Archivum Mathematicum

Miloš Ráb

Asymptotic formulas for solutions of the equation $[p(t)y']' = q(t)y + r(t)$

Archivum Mathematicum, Vol. 10 (1974), No. 2, 79--85

Persistent URL: http://dml.cz/dmlcz/104820

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS,
X: 79—86, 1974

ASYMPTOTIC FORMULAS FOR SOLUTIONS OF THE EQUATION $[p(t)y']' = q(t)y + r(t)$

MILOŠ RÁB, Brno

(Received August 3, 1973)

Let us consider the differential equation

$$[p(t)\,y']' = q(t)\,y + r(t). \tag{1}$$

Throughout the paper we suppose that p, q, r are continuous complex-valued functions defined for $t \in J = [t_0, \infty)$ and $p(t) \neq 0$, $r(t) \not\equiv 0$. In [1] asymptotic formulas for solutions of (1) in the case $r(t) \equiv 0$ have been derived considering (1) for a perturbed equation of $[p(t)\,z']' = 0$. In this paper we shall derive asymptotic formulas for a particular solution of (1) satisfying the integral equation

$$y(t) = \int_{t_1}^{t} \frac{1}{p(\xi)} \int_{t_2}^{\xi} q(\eta)\,y(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi + \int_{t_3}^{t} \frac{1}{p(\xi)} \int_{t_4}^{\xi} r(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi,$$

where t_i, $i = 1, \dots, 4$ are suitable numbers, $t_0 \leqq t_i \leqq \infty$. In this way, regarding the results contained in [1], the asymptotic nature of the general solution (1) will be described.

Let us denote

$$\delta(t) = \int_{t_3}^{t} \frac{1}{p(\xi)} \int_{t_4}^{\xi} r(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

and define linear operators $K_n, L_n : C(J) \to C(J)$ where $C(J)$ is the set of all continuous functions $x(t)$ defined on J in the following way

$$K_0x(t) = x(t), \qquad K_nx(t) = \int_{t_1}^{t} \frac{1}{p(\xi)} \int_{t_2}^{\xi} q(\eta)\,K_{n-1}x(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{2}$$

$$L_0x(t) = x(t), \qquad L_nx(t) = \int_{t_2}^{t} q(\xi) \int_{t_1}^{\xi} \frac{1}{p(\eta)} L_{n-1}x(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{3}$$

Then the series $y(t) = \sum_0^\infty K_n\delta(t)$ is a formal solution of (1) and its derivative is given by

$$p(t)\,y'(t) = \int_{t_4}^{t} r(\xi)\,d\xi + \sum_0^\infty L_n \int_{t_2}^{t} q(\xi) \int_{t_3}^{\xi} \frac{1}{p(\eta)} \int_{t_4}^{\eta} r(\sigma)\,d\sigma\,d\eta\,d\xi.$$

Further, the following special cases of (2) and (3) will be investigated

$$T_n x(t) = \int_\infty^t \frac{1}{p(\xi)} \int_\infty^\xi q(\eta)\,T_{n-1}x(\eta)\,d\eta\,d\xi, \qquad \Phi_n x(t) = \int_\infty^t q(\xi) \int_\infty^\xi \frac{1}{p(\eta)}\,\Phi_{n-1}(\eta)\,d\eta\,d\xi,$$

$$\Psi_n x(t) = \int_\infty^t \frac{1}{p(\xi)} \int_{t_0}^\xi q(\eta)\,\Psi_{n-1}x(\eta)\,d\eta\,d\xi, \qquad \Omega_n x(t) = \int_{t_0}^t q(\xi) \int_\infty^\xi \frac{1}{p(\eta)}\,\Omega_{n-1}x(\eta)\,d\eta\,d\xi,$$

$$n = 1, 2, \ldots,$$

$$T_0 x(t) = \Phi_0 x(t) = \Psi_0 x(t) = \Omega_0 x(t) = x(t).$$

Theorem 1. *Suppose*

$$\int_{t_0}^\infty \frac{d\xi}{|p(\xi)|} < \infty, \qquad \int_{t_0}^\infty |q(\xi)|\,d\xi < \infty, \qquad \int_{t_0}^\infty |r(\xi)|\,d\xi < \infty.$$

