:
1) For every x from $\|x - x_0\| \leqq r$ it has the derivation of Frechet, there exists the operator $f'(x_0)^{-1}$ and $\|f'(x_0)^{-1}\| \leqq B$.
2) $\|f'(x) - f'(x_0)\| \leqq C$ for $\|x - x_0\| \leqq r$ and $0 < BC < 1$.

Then the equation (1) has a unique solution x^* in r-neighbourhood of $x_0 \in X$ for every element $y \in Y$ which yields the inequality $\|y - f(x_0)\| \leqq \frac{1}{B} r (1-q)$.

The modified Newton-Kantorowitch iterative process defined by the equality

$$\xi_n = \xi_{n-1} - F'(x_0)^{-1} F(\xi_{n-1})$$

is convergent in the norm of the space X to the solution x^* of the equation (1).

REFERENCES.

[1] I.FENYŐ, Ueber die Lösung der im Banachschen Raume definierten nichtlinearen Gleichungen. Acta

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[2] M.M.Vajnberg, Variacionnyje metody issledovania nelinejnych operatorov, Moskva 1956

[3] L.V.Kantorowitch - G.P.Akilov, Funkcionalnyj analiz v normirovannych prostranstvach, Moskva 1959