Then there exists a solution $y(t)$ of (1) *such that*

(4) $$y(t) = \sum_0^n T_k \int_\infty^t \frac{1}{p(\xi)} \int_\infty^\xi r(\eta)\,d\eta\,d\xi + \varepsilon_1(t)$$

and

$$p(t)\,y'(t) = \sum_0^n \Phi_k \int_\infty^t r(\xi)\,d\xi + \varepsilon_2(t).$$

Here

(5) $$|\varepsilon_1(t)| \leqq \alpha(t)\frac{\tau^{n+1}(t)}{(n+1)!}\exp\{\tau(t)\},$$

$$\alpha(t) = \int_t^\infty \frac{1}{|p(\xi)|} \int_\xi^\infty |r(\eta)|\,d\eta\,d\xi, \qquad \tau(t) = \int_t^\infty \frac{1}{|p(\xi)|} \int_\xi^\infty |q(\eta)|\,d\eta\,d\xi,$$

(6) $$|\varepsilon_2(t)| \leqq \frac{\varphi^{n+1}(t)}{(n+1)!}\exp\{\varphi(t)\} \int_t^\infty |r(\xi)|\,d\xi, \qquad \varphi(t) = \int_t^\infty |q(\xi)| \int_\xi^\infty \frac{1}{|p(\eta)|}\,d\eta\,d\xi.$$

Proof. Let us denote

$$a(t) = \int_{\infty}^{t} \frac{1}{p(\xi)} \int_{\infty}^{\xi} r(\eta)\, d\eta\, d\xi.$$

We shall prove by induction

$$(7) \qquad | T_n a(t) | \leqq \alpha(t) \frac{\tau^n(t)}{n!}.$$

It holds $| T_0 a(t) | = | a(t) | \leqq \alpha(t)$ and by means of (7) we receive

$$| T_{n+1} a(t) | = \left| \int_t^{\infty} \frac{1}{p(\xi)} \int_{\xi}^{\infty} q(\eta)\, T_n a(\eta)\, d\eta\, d\xi \right| \leqq \int_t^{\infty} \frac{1}{| p(\xi) |} \int_{\xi}^{\infty} | q(\eta) |\, \alpha(\eta) \frac{\tau^n(\eta)}{n!}\, d\eta\, d\xi \leqq$$

$$\leqq \alpha(t) \int_t^{\infty} \frac{1}{| p(\xi) |} \int_{\xi}^{\infty} | q(\eta) |\, d\eta \frac{\tau^n(\xi)}{n!}\, d\xi = \alpha(t) \int_t^{\infty} - \tau'(\xi) \frac{\tau^n(\xi)}{n!}\, d\xi = \alpha(t) \frac{\tau^{n+1}(t)}{(n+1)!}$$

using the fact that $\alpha(t)$, $\tau(t)$ are nonincreasing functions on J. From this it follows that the series $y(t) = \sum_0^{\infty} T_n a(t)$ is uniformly convergent on J since

$$\sum_0^{\infty} \alpha(t_0) \frac{\tau^n(t_0)}{n!}$$

is its convergent majorant on this interval. Thus $y(t)$ is a solution of (1). If we write $y(t)$ in the form (4) we receive for $\varepsilon_1(t)$ the following estimation

$$| \varepsilon_1(t) | = | \sum_{n+1}^{\infty} T_k a(t) | \leqq \alpha(t) \frac{\tau^{n+1}(t)}{(n+1)!} \left[1 + \frac{\tau(t)}{n+2} + \frac{\tau^2(t)}{(n+2)(n+3)} + \dots \right] \leqq$$

$$\leqq \alpha(t) \frac{\tau^{n+1}(t)}{(n+1)!} \exp \{\tau(t)\}.$$

In the same manner one proves the uniform convergence of the series

$$p(t)\, y'(t) = \sum_0^{\infty} \Phi_k \int_{\infty}^{t} r(\xi)\, d\xi$$

and the estimation (6) for $\varepsilon_2(t) = \sum_{n+1}^{\infty} \Phi_k \int_{\infty}^{t} r(\xi)\, d\xi$.

An easy modification of the preceding proof leads to the following statement.

If

$$\int_{t_0}^{\infty} | q(\xi) | \, d\xi < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | r(\eta) | \, d\eta \, d\xi < \infty$$

then there exists a solution $y(t)$ of (1) *such that*

$$y(t) = \sum_{0}^{n} T_k \int_{\infty}^{t} \frac{1}{p(\xi)} \int_{t_0}^{\xi} r(\eta) \, d\eta \, d\xi + \varepsilon_3(t)$$

and

$$p(t) \, y'(t) = \int_{t_0}^{t} r(\xi) \, d\xi + \sum_{0}^{n-1} \Phi_k \int_{\infty}^{t} q(\xi) \int_{\infty}^{\xi} \frac{1}{p(\eta)} \int_{t_0}^{\eta} r(\sigma) \, d\sigma \, d\eta \, d\xi + \varepsilon_4(t).$$

Here

$$| \varepsilon_3(t) | \leqq \frac{\tau^{n+1}(t)}{(n+1)!} \exp \{\tau(t)\} \int_{t}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | r(\eta) | \, d\eta \, d\xi,$$

$$| \varepsilon_4(t) | \leqq \frac{\varphi^n(t)}{n!} \exp \{\varphi(t)\} \int_{t}^{\infty} | q(\xi) | \int_{\xi}^{\infty} \frac{1}{| p(\eta) |} \int_{t_0}^{\eta} | r(\sigma) | \, d\sigma \, d\eta \, d\xi.$$

Theorem 2. *Suppose*

$$\int_{t_0}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | r(\eta) | \, d\eta \, d\xi < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | q(\eta) | \, d\eta \, d\xi < 1.$$

Then there exists a solution $y(t)$ of (1) *of the form*

$$y(t) = \sum_{0}^{n} \Psi_k \int_{\infty}^{t} \frac{1}{p(\xi)} \int_{t_0}^{\xi} r(\eta) \, d\eta \, d\xi + \varepsilon_5(t), \tag{8}$$

where

$$| \varepsilon_5(t) | \leqq \int_{t_0}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | r(\eta) | \, d\eta \, d\xi \, \frac{\psi^{n+1}}{1 - \psi}, \qquad \psi = \int_{t_0}^{\infty} \frac{1}{| p(\xi) |} \int_{t_0}^{\xi} | q(\eta) | \, d\eta \, d\xi.$$

Adding further assumption

$$\int_{t_0}^{\infty} | r(\xi) | \, d\xi < \infty, \tag{9}$$

then

(10) $$p(t)\,y'(t) = \sum_0^n \Omega_k \int_{t_0}^{t} r(\xi)\,\mathrm{d}\xi + \varepsilon_6(t),$$

and

(11) $$|\,\varepsilon_6(t)\,| \leqq \frac{\omega^{n+1}}{1-\omega} \int_{t_0}^{\infty} |\,r(\xi)\,|\,\mathrm{d}\xi, \qquad \omega = \int_{t_0}^{\infty} |\,q(\xi)\,| \int_{\xi}^{\infty} \frac{1}{|\,p(\eta)\,|}\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

If we suppose instead of (9)

$$\int_{t_0}^{\infty} |\,q(\xi)\,| \int_{\xi}^{\infty} \frac{1}{|\,p(\eta)\,|} \int_{t_0}^{\eta} |\,r(\sigma)\,|\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}\xi < \infty,$$

it holds again (10) *with*

$$|\,\varepsilon_6(t)\,| \leqq \frac{\omega^n}{1-\omega} \int_{t_0}^{\infty} |\,q(\xi)\,| \int_{\xi}^{\infty} \frac{1}{|\,p(\eta)\,|} \int_{t_0}^{\eta} |\,r(\sigma)\,|\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

P r o o f. First of all we shall prove by induction

(12) $$|\,\Psi_n a(t)\,| \leqq \alpha\psi^n$$

$$\text{where} \quad a(t) = \int_{\infty}^{t} \frac{1}{p(\xi)} \int_{t_0}^{\xi} r(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi, \quad \alpha = \int_{t_0}^{\infty} \frac{1}{|\,p(\xi)\,|} \int_{t_0}^{\xi} |\,p(\eta)\,|\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

For $n = 0$ we have $|\,\Psi_0 a(t)\,| = |\,a(t)\,| \leqq \alpha$ and using (12) we receive

$$|\,\Psi_{n+1}a(t)\,| = |\int_{t}^{\infty} \frac{1}{p(\xi)} \int_{t_0}^{\xi} q(\eta)\,\Psi_n a(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,| \leqq \int_{t}^{\infty} \frac{1}{|\,p(\xi)\,|} \int_{t_0}^{\xi} |\,q(\eta)\,|\,\alpha\psi^n\,\mathrm{d}\eta\,\mathrm{d}\xi \leqq$$

$$\leqq \alpha\psi^n \int_{t_0}^{\infty} \frac{1}{|\,p(\xi)\,|} \int_{t_0}^{\xi} |\,q(\eta)\,|\,\mathrm{d}\eta\,\mathrm{d}\xi = \alpha\psi^{n+1}.$$

Hence, the series $y(t) = \sum_0^{\infty} \Psi_n a(t)$ converges uniformly on J since $\sum_0^{\infty} \alpha\psi^n$ is its convergent majorant on J. Thus $y(t)$ is a solution of (1). If we write $y(t)$ in the form (8) we have

$$|\,\varepsilon_5(t)\,| = |\sum_{n+1}^{\infty} \Psi_n a(t)\,| \leqq \alpha\psi^{n+1}[1 + \psi + \psi^2 + \ldots] = \alpha\frac{\psi^{n+1}}{1-\psi}\,.$$

This is the first part of the theorem.

Now, let us suppose (9). Using the fact that the assumption

$$\int_{t_0}^{\infty} \frac{1}{|p(\xi)|} \int_{t_0}^{\xi} |q(\eta)| \, d\eta \, d\xi < 1 \quad \text{implies} \quad \int_{t_0}^{\infty} |q(\xi)| \int_{\xi}^{\infty} \frac{1}{|p(\eta)|} \, d\eta \, d\xi < 1$$

and that the function $\omega(t) = \int_{t_0}^{t} |q(\xi)| \int_{\xi}^{\infty} \frac{1}{|p(\eta)|} \, d\eta \, d\xi$ is nondecreasing we verify easily by induction

$$\left| \Omega_n \int_{t_0}^{\infty} r(\xi) \, d\xi \right| \leqq \omega^n \int_{t_0}^{\infty} |r(\xi)| \, d\xi. \tag{13}$$

It is namely $\left| \Omega_0 \int_{t_0}^{\infty} r(\xi) \, d\xi \right| \leqq \int_{t_0}^{\infty} |r(\xi)| \, d\xi$ and by means of (13) we receive

$$\left| \Omega_{n+1} \int_{t_0}^{t} r(\xi) \, d\xi \right| \leqq \int_{t_0}^{t} |q(\xi)| \int_{\xi}^{\infty} \frac{1}{|p(\eta)|} \omega^n \int_{t_0}^{\infty} |r(\sigma)| \, d\sigma \, d\eta \, d\xi \leqq$$

$$\leqq \omega^n \int_{t_0}^{\infty} |r(\xi)| \, d\xi \int_{t_0}^{t} |q(\xi)| \int_{\xi}^{\infty} \frac{1}{|p(\eta)|} \, d\eta \, d\xi \leqq \omega^{n+1} \int_{t_0}^{\infty} |r(\xi)| \, d\xi.$$

From this inequality it follows the uniform convergence of the series $\sum_0^{\infty} \Omega_n \int_{t_0}^{t} r(\xi) \, d\xi$ and the estimate (11) for $\epsilon_6(t)$ in (10).

In the same manner we obtain the last part of the theorem.

Note. *Let us define under the assumption*

$$\int_{t_0}^{\infty} \frac{1}{|p(\xi)|} \int_{t_0}^{\xi} [|q(\eta)| + |r(\eta)|] \, d\eta \, d\xi < \infty,$$

the operator Θ_n

$$\Theta_0 x(t) = x(t), \qquad \Theta_n x(t) = \int_{t_0}^{t} \frac{1}{p(\xi)} \int_{t_0}^{\xi} q(\eta) \, \Theta_{n-1} x(\eta) \, d\eta \, d\xi.$$

Then there is a solution $y(t)$ *of* (1) *such that*

$$y(t) = \sum_0^{n} \Theta_k \int_{t_0}^{t} \frac{1}{p(\xi)} \int_{t_0}^{\xi} r(\eta) \, d\eta \, d\xi + \varepsilon_7(t)$$

and

$$|\varepsilon_7(t)| \leqq \frac{\vartheta^{n+1}(t)}{(n+1)!} e^{\vartheta(t)} \int_{t_0}^{t} \frac{1}{|p(\xi)|} \int_{t_0}^{\xi} |r(\eta)| \, d\eta \, d\xi,$$

$$\vartheta(t) = \int_{t_0}^{t} \frac{1}{|p(\xi)|} \int_{t_0}^{\xi} |q(\eta)| \, d\eta \, d\xi.$$

The proof of this statement is similar to that of Theorem 1 and will be omitted here.

REFERENCES

M. Ráb: *Asymptotic expansions of solutions of the equation* $[p(x)y']' - q(x)y = 0$ with complex-valued coefficients.
Arch. Math. (Brno), 8 (1972) No 1, 1—15.

M. Ráb
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia