Jan Hanák

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Archivum Mathematicum, Vol. 10 (1974), No. 1, 9--26

Persistent URL: `http://dml.cz/dmlcz/104814`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT; UJEP BRUNENSIS,
X: 9—26, 1974

ON PLAIN ABSOLUTE EQUILIBRIUM POINTS IN GENERAL NON-ORDERED GAMES WITH PERFECT INFORMATION [II]*)

JAN HANÁK, Brno

(Received April 13, 1973)

§ 4. PROOF OF THE MAIN THEOREM

4.0. We consider the situation described in § 3.0 and use the introduced *conventions*, see §§ 1.0, 1.7, 2.1, 2.6.

We shall assume (without loss of generality; cf., e.g, [9] – the construction in Ch. IV, § 1.9) that *all the chains $\mathscr{V}_j (j \in J)$ are complete* (we shall need this for the construction of elements c_j, cf. below; nevertheless, it would be sufficient to use only certain one-element extensions of the chains $\mathscr{V}_j$).

For any $j \in J$, let $c_j \in V_j$ be such that (for each $\boldsymbol{x} \in \boldsymbol{X}$)

$$f_j^*(\boldsymbol{x}) \left\{\begin{smallmatrix} \geqq \\ = \\ \leqq \end{smallmatrix}\right\} 0 \Rightarrow f_j(\boldsymbol{x}) \left\{\begin{smallmatrix} \geqq \\ = \\ \leqq \end{smallmatrix}\right\} c_j.$$

[Such c_j exists. Namely, if $c_j^- = \sup \{f_j(\boldsymbol{x});\ \boldsymbol{x} \in \boldsymbol{X},\ f_j^*(\boldsymbol{x}) = -1\}$, $c_j^+ = \inf \{f_j(\boldsymbol{x});\ \boldsymbol{x} \in \boldsymbol{X}, f^*(\boldsymbol{x}) = 1\}$ (of course, $\sup \emptyset = \inf V_j$, $\inf \emptyset = \sup V_j$), $C_j = \{f_j(\boldsymbol{x});\ \boldsymbol{x} \in \boldsymbol{X},\ f_j^*(\boldsymbol{x}) = 0\}$, then C_j contains at most one element (see § 1.19, D $(\overline{4.2})$), $c_j^- \leqq c_j^+$ (see D$(\bar{2})$), and $c_j^- \leqq c_j^0 \leqq c_j^+$ if $\{c_j^0\} = C_j$ (see D$(\bar{2})$); thus, we put $c_j = c_j^0$ if $\{c^\circ\} = C_j$, and we choose, e.g., $c_j \in \{c_j^-, c_j^+\}$ if $C_j = \emptyset$.]

Let ∞^{**} be some (auxiliary) element, $\infty^{**} \notin W^*$, let $\mathscr{W}^{**} = (W^{**}, \leqq^{**})$, where $W^{**} = W^{**} \cup \{\infty^{**}\}$, $\leqq^{**} = \leqq^* \cup (W^* \times \{\infty^{**}\})$; therefore, $\mathscr{W}^{**}$ is a chain, ∞^{**} is its greatest element, and $\leqq^{**}$ restricted to W^* gives $\leqq^*$.

Let $\mathrm{L}^{**} : \boldsymbol{X} \to W^{**}$ be such that (for each $\boldsymbol{x} \in \boldsymbol{X}$)

$$\mathrm{L}^{**}(\boldsymbol{x}) = \begin{Bmatrix} \mathrm{L}^*(\boldsymbol{x}) \\ \infty^{**} \end{Bmatrix} \quad \text{if} \quad \mathrm{L}(\boldsymbol{x}) \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix}.$$

Thus, always $\mathrm{L}^{**}(\boldsymbol{x}) \geqq \mathrm{L}^*(\boldsymbol{x})$, and there holds:

$$x \in P, \quad y \in \Gamma x, \quad \boldsymbol{y} \in \Gamma y \Rightarrow \mathrm{L}^{**}(x \oplus \boldsymbol{y}) \geqq \mathrm{L}^{**}(\boldsymbol{y})$$

(see D′(3) in § 1.17: if $\mathrm{L}^{**}(x \oplus \boldsymbol{y}) < \infty^{**}$, then $\mathrm{L}^*(\boldsymbol{y}) \leqq \mathrm{L}^*(x \oplus \boldsymbol{y}) = \mathrm{L}^{**}(x \oplus \boldsymbol{y}) < \infty^{**}$, hence $\mathrm{L}^{**}(\boldsymbol{y}) = \mathrm{L}^*(\boldsymbol{y})$).

*) The first part of this article appeared in Archivum mathematicum *9* (1973), F 1, pp. 151—170.

4.1. By transfinite induction (see § 4.3) *we shall construct, for any ordinal number* ξ, *set* Q^ξ *and mapping* σ^ξ in such a way that, among others, it will hold (for any ordinal number ξ)

$$Q^\xi \subseteq P \tag{0}$$

$$\sigma^\xi \in \mathop{\times}_{z \in Z \cap Q^\xi} \Gamma z \tag{1}$$

$$\operatorname{im} \sigma^\xi \subseteq Q^\xi \tag{2}$$

$$\eta \leqq \xi \Rightarrow Q^\eta \subseteq Q^\xi, \quad \sigma^\eta \subseteq \sigma^\xi \tag{3}$$

Thus, $\sigma^\xi \in \mathrm{T}(\Gamma)$, $\operatorname{im} \sigma^\xi \subseteq Q^\xi \subseteq P_0 \cup (Z \cap Q^\xi) = P_0 \cup \operatorname{dom} \sigma^\xi$. Therefore, σ^ξ is a conservative Γ-transformation (§ 1.9), and $\mathfrak{p}(x, \sigma^\xi)$ (the only Γ-play complying with σ^ξ and starting from x, see § 1.14) is defined for each $x \in P_0 \cup \operatorname{dom} \sigma^\xi (\supseteq Q^\xi)$. Clearly (cf. (3)),

$$\mathfrak{p}(x, \sigma^\eta) = \mathfrak{p}(x, \sigma^\xi) \quad \text{if} \quad x \in P_0 \cup \operatorname{dom} \sigma^\eta (\supseteq Q^\eta).$$

4.2. If the set Q^ξ and the mapping σ^ξ are defined for some ξ (cf. § 4.1), we shall introduce, moreover, mappings χ_x^ξ, F_x^ξ, L_x^ξ, elements y_z^ξ, λ_x^ξ, λ^ξ, and set $\Delta^\xi (x \in P, z \in Z)$ in the following way.

For $x \in P$, let the mappings χ_x^ξ, F_x^ξ and L_x^ξ have domain Γx and be defined in this way:

$$\text{if} \quad y \in \begin{Bmatrix} \Gamma x \cap Q^\xi \\ \Gamma x \setminus Q^\xi \end{Bmatrix}, \quad \text{then}$$

$$\chi_x^\xi(y) = \begin{Bmatrix} -\frac{1}{2} + f^*_{j(x)}(x \oplus \mathfrak{p}(y, \sigma^\xi)) \, [= f^*_{j(x)}(\mathfrak{p}(y, \sigma^\xi)) - \frac{1}{2}] \text{ (cf. D(iii)!)} \\ 0 \end{Bmatrix},$$

$$F_x^\xi(y) = \begin{Bmatrix} f_{j(x)}(x \oplus \mathfrak{p}(y, \sigma^\xi)) \\ c_{j(x)} \end{Bmatrix}, \qquad \mathrm{L}_x^\xi(y) = \begin{Bmatrix} \mathrm{L}^{**}(x \oplus \mathfrak{p}(y, \sigma^\xi)) \\ \infty^{**} \end{Bmatrix}.$$

Now, let $z \in Z$.

Evidently, there exists $\bar\sigma \in \mathrm{T}_F(\Gamma)$ such that $\bar\sigma \mid (Z \cap Q^\xi) = \sigma^\xi$. Let $(\mathbf{y}(y);\ y \in \Gamma z) = (\mathfrak{p}(y, \bar\sigma);\ y \in \Gamma z)$. Then $(\mathbf{y}(y);\ y \in \Gamma z) \in \mathop{\times}_{y \in \Gamma z} \Gamma y$, and $\{\mathbf{y}(y);\ y \in \Gamma z\}$ is plain. Therefore (cf. § 3.0, (B/1)), $\{f_{j(z)}(z \oplus \mathbf{y}(y));\ y \in \Gamma z\}$ is inversely well-ordered in $\mathscr{V}_{j(z)}$. But $\mathbf{y}(y) = \mathfrak{p}(y, \bar\sigma) = \mathfrak{p}(y, \sigma^\xi)$ if $y \in Q^\xi \cap \Gamma z$ (cf. §§ 1.9, 1.14); hence, $\operatorname{im} F_z^\xi (\subseteq \{c_{j(z)}\} \cup \{f_{j(z)}(z \oplus \mathbf{y}(y));\ y \in Q^\xi \cap \Gamma z\} = \{c_{j(z)}\} \cup \{f_{j(z)}(z \oplus \mathfrak{p}(y, \sigma^\xi));\ y \in Q^\xi \cap \Gamma z\})$ is inversely well-ordered in $\mathscr{V}_{j(z)}$. Further, $\operatorname{im} \chi^\xi (\subseteq \{-\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}\})$ is finite. Consequently,

$$\chi^* = \max \{\chi_z^\xi(y);\ y \in \Gamma z\} \qquad \text{and} \qquad F_Y^* = \max \{F^\xi(y);\ y \in Y\}$$

exist for any Y such that $0 \neq Y \subseteq \Gamma z$.

We shall choose element $y_z^\xi = y^* \in \Gamma z$ in the following way (we consider four cases, according to the possible values of χ^*):

a) Let $\chi^* = \frac{1}{2}$. Let $Y = \{y;\ y \in \Gamma z \cap Q^\xi,\ f^*_{j(z)}(\boldsymbol{p}(y, \sigma^\xi)) = 1\}$ $(= \{y;\ y \in \Gamma z,\ \chi^\xi_z(y) = \chi^*\})$; thus, Y is a nonempty subset of Γz, and $f^*_{j(z)}(\boldsymbol{y}(y)) = 1$ for each $y \in Y$. By means of (B/2) (§ 3.0) we conclude that $\{L^*(z \oplus \boldsymbol{y}(y));\ y \in \Gamma z\}$ is well-ordered. Therefore, $\{L^\xi_z(y);\ y \in Y,\ F^\xi_z(y) = F^*_Y\}$ $(\subseteq \{\infty^{**}\} \cup \{L^*(z \oplus \boldsymbol{y}(y));\ y \in \Gamma z\}$, cf. above!) is well-ordered (in $\mathscr{W}^{**}$) and nonempty. Hence, $\min \{L^\xi_z(y);\ y \in Y,\ F^\xi_z(y) = F^*_Y\}$ exists, and we choose $y^* \in Y$ in such a way that $L^\xi_z(y^*)$ equals this minimum.

b) Let $\chi^* = 0$. Then $\Gamma z \setminus Q^\xi \neq 0$, and we choose $y^* \in \Gamma z \setminus Q^\xi$ arbitrarily.

c) Let $\chi^* = -\frac{1}{2}$. We choose any $y^* \in \Gamma z$ such that $L(\boldsymbol{p}(y^*, \sigma^\xi)) = \infty$ (nevertheless, this case could be excluded in the following) if it is possible; otherwise let $y^* \in \{y;\ y \in \Gamma z,\ \chi^\xi_z(y) = \chi^*\}$ $(\neq \emptyset!)$.

d) Let $\chi^* = -\frac{3}{2}$. The definitions of χ^* and χ^ξ_z show that $\Gamma z \subseteq Q^\xi$ and $f^*_{j(z)}(\boldsymbol{p}(y, \sigma^\xi)) = -1$ for each $y \in \Gamma z$. By means of (B/2) we conclude that $\{L^*(z \oplus \boldsymbol{y}(y));\ y \in \Gamma z\}$ is inversely well-ordered. Therefore (cf. case a)!), $\{L^\xi_z(y);\ y \in Y\}$, with $Y = \Gamma z$, is inversely well-ordered and nonempty. Hence, $\max \{L^\xi_z(y);\ y \in Y,\ F^\xi_z(y) = F^*_Y\}$ exists, and we choose $y^* \in Y$ in such a way that $L^\xi_z(y^*)$ equals this maximum.

In any case, *for each* $y \in \Gamma z$ *there holds*:

either
$$\chi^\xi_z(y) < \chi^\xi_z(y^*),$$
or
$$\chi^\xi_z(y) = \chi^\xi_z(y^*), \quad F^\xi_z(y) < F^\xi_z(y^*),$$
or
$$\chi^\xi_z(y) = \chi^\xi_z(y^*) \left\{ \begin{matrix} > \\ \leqq \end{matrix} \right\} 0, \quad F^\xi_z(y) = F^\xi_z(y^*), \quad L^\xi_z(y) \left\{ \begin{matrix} \geqq \\ \leqq \end{matrix} \right\} L^\xi_z(y^*).$$

[This can be proved simply in cases a), d), and also b). – If $\chi^* = -\frac{1}{2}$ (case c), then $\Gamma z \subseteq Q^\xi$ (cf. the definitions of χ^* and χ^ξ_z), and always $f^*_{j(z)}(\boldsymbol{p}(y^*, \sigma^\xi)) = 0$ (see case c) and D(i)), thus $\chi^\xi_z(y^*) = -\frac{1}{2} = \chi^*$ (as $\Gamma z \subseteq Q^\xi$). Further we conclude (using $\Gamma z \subseteq Q^\xi$ again): either $L(\boldsymbol{p}(y^*, \sigma^\xi)) = \infty$, then $L^\xi_z(y^*) = \infty^{**} = \max \{L^\xi_z(y);\ y \in \Gamma z\}$ (cf. § 4.0), or $L(\boldsymbol{p}(y, \sigma^\xi)) < \infty$ for each $y \in \Gamma z$, and then $L^\xi_z(y^*) = \infty^* = \max \{L^\xi_z(y);\ y \in \Gamma z\}$ (cf. above, D(ii) and § 4.0). Thus, always $L^\xi_z(y^*) = \max \{L^\xi_z(y); y \in \Gamma z\}$. Hence, if $y \in \Gamma z$, then $L^\xi_z(y) \leqq L^\xi_z(y^*)$, $\chi^\xi_z(y) \leqq \chi^* = \chi^\xi_z(y^*) = -\frac{1}{2}$, and if $\chi^\xi_z(y) = \chi^\xi_z(y^*)$, then $f^*_{j(z)}(\boldsymbol{p}(y, \sigma^\xi)) = 0 = f^*_{j(z)}(\boldsymbol{p}(y^*, \sigma^\xi))$ (cf. above), thus $f^*_{j(z)}(z \oplus \boldsymbol{p}(y, \sigma^\xi)) = 0 = f^*_{j(z)}(z \oplus \boldsymbol{p}(y^*, \sigma^\xi))$ (cf. D(iii)), but this implies that $F^\xi_z(y) = f_{j(z)}(z \oplus \boldsymbol{p}(y, \sigma^\xi)) = f_{j(z)}(z \oplus \boldsymbol{p}(y^*, \sigma^\xi)) = F^\xi_z(y^*)$ (see § 1.19, D(4.2)).]

Now, we put
$$\lambda^\xi_x = \left\{ \begin{matrix} L^\xi_x(y^\xi_x) \\ L^{**}((x)) \end{matrix} \right\} \quad \text{if} \quad x \in \left\{ \begin{matrix} P_0 \\ Z \end{matrix} \right\},$$
and
$$\lambda^\xi = \min \{\lambda^\xi_x;\ x \in P \setminus Q^\xi\}$$
with
$$\min \emptyset = \infty^{**}.$$

The correctness of the definition of λ^ξ follows from the property (A) (§ 3.0). [Namely, if $P \setminus Q^\xi \neq \emptyset$, we put $\nabla = \{z;\ z \in Z \setminus Q^\xi,\ y_z^\xi \in Q^\xi\}$, and for any $z \in \nabla$ we denote $y(z) = y_z^\xi$, $\mathbf{y}(z) = \mathbf{p}(y_z^\xi, \sigma^\xi)$. It is clear (cf. § 4.1) that $(y(z);\ z \in \nabla) \in \bigtimes_{z \in \nabla} (\Gamma z \setminus \nabla)$, $(\mathbf{y}(z);\ z \in \nabla) \in \bigtimes_{z \in \nabla} \Gamma y(z)$, and that set $\{\mathbf{y}(z);\ z \in \nabla\}$ is plain and each of its elements passes in $Q^\xi \subseteq P \cup \nabla$. Further, $\{\lambda_x^\xi;\ x \in P \setminus Q^\xi\} \subseteq \{\mathrm{L}^{**}((x));\ x \in P_0 \setminus Q^\xi\} \cup \cup \{\mathrm{L}_z^\xi(y_z^\xi);\ z \in \nabla\} \cup \{\infty^{**}\} \subseteq \{\infty^{**}\} \cup \{\mathrm{L}^*((x));\ x \in P_0\} \cup \{\mathrm{L}^*(z \oplus \mathbf{y}(z));\ z \in \nabla\}$ (as $\mathrm{L}^{**}(\mathbf{x}) \in \{\mathrm{L}^*(\mathbf{x}), \infty^{**}\}$ for any $\mathbf{x} \in \mathbf{X}$, see § 4.0), but the three sets $\{\mathrm{L}^*((x));\ x \in P_0\}$, $\{\mathrm{L}^*(z \oplus \mathbf{y}(z));\ z \in \nabla\}$ (cf. (A.1) and (A.2), respectively) and $\{\infty^{**}\}$ are well-ordered in $\mathscr{W}^{**}$, hence the union of them is well-ordered, too, and, therefore, the nonempty subset $\{\lambda_x^\xi;\ x \in P \setminus Q^\xi\}$ of this union has the smallest element.]

Finally, we define

$$\Delta^\xi = \{x;\ x \in P \setminus Q^\xi,\ \lambda_x^\xi = \lambda\}.$$

4.3. (The introduction of the sets Q^ξ and the mappings σ^ξ.)

a) Firstly, we put

$$Q^\circ = \emptyset, \qquad \sigma^\circ = \emptyset;$$

then the conditions (0) – (3) (§ 4.1) are satisfied (for $\xi = 0$) in a trivial way.

b) Let $\xi \geqq 0$ be an ordinal number, and let Q^η and σ^η be defined for all $\eta \leqq \xi$ in such a way that the conditions (0) – (3) (§ 4.1) are satisfied. We put

$$Q^{\xi+1} = Q^\xi \cup \Delta^\xi$$

and we define $\sigma^{\xi+1} : Z \cap Q^{\xi+1} \to P$:

$$\sigma^{\xi+1} z = \begin{Bmatrix} \sigma^\xi z \\ y_z^\xi \end{Bmatrix} \quad \text{if} \quad z \in Z \cap \begin{Bmatrix} Q^\xi \\ \Delta^\xi \end{Bmatrix};$$

the definition is correct, as $Q^\xi \cap \Delta^\xi = \emptyset$.

Note that if $\lambda^\xi = \infty^{**}$, then $\infty^{**} \geqq \lambda_x^\xi \geqq \lambda^\xi = \infty^{**}$ for each $x \in P \setminus Q^\xi$, therefore $\lambda_x^\xi = \lambda^\xi$, and hence $\Delta^\xi = P \setminus Q^\xi$, $Q^{\xi+1} = P$.

The conditions (0), (1) and (3) (§ 4.1) are satisfied, of course, also with $\xi + 1$ in the place of ξ. Further, let $z \in Z \cap Q^{\xi+1}$; then: either $z \in Z \cap Q^\xi$, then $\sigma^{\xi+1} z = \sigma^\xi z \in Q^\xi \subseteq Q^{\xi+1}$; or $z \in (Q^{\xi+1} \setminus Q^\xi) \cap Z = \Delta^\xi \cap Z$, and then if $\lambda^\xi < \infty^{**}$, then $\mathrm{L}_z^\xi(y_z^\xi) = \lambda_z^\xi = \lambda^\xi < \infty^{**}$, $y_z^\xi \in Q^\xi \cap \Gamma z$, $\sigma^{\xi+1} z = y_z^\xi \in Q^\xi \subseteq Q^{\xi+1}$, while if $\lambda^\xi = \infty^{**}$, then (cf. above) $\sigma^{\xi+1} z \in P = Q^{\xi+1}$. Thus, also the condition (2) is satisfied with $\xi + 1$ in the place of ξ.

c) Let $\xi > 0$ be a limit ordinal number, and let Q^ζ and σ^ζ be defined for all $\zeta < \xi$ in such a way that the conditions (0) – (3) are satisfied with ζ in the place of ξ, for each $\zeta < \xi$. We put

$$Q^\xi = \bigcup_{\eta < \xi} Q^\eta, \qquad \sigma^\xi = \bigcup_{\eta < \xi} \sigma^\eta;$$

from the suppositions it immediately follows that the conditions (0) – (3) are satisfied for the considered ξ.

4.4. In such a way Q^ξ and σ^ξ are defined for all ordinal numbers ξ, and the conditions (0) – (3) are satisfied for any ξ. Further, for any ordinal numbers ξ_1, ξ_2

$$\xi_1 \neq \xi_2 \Rightarrow \Delta^{\xi_1} \cap \Delta^{\xi_2} = \emptyset$$

(namely, if, e.g., $\xi_1 < \xi_2$, then $\Delta^{\xi_1} \cap \Delta^{\xi_2} \subseteq Q^{\xi_1+1} \cap \Delta^{\xi_2} \subseteq Q^{\xi_2} \cap \Delta^{\xi_2} = \emptyset$, see (3) in § 4.1, and § 4.2), and for any ξ

$$Q^\xi = \bigcup_{\eta<\xi} \Delta^\eta$$

(it can be easily proved by means of transfinite induction). Thus, we conclude that (for any ξ)

$$\operatorname{card} Q^\xi = \sum_{\eta<\xi} \operatorname{card} \Delta^\eta;$$

but, evidently (cf. § 4.2),

$$Q^\xi \neq P \Leftrightarrow \Delta^\xi \neq \emptyset \Leftrightarrow Q^\xi \subsetneqq Q^{\xi+1}.$$

Hence, supposing that $Q^\xi \subsetneqq Q^{\xi+1}$ for any ξ, we have $\operatorname{card} P < \operatorname{card} \xi \leqq \sum_{\eta<\xi} \operatorname{card} \Delta^\eta = \operatorname{card} Q^\xi \leqq \operatorname{card} P$ for sufficiently great ξ, but this is impossible. Therefore, there exists ξ such that $Q^\xi = Q^{\xi+1} (= P)$, and we may define

$\mu = \min \{\xi;\ \xi \text{ is an ordinal number, } Q^\xi = Q^{\xi+1}\}$.

Of course (cf. above),

$$\Delta^\eta \left\{ \begin{matrix} \neq \\ = \end{matrix} \right\} \emptyset \qquad \text{for any} \qquad \eta \left\{ \begin{matrix} < \\ \geqq \end{matrix} \right\} \mu.$$

Thus (cf. above),

$$P = Q^\mu = \bigcup_{\eta<\mu} \Delta^\eta,$$

and Δ^η are mutually disjoint; consequently, for any $x \in P$ there exists exactly one ordinal number $\xi(x)$ such that

$$x \in \Delta^{\xi(x)}$$

(and, of course, $\xi(x) < \mu$).

4.5.0. We denote

$$\sigma = \sigma^\mu, \qquad \boldsymbol{p} = (\boldsymbol{p}(x);\ x \in P) = (\boldsymbol{p}(x, \sigma);\ x \in P);$$

thus, $\sigma \in \mathrm{T}_F(\Gamma)$, $\boldsymbol{p} \in \bigtimes_{x \in P} \Gamma x$, $\sigma^\xi \subseteq \sigma$ for each ξ, and (see §§ 4.3-4)

$$\sigma z = y_z^{\xi(z)} \qquad \text{for each} \qquad z \in Z.$$

4.5.1. Let $x \in P$ and $z \in Z$ in § 4.5.1.

Evidently,

$$(\alpha) \qquad \boldsymbol{p}(x) = \begin{Bmatrix} x \oplus \boldsymbol{p}(\sigma x) \\ (x) \end{Bmatrix} \quad \text{if} \quad x \in \begin{Bmatrix} Z \\ P_0 \end{Bmatrix},$$

$$(\beta') \qquad \sigma z \in Q^{\xi(z)} \Leftrightarrow \xi(\sigma z) < \xi(z),$$

$$(\gamma) \qquad \lambda^{\xi(x)} = \lambda_x^{\xi(x)} = \begin{Bmatrix} \mathrm{L}_x^{\xi(x)}(\sigma x) \\ \mathrm{L}^{**}((x)) \end{Bmatrix} \quad \text{if} \quad x \in \begin{Bmatrix} Z \\ P_0 \end{Bmatrix};$$

hence (cf. (γ), (α), and § 4.2)

$$(\delta') \qquad \lambda^{\xi(x)} = \begin{Bmatrix} \mathrm{L}^{**}(\boldsymbol{p}(x)) \\ \infty^{**} \end{Bmatrix} \quad \text{if} \quad \begin{Bmatrix} x \in P_0 \vee (x \in Z \wedge \sigma x \in Q^{\xi(x)}) \\ x \in Z \wedge \sigma x \notin Q^{\xi(x)} \end{Bmatrix};$$

consequently,

$$(\delta'') \qquad \lambda^{\xi(x)} < \infty^{**} \Rightarrow \lambda^{\xi(x)} = \mathrm{L}^{**}(\boldsymbol{p}(x)).$$

Further, for any ξ

$$(\varepsilon') \qquad \lambda^{\xi} = \infty^{**}, \qquad \Delta^{\xi} \begin{Bmatrix} = \\ \neq \end{Bmatrix} \emptyset \Rightarrow \begin{Bmatrix} \xi \geqq \mu \\ \xi + 1 = \mu \end{Bmatrix}, \qquad \lambda^{\eta} = \infty^{**} \qquad \text{for all } \eta \geqq \xi;$$

namely, the upper alternative follows trivially from §§ 4.4-2, while if $\lambda^{\xi} = \infty^{**}$, $\Delta^{\xi} \neq \emptyset$, then (cf. § 4.3, case b)) $Q^{\xi+1} = P$, hence $\Delta^{\xi+1} = \emptyset \neq \Delta^{\xi}$, and, consequently (cf. § 4.4), $\xi + 1 = \mu$.

If $\lambda^{\xi(x)} = \infty^{**}$, then (cf. ($\varepsilon'$)) $\xi(x) + 1 = \mu$ (as $\Delta^{\xi(x)} \supseteq \{x\} \neq \emptyset$), hence $\mu > 0$, μ is isolated, and $\xi(x) = \mu - 1$; further, $x \in Z$ (namely, if $x \in P_0$, then (γ) and § 4.0 give $\infty^{**} = \lambda^{\xi(\sigma x)} = \mathrm{L}^{**}((x)) < \infty^{**}$, which is a contradiction), and $\lambda^{\xi(\sigma x)} = \infty^{**}$ [namely, if $\lambda^{\xi(\sigma x)} < \infty^{**}$, then: $\xi(\sigma x) < \xi(x)$ (see (ε')), $\sigma x \in Q^{\xi(x)}$ (see (β')), $\infty^{**} = \lambda^{\xi(x)} = \mathrm{L}^{**}(\boldsymbol{p}(x)) = \mathrm{L}^{**}(x \oplus \boldsymbol{p}(\sigma x))$ (see (δ') and (α)); consequently, $\infty^{**} = \mathrm{L}^{**}(\boldsymbol{p}(\sigma x)) = \lambda^{\xi(\sigma x)} < \infty^{**}$ (see § 4.0 and D(1), and (δ'')), but this is a contradiction], therefore, again $\xi(\sigma x) = \mu - 1$ (cf. the beginning of this consideration). Thus, we have proved

$$(\varepsilon) \qquad \lambda^{\xi(x)} = \infty^{**} \Rightarrow \mu \ \text{ is isolated}, \ \ \mu > 0, \ \ x \in Z, \qquad \xi(\sigma x) = \xi(x) = \mu - 1.$$

Therefore, if $\lambda^{\xi(x)} = \infty^{**}$ and $\boldsymbol{p}(x) = (x_k;\ k \in W_l)$, then $x_{k+1} = \sigma x_k$ for each $k < l$ (§ 4.5.0) and (by means of induction, using (ε)) we obtain that $x_k \in Z$ for each $k \in W_l$, but then $l = \infty$, $\mathrm{L}^{**}(\boldsymbol{p}(x)) = \infty^{**} = \lambda^{\xi(x)}$ (§ 4.0). This and (δ'') give

$$(\delta) \qquad \lambda^{\xi(x)} = \mathrm{L}^{**}(\boldsymbol{p}(x)).$$

If $\xi(\sigma z) \geqq \xi(z)$, then $\sigma z \notin Q^{\xi(z)}$ (see (β')), $\lambda^{\xi(z)} = \infty^{**}$ (see (δ')), and $\xi(\sigma z) = \xi(z)$ (see (ε)); if $\xi(\sigma z) < \xi(z)$, then $\sigma z \in Q^{\xi(z)}$ (see (β')), and $\lambda^{\xi(z)} < \infty^{**}$ (see (ε)); therefore

$$(\beta) \qquad \sigma z \begin{Bmatrix} \in \\ \notin \end{Bmatrix} Q^{\xi(z)} \Leftrightarrow \xi(\sigma z) \begin{Bmatrix} < \\ = \end{Bmatrix} \xi(z) \Rightarrow \lambda^{\xi(z)} \begin{Bmatrix} < \\ = \end{Bmatrix} \infty^{**}.$$

4.5.2.1. Let

$$z \in Z, \qquad \xi(z) < \infty^*.$$

Then (cf. § 4.5.1 (β)) $\sigma z \in Q^{\xi(z)}$, $\xi(\sigma z) < \xi(z)$; by means of § 4.5.1 (α), (δ) and § 4.0 we obtain $\lambda^{\xi(\sigma x)} = L^{**}(\boldsymbol{p}(\sigma z)) \leqq L^{**}(z \oplus \boldsymbol{p}(\sigma z)) = L^{**}(\boldsymbol{p}(z)) = \lambda^{\xi(z)} \angle \infty^*$, hence (see § 4.0, and D(3)) $L^{**}(\boldsymbol{p}(\sigma z)) = L^*(\boldsymbol{p}(\sigma z)) < L^*(z \oplus \boldsymbol{p}(\sigma z)) = L^{**}(z \oplus \boldsymbol{p}(\sigma z))$. Thus

$$\lambda^{\xi(\sigma z)} < \lambda^{\xi(z)}.$$

Considering that $\infty^* > \lambda^{\xi(z)} = L^{**}(\boldsymbol{p}(z)) = L^*(\boldsymbol{p}(z))$ (§ 4.5.1 (ε), and § 4.0), we have

$$f^*_{j(z)}(\boldsymbol{p}(z)) \neq 0$$

(cf. D(ii)), and $\sigma z \in Q^{\xi(z)}$ gives (cf. §§ 4.2, 4.5.0)

$$\chi_z^{\xi(z)}(\sigma z) = -\frac{1}{2} + f^*_{j(z)}(\boldsymbol{p}(z)) \neq -\frac{1}{2}.$$

4.5.2.2. *Let $x \in P$, let η be an ordinal number. Then*

(a) $\quad f^*_{j(x)}(\boldsymbol{p}(x)) = -1 \Rightarrow \Gamma x \subseteq Q^{\xi(x)}$, *and* $f^*_{j(x)}(\boldsymbol{p}(y)) = -1$ *for each* $y \in \Gamma x$.

(b) $\quad f^*_{j(x)}(\boldsymbol{p}(y)) = 1$ *for some* $y \in \Gamma x \cap Q^\eta \Rightarrow y_x^\eta \in Q^\eta$.

Proof.

(a) If $x \in P_0$, then $\Gamma x = \emptyset$, and the statement holds trivially. Let $x \in Z, f^*_{j(x)}(\boldsymbol{p}(x)) = -1$. Then (see D(i), § 4.0, § 4.5.1 (δ), (β)) $L(\boldsymbol{p}(x)) < \infty, \infty^{**} > L^{**}(\boldsymbol{p}(x)) = \lambda^{\xi(x)}$, $\sigma x \in Q^{\xi(x)}$, hence (cf. §§ 4.2, 4.5.0) $-\frac{3}{2} \leqq \chi_x^{\xi(x)}(y) \leqq \chi_x^{\xi(x)}(y_x^{\xi(x)}) = \chi_x^{\xi(x)}(\sigma x) = -\frac{1}{2} + f^*_{j(x)}(\boldsymbol{p}(x)) = -\frac{3}{2}$ holds for each $y \in \Gamma x$; therefore, $\chi_x^{\xi(x)}(y) = -\frac{3}{2}$, and (cf. § 4.2) $y \in Q^{\xi(x)}, f^*_{j(x)}(\boldsymbol{p}(y)) = \frac{1}{2} + \chi_x^{\xi(x)}(y) = -1$.

b) Let $f^*_{j(x)}(\boldsymbol{p}(y)) = 1$ for some $y \in \Gamma x \cap Q^\eta$. Then $\frac{1}{2} = -\frac{1}{2} + f^*_{j(x)}(\boldsymbol{p}(y)) = \chi_x^\eta(y) \leqq \chi_x^\eta(y_x^\eta) \leqq \frac{1}{2}$ (see § 4.2), hence $\chi_x^\eta(y_x^\eta) = \frac{1}{2}$ and (cf. § 4.2) $y_x^\eta \in Q^\eta$.

4.5.2.3. *Let $x \in P$, let η_1 and η_2 be ordinal numbers. Then*

(a) $\quad x \in Z, \quad \eta_1 \leqq \eta_2, \quad y_x^{\eta_1}, y_x^{\eta_2} \in Q^{\eta_1} \Rightarrow \lambda_x^{\eta_1} = \lambda_x^{\eta_2}$.

(b) $\quad x \in P_0 \Rightarrow \lambda_x^{\eta_1} = \lambda_x^{\eta_2}$.

Proof.

(a) Let $\eta_1 \leqq \eta_2$. Then $Q^{\eta_1} \subseteq Q^{\eta_2}$ (§ 4.1 (3) etc.), and, for $x \in Z$, $\chi_x^{\eta_1} \mid Q^{\eta_1} \cap \Gamma x = \chi_x^{\eta_2} \mid Q^{\eta_1} \cap \Gamma x$, $F_x^{\eta_1} \mid Q^{\eta_1} \cap \Gamma x = F_x^{\eta_2} \mid Q^{\eta_1} \cap \Gamma x$, $L_x^{\eta_1} \mid Q^{\eta_1} \cap \Gamma x = L_x^{\eta_2} \mid Q^{\eta_1} \cap \Gamma x$ (see §§ 4.2, 4.5.0), but this and the definition of $y_x^{\eta_1}$ and $y_x^{\eta_2}$ (in § 4.2) give, under the supposition $y_x^{\eta_1}, y_x^{\eta_2} \in Q^{\eta_1}$ $(\subseteq Q^{\eta_2})$, that $\chi_x^{\eta_2}(y_x^{\eta_2}) = \chi_x^{\eta_1}(y_x^{\eta_2}) \leqq \chi_x^{\eta_1}(y_x^{\eta_1}) = \chi_x^{\eta_2}(y_x^{\eta_1}) \leqq \chi_x^{\eta_2}(y_x^{\eta_2})$, hence $\chi_x^{\eta_k}(y_x^{\eta_1}) = \chi_x^{\eta_k}(y_x^{\eta_2})$ for $k = 1, 2$; now $F_x^{\eta_k}(y_x^{\eta_1}) = F_x^{\eta_k}(y_x^{\eta_2})$ $(k = 1, 2)$ follows from § 4.2 by means of an analogical way, and then also $\lambda_x^{\eta_1} = L_x^{\eta_1}(y_x^{\eta_1}) = L_x^{\eta_2}(y_x^{\eta_2}) = \lambda_x^{\eta_2}$ follows analogously from § 4.2.

(b) If $x \in P_0$, then $\lambda_x^{\eta_1} = \mathrm{L}^{**}((x)) = \lambda_x^{\eta_2}$ (see § 4.2).

4.5.3. *Let ξ_1 and ξ_2 be ordinal numbers. Then*

$$(*) \qquad \begin{cases} \xi_1 \leqq \xi_2 \Rightarrow \lambda^{\xi_1} \leqq \lambda^{\xi_2}, \\ \xi_1 < \xi_2,\ \lambda^{\xi_2} < \infty^* \Rightarrow \lambda^{\xi_1} < \lambda^{\xi_2}. \end{cases}$$

Proof. For any ordinal number ξ, let $\mathrm{V}(\xi)$ be the following statement: *if ξ_1 and ξ_2 are ordinal numbers and $\xi_2 \leqq \xi$, then (*) holds.*

Evidently, it is sufficient to prove the following assertion:

if ξ_0 is an ordinal number and if $\mathrm{V}(\xi)$ *holds for any $\xi < \xi_0$, then also* $\mathrm{V}(\xi_0)$ *holds* (as then $\mathrm{V}(\eta)$ holds for each η).

Thus, let ξ_0 be an ordinal number and let $\mathrm{V}(\xi)$ hold for any $\xi < \xi_0$. We shall prove that $\mathrm{V}(\xi_0)$ holds; evidently, it is sufficient to put $\xi_2 = \xi_0$ and to prove (*) with this ξ_2 for each $\xi_1 \neq \xi_2$.

Thus, *let*

$$\xi_1 < \xi_2 = \xi_0;$$

we are proving that $\lambda^{\xi_1} \leqq \lambda^{\xi_2}$ and that, moreover, $\lambda^{\xi_1} < \lambda^{\xi_2}$ whenever $\lambda^{\xi_2} < \infty^*$.

α) Let $\lambda^{\xi_2} \geqq \infty^*$. If $\lambda^{\xi_1} > \lambda^{\xi_2}$, then $\lambda^{\xi_1} = \infty^{**} = \lambda^{\xi_2}$ (see §§ 4.0 and 4.5.1 (ε')), which is impossible. Hence $\lambda^{\xi_1} \leqq \lambda^{\xi_2}$.

β) Now, let $\lambda^{\xi_2} < \infty^*$. Then $\Delta^{\xi_2} \neq \emptyset$ (as $\Delta^{\xi_2} = \emptyset$ gives, by means of § 4.2, $P = Q^{\xi_2}$, $\lambda^{\xi_2} = \infty^{**}$, which is a contradiction); we choose some $x \in \Delta^{\xi_2}$. Thus (see, e.g., §§ 4.4, 4.5.1 (γ))

$$(^+) \qquad \xi_2 = \xi(x),\ \lambda^{\xi_2} = \lambda_x^{\xi_2},$$

and, of course, $x \notin Q^{\xi_1} \cup \Delta^{\xi_1}$ $(= \bigcup_{\eta \leqq \xi_1} \Delta^\eta$, see § 4.4), hence (see § 4.2)

$$(^{++}) \qquad \lambda^{\xi_1} < \lambda_x^{\xi_1}.$$

If $x \in P_0$, then (see (+), (++), and § 4.5.2.3 (b)) $\lambda^{\xi_1} < \lambda_x^{\xi_1} = \lambda_x^{\xi_2} = \lambda^{\xi_2}$.

Thus, let $x \in Z$ in the following.

If $\sigma x \in Q^{\xi_2} \setminus Q^{\xi_1}$, then (see §§ 4.4 and 4.5.2.1) $\xi_1 \leqq \xi(\sigma x) < \xi(x) = \xi_2$, and now (cf. $\mathrm{V}(\xi(\sigma x))$ and § 4.5.2.1) $\lambda^{\xi_1} \leqq \lambda^{\xi(\sigma x)} < \lambda^{\xi(x)} = \lambda^{\xi_2}$.

Thus, let $\sigma x \in Q^{\xi_1}$ (the other possibility) in the following. It is sufficient to prove either $\lambda^{\xi_1} < \lambda^{\xi_2}$ or $\lambda_x^{\xi_1} = \lambda_x^{\xi_2}$, as in the latter case again $\lambda^{\xi_1} < \lambda^{\xi_2}$ (see (+) and (++)).

There holds $f^*_{j(x)}(\boldsymbol{p}(x)) \in \{-1, +1\}$ (see § 4.5.2.1 and D(o)).

a) Let $f^*_{j(x)}(\boldsymbol{p}(x)) = 1$. Then $f^*_{j(x)}(\boldsymbol{p}(\sigma x)) = 1$ (see § 4.5.1 (α) and D(iii)), $\sigma x \in Q^{\xi_1} \cap \Gamma x$ (the supposition), hence $y_x^{\xi_1} \in Q^{\xi_1}$, $y_x^{\xi_2} = \sigma x \in Q^{\xi_1}$ (see § 4.5.2.2 (b) and § 4.5.0), but then § 4.5.2.3 (a) gives $\lambda_x^{\xi_1} = \lambda_x^{\xi_2}$.

b) Let $f^*_{j(x)}(\boldsymbol{p}(x)) = -1$. Then $\Gamma x \subseteq Q^{\xi(x)} = Q^{\xi_2}$, and $f^*_{j(x)}(\boldsymbol{p}(x)) = -1$ for each $y \in \Gamma x$ (§ 4.5.2.2 (a)), hence $\chi_x^{\xi_2}(y) = -\frac{1}{2}$, $F_x^{\xi_2}(y) = f_{j(x)}(x \oplus \boldsymbol{p}(y))$, and $\mathrm{L}_x^{\xi_2}(y) = \mathrm{L}^{**}(x \oplus \boldsymbol{p}(y))$ for each $y \in \Gamma x$ (§§ 4.2, 4.5.0). If $\Gamma x \subseteq Q^{\xi_1}$, then, of course,

$y_x^{\xi_1}$, $y_x^{\xi_2} \in Q^{\xi_1}$, and $\lambda_x^{\xi_1} = \lambda_x^{\xi_2}$ (§ 4.5.2.3 (a)). Thus, let there exist $y \in \Gamma x \setminus Q^{\xi_1} \subseteqq Q^{\xi_2} \setminus Q^{\xi_1}$; then $\xi_1 \leqq \xi(y) < \xi_2$ (§ 4.4). But (cf. above) $\chi_x^{\xi_2}(y) = -\frac{1}{2} = \chi_x^{\xi_2}(\sigma x)$ hence (see § 4.2) either $F_x^{\xi_2}(y) = F_x^{\xi_2}(\sigma x)$, and then $\mathrm{L}_x^{\xi_2}(y) \leqq \mathrm{L}_x^{\xi_2}(\sigma x)$ or $\boldsymbol{f}_{j(x)}(x \oplus \boldsymbol{p}(y)) = F_x^{\xi_2}(y) < F_x^{\xi_2}(\sigma x) = \boldsymbol{f}_{j(x)}(x \oplus \boldsymbol{p}(\sigma x))$, but $\boldsymbol{f}^*_{j(x)}(x \oplus \boldsymbol{p}(y)) = -1 = \boldsymbol{f}^*_{j(x)}(x \oplus \boldsymbol{p}(\sigma x))$ (cf. above and D(iii)), hence (cf. above and D(3)) $\mathrm{L}^*(x \oplus \boldsymbol{p}(y)) \leqq \mathrm{L}^*(x \oplus \boldsymbol{p}(\sigma x)) = \mathrm{L}^*(\boldsymbol{p}(x)) = \lambda^{\xi_2} < \infty^*$ (see § 4.5.2.1), therefore (cf. § 4.0) $\mathrm{L}_x^{\xi_2}(y) = \mathrm{L}^{**}(x \oplus \boldsymbol{p}(y)) = \mathrm{L}^*(x \oplus \boldsymbol{p}(y)) \leqq \lambda^{\xi_2} = \mathrm{L}_x^{\xi_2}(\sigma x)$. Consequently, always $(\mathrm{L}_x^{\xi_2}(y) =)\, \mathrm{L}^{**}(x \oplus \boldsymbol{p}(y)) \leqq \lambda^{\xi_2}\ (= \mathrm{L}_x^{\xi_2}(\sigma x)) < \infty^*$, hence (cf. § 4.0) $\mathrm{L}^{**}(x \oplus \boldsymbol{p}(y)) = \mathrm{L}^*(\boldsymbol{p}(y))$, $\mathrm{L}^{**}(\boldsymbol{p}(y)) = \mathrm{L}^*(\boldsymbol{p}(y))$, and from this and using $\xi_1 \leqq \xi(y) < \xi_2$ (cf. $\mathrm{V}(\xi(y))$, § 4.5.1 (δ) and D(3)) we conclude that $\lambda^{\xi_1} \leqq \lambda^{\xi(y)} = \mathrm{L}^*(\boldsymbol{p}(y)) < \mathrm{L}^*(x \oplus \boldsymbol{p}(y)) \leqq \lambda^{\xi_2} < \infty^*$, and hence $\lambda^{\xi_1} < \lambda^{\xi_2}$.

4.5.4. *Let $z \in Z$, $y \in \Gamma z$. Then* (cf. § 4.5.1 (α)!)

$$(*) \qquad \boldsymbol{f}_{j(z)}(z \oplus \boldsymbol{p}(y)) \leqq \boldsymbol{f}_{j(z)}(z \oplus \boldsymbol{p}(\sigma z)) = \boldsymbol{f}_{j(z)}(\boldsymbol{p}(z)).$$

Proof. We denote $j = j(z)$, $\xi = \xi(z)$; thus $\sigma z = y_z^\varepsilon$ (§ 4.5.0). There occurs exactly one of the following three cases:

(*i*) $y^\xi \in Q^\xi$, $\sigma z \in Q^\xi$. Then (cf. §§ 4.2, 4.5.1 (α)) there occurs one of the following two subcases:

(*i/1*) $-\dfrac{1}{2} + \boldsymbol{f}_j^*(z \oplus \boldsymbol{p}(y)) = \chi_z^\xi(y) < \chi_z^\xi(\sigma z) = -\dfrac{1}{2} + \boldsymbol{f}_j^*(\boldsymbol{p}(z))$;

(*i/2*) $\boldsymbol{f}_j(z \oplus \boldsymbol{p}(y)) = F_z^\xi(y) \leqq F_z^\xi(\sigma z) = \boldsymbol{f}_j(\boldsymbol{p}(z))$.

If (i/2) occurs, then (*) holds; if (i/1) occurs, then $\boldsymbol{f}_j^*(z \oplus \boldsymbol{p}(y)) < \boldsymbol{f}_j^*(\boldsymbol{p}(z))$, and hence (see $\mathrm{D}(\bar{2})$) again (*) holds.

(*ii*) $y \notin Q^\xi$, $\sigma z \in Q^\xi$. Then $0 = \chi_z^\xi(y) \leqq \chi_z^\xi(\sigma z) \neq 0$ (see § 4.2), hence $\chi_z^\xi(y) = 0 < \dfrac{1}{2} = \chi_z^\xi(\sigma z) = -\dfrac{1}{2} + \boldsymbol{f}_j^*(\boldsymbol{p}(z))$,

$$(+/1) \qquad \boldsymbol{f}_j^*(\boldsymbol{p}(z)) = 1.$$

If $\boldsymbol{f}_j^*(z \oplus \boldsymbol{p}(y)) \leqq 0$, then (*) holds (see D(2)). Thus, let

$$(+/2) \qquad \boldsymbol{f}_j^*(z \oplus \boldsymbol{p}(y)) = 1$$

(in the following in case (ii)). Thus (see (+/1), §§ 4.5.1 (β), 4.0)

$$(+/3) \qquad \mathrm{L}^*(\boldsymbol{p}(z)) = \lambda^\xi \leqq \infty^*,$$

and by means of (+/2), D(i) and § 4.0 we conclude

$$(+/4) \qquad \mathrm{L}^*(z \oplus \boldsymbol{p}(y)) = \mathrm{L}^{**}(z \oplus \boldsymbol{p}(y)) \leqq \infty^*.$$

If $\mathrm{L}^*(z \oplus \boldsymbol{p}(y)) = \infty^*$ ($\geqq \mathrm{L}^*(\boldsymbol{p}(z))$, see (+/3)), then: if $\infty^* > \mathrm{L}^*(\boldsymbol{p}(z))$, then (*) follows from (+/1), (+/2) and $\mathrm{D}(\bar{3})$, while if $\infty^* = \mathrm{L}^*(\boldsymbol{p}(z))$, then (*) follows from (+/1), (+/2) and $\mathrm{D}(\bar{4})$.

Thus, let now

$$(+/5) \qquad \mathrm{L}^*(z \oplus \boldsymbol{p}(y)) < \infty^*;$$

then $\xi(y) \geqq \xi$ (as $y \notin Q^\xi$, cf. § 4.4), and we obtain $\infty^* > \mathrm{L}^*(z \oplus \boldsymbol{p}(y)) > \mathrm{L}^*(\boldsymbol{p}(y)) = \lambda^{\xi(y)} \geqq \lambda^\xi = \mathrm{L}^*(\boldsymbol{p}(z))$ (see $(+/5)$, D(3), §§ 4.0, 4.5.1 (δ), 4.5.3, and $(+/3)$), hence $\mathrm{L}^*(z \oplus \boldsymbol{p}(y)) > \mathrm{L}^*(\boldsymbol{p}(z))$, but this together with $(+/2)$, $(+/1)$ and D(3) again gives (*).

(*iii*) $\sigma z \notin Q^\xi$. Then (see §§ 4.5.1 (β), (δ), 4.0, and D(i)) $\infty^{**} = \lambda^\xi = \mathrm{L}^{**}(\boldsymbol{p}(z))$, $\mathrm{L}(\boldsymbol{p}(z)) = \infty$,

$$(+) \qquad f_j^*(\boldsymbol{p}(z)) = 0.$$

Further, $z \in \Delta^{\xi(z)}$ (§ 4.4), thus (§ 4.5.1 (ε')) $\xi + 1 = \mu$, hence (cf. § 4.4) $P = Q^\mu = Q^{\xi+1} = Q^\xi \cup \Delta^\xi$. If $y \in \Gamma x \cap \Delta^\xi$, then $\infty^{**} = \lambda^\xi = \lambda^{\xi(y)} = \mathrm{L}^{**}(\boldsymbol{p}(y))$, $\mathrm{L}(\boldsymbol{p}(y)) = \infty = \mathrm{L}(z \oplus \boldsymbol{p}(y))$, $f_j^*(z \oplus (\boldsymbol{p}y)) = 0 = f_j^*(\boldsymbol{p}(z))$ (see (+), and again §§ 4.5.1 (δ) 4.0, and D(i)), therefore (*) follows from DD(ii), $(\bar{4})$. If $y \in \Gamma x \cap Q^\xi$ (the other possibility), then $-\frac{1}{2} + f_j^*(z \oplus \boldsymbol{p}(y)) = \chi_z^\xi(y) \leqq \chi_z^\xi(\sigma z) = 0$, $f_j^*(z \oplus \boldsymbol{p}(y)) \leqq \underline{0} = f_j^*(\boldsymbol{p}(z))$ (see §§ 4.5.0, 4.2, and (+)), and now (*) follows from $\mathrm{D}(\bar{2})$, or DD(ii), $(\bar{4})$.

4.5.5. *Let* $z \in Z$, $j \in J$, $f_j^*(\boldsymbol{p}(z)) = -1$, *let*

$$y \begin{Bmatrix} = \sigma z \\ \in \Gamma z \end{Bmatrix} \quad \textit{if} \quad j(z) \begin{Bmatrix} \neq \\ = \end{Bmatrix} j.$$

Then $f_j^*(\boldsymbol{p}(y)) = -1$, *and* $\xi(y) < \xi(z)$.

Proof.

a) If $j \neq j(z)$, then $y = \sigma z$ (the supposition), hence (see § 4.5.1 (α) and D(iii)) $f_j^*(\boldsymbol{p}(y)) = f_j^*(\sigma z)) = f_j^*(\boldsymbol{p}(z)) = -1$; further, $\sigma z \notin Q^{\xi(z)}$ (namely, if $\sigma z \in Q^{\xi(z)}$, then by means of §§ 4.5.1 (β), (δ), 4.0 and D(i) we obtain $\infty^{**} = \lambda^{\xi(z)} = \mathrm{L}^{**}(\boldsymbol{P}(z))$, $\mathrm{L}(\boldsymbol{p}(z)) = \infty, f_j^*(\boldsymbol{p}(z)) = 0$, but this is a contradiction). Therefore (§ 4.5.1 (β)) $\xi(y) = \xi(\sigma z) < \xi(z)$.

b) If $j = j(z)$, then (see §§ 4.5.2.2(a) and 4.4) $f_j^*(\boldsymbol{p}(y)) = -1$, $y \in \Gamma z \subseteq Q^{\xi(z)} = \bigcup_{\eta < \xi(z)} \Delta^\eta$, hence $\xi(y) < \xi(z)$.

4.6. σ *is a plain absolute equilibrium point of* $\mathscr{G}$.
(Thus, the main theorem is proved.)

Proof. $\sigma \in \mathrm{T_F}(\Gamma)$ (§ 4.5.0). Let $x \in P$, $j \in J$, let $\boldsymbol{x} = (x_k; k \in W_l) \in \boldsymbol{\Gamma} x$ comply with $\sigma \mid (Z \setminus Z(j))$. It is sufficient to prove (cf. §§ 2.9, 4.5.0) that

$$(*) \qquad f_j(\boldsymbol{x}) \leqq f_j(\boldsymbol{p}(\boldsymbol{x})).$$

Let

$$K = \{k; k < l, x_{k+1} \neq \sigma x_k\}.$$

Of course,

$$(+) \qquad \{x_k; k \in K\} \subseteq Z(j),$$

$$(++) \qquad k < l, \quad j(x_k) \neq j \Rightarrow x_{k+1} = \sigma x_k.$$

If $f_j^*(\boldsymbol{p}(x)) = -1$, then by means of $(++)$ and § 4.5.5 (and by induction) we obtain that for each $k < l$ there holds $f_j^*(\boldsymbol{p}(x_k)) = -1 = f_j^*(\boldsymbol{p}(x_{k+1}))$ and $\xi(x_{k+1}) < \xi(x_k)$; thus, $(\xi(x_k); k \in W_l)$ is a decreasing sequence of ordinal numbers, hence it must be finite. Therefore, $l < \infty$, and K is finite in the considered case.

Thus, if K is infinite, then $f_j^*(\boldsymbol{p}(x)) \geqq 0$ and, of course, $\mathrm{L}(\mathbf{x}) = \infty$, hence (see D(i)) $f_j^*(\mathbf{x}) = 0 \leqq f_j^*(\boldsymbol{p}(x))$, and now (*) follows from DD($\bar{2}$), (ii), ($\bar{4}$).

Let K be finite in the following. Then there exists $k_0 \in W_l$ such that $k < k_0$ for each $k \in K$ (see $(+)$). Clearly, $\mathbf{x}^{[k_0]} = \boldsymbol{p}(x_{k_0})$; thus, of course $f_j(\mathbf{x}^{[k_0]}) = f_j(\boldsymbol{p}(x_{k_0}))$. Let $f_j(\mathbf{x}^{[k]}) \leqq f_j(\boldsymbol{p}(x_j))$ hold for some $k > 0$, $k \in W_{k_0}$. Then $f_j(\mathbf{x}^{[k-1]}) = f_j(x_{k-1} \oplus \oplus\, \mathbf{x}^{[k]}) \leqq f_j(x_{k-1} \oplus \boldsymbol{p}(x_k)) \leqq f_j(x_{k-1} \oplus \boldsymbol{p}(\sigma x_{k-1})) = f_j(\boldsymbol{p}(x_{k-1}))$ (cf. the induction supposition and D($\bar{1}$) – namely, $x_k \in \Gamma x_{k-1}$, $\mathbf{x}^{[k]}$, $\boldsymbol{p}(x_k) \in \Gamma x_k$; further, see § 4.5.4 if $j(x_{k-1}) = j$, and $(++)$ if $j(_{k-1}) \neq j$, and cf. § 4.5.1 (α)). Consequently, $f_j(\mathbf{x}^{[k]}) \leqq \leqq f_j(\boldsymbol{p}(x_k))$ holds for each $k \in W_{k_0}$; but for $k = 0$ we obtain $f_j(\mathbf{x}) = f_j(\mathbf{x}^{[0]}) \leqq \leqq f_j(\boldsymbol{p}(x_0)) = f_j(\boldsymbol{p}(x))$, hence (*) holds.

§ 5. REMARKS

In this section, we present an extension of our results (§ 3) to the case of poset-valued pay-off functions (we shall need some auxiliary definitions and propositions, see § 5.0), and we comment on the meaning of various notions of equilibrium point, on the results of this paper, on the main proof (§ 4) and on the construction used in it.

5.0.0. Definitions, remarks. Let $\mathscr{V} = (V, \leqq)$ be a *poset*.

Under a *linear extension* of $\mathscr{V}$ [or $\leqq$] one means $\hat{\mathscr{V}} = (V, \hat{\leqq})$ [or $\hat{\leqq}$] such that $(V, \hat{\leqq})$ is a chain and $\leqq \subseteq \hat{\leqq}$.

One says that $\mathscr{V}$ satisfies the *maximum condition* iff any nonempty subset of V has a maximal element. (There are well-known equivalent conditions, especially the *increasing chain condition*. Cf., e.g., [9], ch. I, § 5. Of course, if $\mathscr{V}$ is a chain, then the maximum condition for $\mathscr{V}$ is satisfied iff $\mathscr{V}$ is inversely well-ordered.)

We say that $\mathscr{V}$ satisfies the *finite antichain condition* iff any infinite subset of V contains some two distinct but $\leqq$-comparable elements. Of course, this condition is satisfied if $\mathscr{V}$ is a chain or if V is finite.

5.0.1. Theorem. (Szpilrajn, [11]) *Every poset has a linear extension.*

5.0.2. Lemma. *Let $\mathscr{V} = (V, \leqq)$ be a poset satisfying both the maximum condition and the finite antichain condition. Then any linear extension of $\mathscr{V}$ is inversely well-ordered.*

Proof. Let $\hat{\mathscr{V}} = (V, \hat{\leqq})$ be a linear extension of $\mathscr{V}$. Let $\emptyset \neq V' \subseteq V$. The set V'' of all $\leqq$-maximal elements of V' is nonempty. V'' is finite (as, of course, each two distinct elements of V'' are $\leqq$-incomparable), hence it has the $\hat{\leqq}$-greatest element; denote this element by v^*. If $v \in V'$, then $\{v'; v' \in V', v' \geqq v\}$ is nonempty and has a $\leqq$-maximal element; let v'' be such an element. Evidently, $v'' \in V''$, thus $v^* \hat{\geqq} v''$, but $v'' \geqq v$, hence $v'' \hat{\geqq} v$, and $v^* \hat{\geqq} v$. Therefore, V' has the $\hat{\leqq}$-greatest element (namely v^*).

5.0.3. Remark. Connected questions are investigated by V. Novák in [16] and [15]; we shall need especially the following result, which is the dual formulation of theorem 2.3 in [15] (cf. § 5.0.0).

5.0.4. Theorem. *Let $\mathscr{V}$ be a poset. Then $\mathscr{V}$ has an inversely well-ordered linear extension iff $\mathscr{V}$ satisfies the maximum condition.*

5.1. A ***poset-valued pay-off function*** on a graph Γ is given by a poset $\mathscr{V} = (V, \leqq)$ and a mapping $f: \mathbf{X}_\Gamma \to V$ (cf. § 1.16). A ***game with perfect information and with poset-valued pay-off functions*** (we also say only: a ***generalized g. p. i.***) is defined in the same way as a game with perfect information in § 2.2 but with the exception that $\mathscr{V}_j$ are posets.

5.2. Definition, remarks. Under a ***plain absolute equilibrium point*** [***strong plain absolute equilibrium point***] of a generalized g.p.i.

$$\mathscr{G} = (\Gamma, (P(j);\ j \in J),\ \ (\mathscr{V}_j = (V_j, \leqq_j);\ j \in J),\ \ (f_j; j \in J))$$

we mean (cf. §§ 2.9, 2.8) $\sigma \in \mathrm{T}_{\mathrm{F}}(\Gamma)$ such that for each $x \in P$, $j \in J$, and for any $\mathbf{x} \in \Gamma x$ complying with $\sigma \mid (Z \setminus Z(j))$ there holds

$$f_j(\mathbf{x}) \not>_j f_j(\mathbf{p}(x, \sigma))$$
$$[f_j(\mathbf{x}) \leqq_j f_j(\mathbf{p}(x, \sigma))];$$

(respectively). Of course, if σ is a strong plain absolute equilibrium point of $\mathscr{G}$, then σ is a plain absolute equilibrium point of $\mathscr{G}$; if $\mathscr{G}$ is a g.p.i. in the sense of § 2.2 (i.e. if all the posets $\mathscr{V}_j$ are chains), then both the introduced notions are equivalent to the notion of plain absolute equilibrium point of $\mathscr{G}$ in the sense of § 2.9. Thus, the terminology is correct.

5.3.1. Definition, remarks. Let Γ be a graph, let L^* with a chain W^* be a pseudo-length (on Γ), let $\boldsymbol{f}^*$ be an L^*-qualitative pay-off function (on Γ). Let $\boldsymbol{f}: \mathbf{X}_\Gamma \to V$ with a poset $\mathscr{V} = (V, \leqq)$ be a poset-valued pay-off function. We say that $\boldsymbol{f}$ is an L^****-quasiqualitative (poset-valued) pay-off function complying with*** $\boldsymbol{f}^*$ iff the conditions $\mathrm{DD}(\bar{1}) - (\bar{4})$ (§ 1.19) are satisfied and, moreover,

$\mathrm{D}(\bar{0})$ $\quad y \in \Gamma x,\ \ \mathbf{y}_1, \mathbf{y}_2 \in \Gamma y,\ \ f(\mathbf{y}_1) \,||\, f(\mathbf{y}_2) \Rightarrow$ either $f(x \oplus \mathbf{y}_1) = f(x \oplus \mathbf{y}_2)$,

or $f(x \oplus \mathbf{y}_1) = f(\mathbf{y}_1)$, $f(x \oplus \mathbf{y}_2) = f(\mathbf{y}_2)$

(for each $x \in P = \operatorname{dom} \Gamma$). Of course, $\mathrm{D}(\bar{0})$ is satisfied trivially if $\mathscr{V}$ is a chain; thus, the terminology is correct (cf. § 1.19).

5.3.2. Lemma. *Let Γ be a graph, let L^* with a chain $\mathscr{W}^*$ be a pseudolength on Γ. Let f^* be an L^*-qualitative pay-off function on Γ.*

Let $f: \mathbf{X}_\Gamma \to V$ with a poset $\mathscr{V} = (V, \leqq)$ be an L^*-quasiqualitative (poset-valued) pay-off function complying with f^*.

If $\hat{\mathscr{V}} = (V, \hat{\leqq})$ is a linear extension of $\mathscr{V}$, then f with $\mathscr{V}$ is an L^*-quasiqualitative pay-off function complying with f^*.

Proof. Let $\hat{\leqq}$ be an arbitrary linear extension of $\leqq$. Evidently, $\mathrm{DD}(\bar{2}) - (\bar{4})$ are satisfied with $\hat{\leqq}$ in the place of $\leqq$. Let $x \in P$, $y \in \Gamma x$, $\mathbf{y}_1, \mathbf{y}_2 \in \boldsymbol{\Gamma} y$, $f(\mathbf{y}_1) \hat{\leqq} f(\mathbf{y}_2)$. If $f(\mathbf{y}_1) \leqq f(\mathbf{y}_2)$, then $f(x \oplus \mathbf{y}_1) \leqq f(x \oplus \mathbf{y}_2)$ (as $\mathrm{D}(\bar{1})$ holds), hence $f(x \oplus \mathbf{y}_1) \hat{\leqq} \hat{\leqq} f(x \oplus \mathbf{y}_2)$. If $f(\mathbf{y}_1) \| f(\mathbf{y}_2)$ (the other possibility), then (cf. $\mathrm{D}(\bar{0})$) either $f(x \oplus \mathbf{y}_1) = = f(\mathbf{y}_1) \hat{<} f(\mathbf{y}_2) = f(x \oplus \mathbf{y}_2)$, or $f(x \oplus \mathbf{y}_1) = f(x \oplus \mathbf{y}_2)$. Thus, also $\mathrm{D}(\bar{1})$ holds with $\hat{\leqq}$ in the place of $\leqq$. Therefore, the lemma is proved (cf. § 1.19).

5.4. ("Meta-") theorem. *The main theorem* (§ 3.0) *will hold, too, after the following re-formulation*:

"*game with perfect information and with poset-valued pay-off functions*" is to be written instead of "*game with perfect information*";

"$\{f_{j(z)}(z \oplus \mathbf{y}(y));\ y \in \Gamma z\}$ *satisfies both the maximum condition and the finite antichain condition*" is to be written instead of "$\{f_{j(z)}(z \oplus \mathbf{y}(y));\ y \in \Gamma z\}$ *is inversely well-ordered*";

denotations *(0)*, *(B)*, *(B/1)* are to be replaced by $(\tilde{O})$, $(\tilde{B})$, $(\widetilde{B/1})$ (respectively).

Proof. This follows immediately from §§ 5.0.1, 5.0.2, and 5.3.2: namely, if $\tilde{\mathscr{G}}$ arises from $\mathscr{G}$ by the replacement of each poset $\mathscr{V}_j$ by its linear extension $\hat{\mathscr{V}}_j$ (cf. § 5.0.1), then $\tilde{\mathscr{G}}$ satisfies the conditions of the main theorem (cf. §§ 5.3.2, 5.0.2, 3.0), and hence $\tilde{\mathscr{G}}$ has a plain absolute equilibrium point σ, but it is clear (see § 5.2) that σ is also a plain absolute equilibrium point of $\mathscr{G}$.

5.5. Remark. By a quite analogical replacement it is possible to obtain a new result from § 3.3 (case (II^+)).

The new results obtained from §§ 3.0 and 3.3 are based, among others, on §§ 5.0.1-2. A simpler replacement (namely, without rewritings concerning the functions f_j, and, hence, without applying §§ 5.0.1-2) would give new results from the cases (I) – (II) of § 3.2.

Further, new results can be obtained from the case (III) of § 3.2 and from § 3.4 by means of a replacement (namely, "generalized g.p.i.", "im f_j satisfies the maximum condition", and (only in § 3.4) "$\mathrm{DD}(\bar{0}), (\bar{1})$" is to be written instead of "g.p.i.", "im f_j is inversely well-ordered", and "$\mathrm{D}(\bar{1})$", respectively) based on § 5.0.4 (instead of §§ 5.0.1-2); of course, at deriving the new result from § 3.4, we also use § 1.21

(cf. the remark in § 3.4) and the fact (following from the proof of lemma 5.3.2) that if a poset-valued pay-off function f with a poset $\mathscr{V}$ satisfies DD($\bar{0}$), ($\bar{1}$), then f with any linear extension $\hat{\mathscr{V}}$ of $\mathscr{V}$ satisfies D($\bar{1}$).

5.6.0. Remark. Under our conception, we have admitted that in games with perfect information players know the preceding course of play at any moment. Then the corresponding variant of the notion of pure strategy can be introduced in the following way.

Supposition. In §§ 5.6.1-3, let Γ be a graph, $P = \operatorname{dom} \Gamma$ (etc., cf. § 1.7), $\mathbf{X} = \mathbf{X}_\Gamma$.

5.6.1. General transformations. (Cf. § 1.9) Under a *Γ-segment* we mean any finite sequence $\mathbf{z} = (z_0, \ldots, z_m)$ $(m \geqq 0)$ of the elements of Z such that $z_i \in \Gamma z_{i-1}(i = 1, \ldots, m)$; for such a Γ-segment $\mathbf{z}$, $\varkappa(\mathbf{z})$ will be the *last element* of $\mathbf{z}$ (here $\varkappa(\mathbf{z}) = z_m$). $\mathbf{Z}_\Gamma$ will be *the set of Γ-segments.* We put

$$\mathbf{T}(\Gamma) = \bigcup_{\mathbf{Y} \subseteq \mathbf{Z}_\Gamma} \bigtimes_{\mathbf{z} \in \mathbf{Y}} \Gamma\, \varkappa(\mathbf{z}), \quad \mathbf{T}_F(\Gamma) = \bigtimes_{\mathbf{z} \in \mathbf{Z}_\Gamma} \Gamma\, \varkappa(\mathbf{z}).$$

$\mathbf{T}(\Gamma)$ is said to be the set of ***general Γ-transformations,*** and its subset $\mathbf{T}_F(\Gamma)$ is called the set of *full* general Γ-transformations. E.g., $\emptyset \in \mathbf{T}(\Gamma)$; $\emptyset$ is called the *empty* general Γ-transformation. $\boldsymbol{\sigma} \in \mathbf{T}(\Gamma)$ is said to be *conservative* iff $(z_0, \ldots, z_m, \boldsymbol{\sigma}(\mathbf{z})) \notin \mathbf{Z}_\Gamma \setminus \operatorname{dom} \boldsymbol{\sigma}$ for any $\mathbf{z} = (z_0, \ldots, z_m) \in \operatorname{dom} \boldsymbol{\sigma}$. Of course, $\emptyset$ and also all full general Γ-transformations are conservative.

5.6.2. General transformations and plays. (Cf. § 1.14) Let $\boldsymbol{\sigma} \in \mathbf{T}(\Gamma)$, $\mathbf{x} = (x_k; k \in W_l) \in \mathbf{X}$. We say that $\mathbf{x}$ *complies with* $\boldsymbol{\sigma}$ iff

$$k < l, (x_0, \ldots, x_k) \in \operatorname{dom} \boldsymbol{\sigma} \Rightarrow x_{k+1} = \boldsymbol{\sigma}(x_0, \ldots, x_k).$$

It is easy to see that if $\boldsymbol{\sigma}$ is a conservative general Γ-transformation and $x \in P_0 \cup \{z; z \in Z, (z) \in \operatorname{dom} \boldsymbol{\sigma}\}$, then there exists *exactly one* $\mathbf{x} \in \Gamma x$ which complies with $\boldsymbol{\sigma}$; this Γ-play will be denoted by $\boldsymbol{p}(x, \boldsymbol{\sigma})$.

5.6.3. Plain transformations and general transformations. The mapping

$$\Phi_\Gamma : \sigma \mapsto \Phi_\Gamma(\sigma) = (\sigma\varkappa(\mathbf{z}); \quad \mathbf{z} \in \mathbf{Z}_\Gamma, \quad \varkappa(\mathbf{z}) \in \operatorname{dom} \sigma)\ (\sigma \in \mathrm{T}(\Gamma))$$

is an injective mapping of $\mathrm{T}(\Gamma)$ into $\mathbf{T}(\Gamma)$. Φ_Γ is called the *natural injection* (of $\mathrm{T}(\Gamma)$ into $\mathbf{T}(\Gamma)$; $\Phi_\Gamma(\sigma)$ "expresses $\sigma \in \mathrm{T}(\Gamma)$ in terms of general Γ-transformations". This expression has natural properties: if $\sigma \in \mathrm{T}(\Gamma)$, $\boldsymbol{\sigma} = \Phi_\Gamma(\sigma)$, then: if σ is conservative [empty; full], then $\boldsymbol{\sigma}$ is conservative [empty; full] (respectively); if σ is conservative, then $P_0 \cup \operatorname{dom} \sigma = P_0 \cup \{z; z \in Z, (z) \in \operatorname{dom} \boldsymbol{\sigma}\}$ (cf. §§ 1.14, 5.6.2), and $\boldsymbol{p}(x, \sigma) = \boldsymbol{p}(x, \boldsymbol{\sigma})$ for any $x \in P_0 \cup \operatorname{dom} \sigma$; (a play) $\mathbf{x}$ complies with σ if and only if $\mathbf{x}$ complies with $\boldsymbol{\sigma}$.

5.7.0. Remarks. (Cf. §§ 0, 2.8.) As we shall show now, the two variants of the notion of (pure) equilibrium point in general strategies (for games with perfect information) can be introduced similarly as in the "plain case" (§§ 2.8-9).

In §§ 5.7.1-6, let

$$\mathscr{G} = (\Gamma, (P(j);\ j \in J),\ (\mathscr{V}_j = (V_j, \leqq j);\ j \in J),\ (f_j; j \in J))$$

be a game with perfect information. We shall use the introduced conventions; $\boldsymbol{p}$ will have the meaning given by § 5.6.2.

5.7.1. Definitions, remarks. (Cf. §§ 2.6–7.) We put

$$\boldsymbol{Z}_\Gamma(j) = \{\boldsymbol{z};\ \boldsymbol{z} \in \boldsymbol{Z}_\Gamma,\quad \varkappa(\boldsymbol{z}) \in P(j)\}$$

(thus, $\boldsymbol{Z}_\Gamma(j_1) \cap \boldsymbol{Z}_\Gamma(j_2) = \emptyset$ if $j_1, j_2 \in J$, $j_1 \neq j_2$; $\bigcup_{j \in J} \boldsymbol{Z}_\Gamma(j) = \boldsymbol{Z}_\Gamma$),

$$\boldsymbol{S}(j) = \mathop{\times}_{\boldsymbol{z} \in \boldsymbol{Z}_\Gamma(j)} \Gamma\, \varkappa(\boldsymbol{z}), \qquad \boldsymbol{S} = \mathop{\times}_{j \in J} \boldsymbol{S}(j).$$

Elements of $\boldsymbol{S}(j)$ are called *(general) strategies* of j (in $\mathscr{G}$). Elements of $\boldsymbol{S}$ are said to be the *(general) strategic situations.*

There exists a *natural bijection of* $\boldsymbol{S}$ *onto* $\mathbf{T}_F(\Gamma)$, namely

$$(\boldsymbol{\sigma}_j; j \in J) \mapsto \bigcup_{j \in J} \boldsymbol{\sigma}_j$$

(for each $(\boldsymbol{\sigma}_j; j \in J) \in \boldsymbol{S}$).

5.7.2. Definition, remarks. (Cf. § **2.8.**) A *(general) equilibrium point* (of $\mathscr{G}$) *in some* $x^\circ \in P$ is $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_j; j \in J) \in \boldsymbol{S}$ such that, for each $j_0 \in J$ and each $\boldsymbol{x} \in \boldsymbol{\Gamma} x^\circ$ complying with $\boldsymbol{\sigma}_j$ for any $j \in J \setminus \{j_0\}$, there holds $f_j(\boldsymbol{x}) \leqq_j f_j(\boldsymbol{y})$ where $\boldsymbol{y}$ is that element of $\boldsymbol{\Gamma} x^\circ$ which complies with $\boldsymbol{\sigma}_j$ for each $j \in J$. But note here that *for any play* $\boldsymbol{x} \in \boldsymbol{X}_\Gamma$ *and each* $j \in J$ *there exists* $\boldsymbol{\sigma}_j \in \boldsymbol{S}(j)$ *such that* $\boldsymbol{x}$ *complies with* $\boldsymbol{\sigma}_j$ (the proof is simple); of course, this is an essential difference in comparison with the "plain case" in § 2.8.

The direct definition of (general) absolute equilibrium point (cf. § 0) can be given in terms of full general Γ-transformations (cf. §§ 2.8-9):

5.7.3. Definition. Under a ***(general) absolute equilibrium point*** of $\mathscr{G}$ we mean $\boldsymbol{\sigma} \in \mathbf{T}_F(\Gamma)$ such that for each $x \in P$, $j \in J$, and for any $\boldsymbol{x} \in \boldsymbol{\Gamma} x$ complying with $\boldsymbol{\sigma} \mid (\boldsymbol{Z}_\Gamma \setminus \boldsymbol{Z}_\Gamma(j))$ there holds

$$f_j(\boldsymbol{x}) \leqq_j f_j(\boldsymbol{p}(x, \sigma)).$$

5.7.4. Remark. *The notion of absolute equilibrium point could be formulated in terms of preference forms of gs. p.i.*, too (cf. § 2.10).

5.7.5. Remark. *§ 2.11 (saddle points etc.) is to be related also to the "general strategies case".*

5.7.6. Remark. Evidently, if $j \in J$, $\sigma \in \mathring{S}(j)$, then $\Phi_\Gamma(\sigma) \in S(j)$. By means of § 5.6.3 we immediately conclude: if $\mathring{\sigma} = (\sigma_j; j \in J) \in \mathring{S}$, $x^\circ \in P$, then $\mathring{\sigma}$ is a plain equilibrium point (of $\mathscr{G}$) in x° if and only if $(\Phi_\Gamma(\sigma_j); j \in J)$ is an equilibrium point in x°; if $\sigma \in T_F(\Gamma)$, then σ is a plain absolute equilibrium point of $\mathscr{G}$ if and only if $\Phi_\Gamma(\sigma)$ is an absolute equilibrium point of $\mathscr{G}$. (Cf. § 0!) Note that it may happen that there is a weak plain absolute equilibrium point σ of $\mathscr{G}$ (see § 3.6) such that (σ is not a plain absolute equilibrium point of $\mathscr{G}$, and hence) $\Phi_\Gamma(\sigma)$ is not a (general) absolute equilibrium point of $\mathscr{G}$ (see § 5.8.4).

5.8.0. Remark. Now we present several examples (the unproved propositions contained in them are simply provable); the aim is to illustrate the meanings of the various notions of equilibrium point, which are considered in this article.

In each of the examples (§§ 5.8.1-4), we construct some *(two-player) finite antagonistic game with real-valued pay-off functions*; thus, we might say "*saddle point*" instead of "equilibrium point" (see § 2.11). The constructed game $\mathscr{G}$ will be denoted in the usual way (cf. § 2.2 etc.); *always* we shall choose $J = \{1, 2\}$, and $\mathscr{V}_j = \mathscr{R}$ (the chain of real numbers) for $j = 1, 2$.

5.8.1. Example. Let $P = \{1, 4\} \times \{0\} \cup \{0, 2, 3\} \times \{1, 2\}$, let $\Gamma(i, k) = (\{i - 1\} \times \{0, 1, 2\}) \cap P$ for each $(i, k) \in P$. Then $P_0 = \{0\} \times \{1, 2\}$. Let $h : P \to \{0, 1\}$ be such that $h((2, 2)) = h((0, 1)) = 1$, $h(x) = 0$ for each $x \in P \setminus \{(2, 2), (0, 1)\}$. If $\mathbf{x} = (x_0, \ldots, x_m) \in \mathbf{X}_\Gamma$ (of course, $m \leqq 4$), then we choose $f(\mathbf{x}) \in \{0, 1\}$ such that $f(\mathbf{x}) \equiv h(x_0) + \ldots + h(x_m)$ (mod 2). This defines $f: \mathbf{X}_\Gamma \to \{0, 1\}$ uniquely. Let $f_1 = f, f_2 = -f$, let $P(1) = P \cap \{4, 2, 0\} \times \{0, 1, 2\}$, $P(2) = P \setminus P(1)$.

Thus, $\mathscr{G}$ has the properties mentioned in § 5.8.0. Further, $\mathscr{G}$ is *locally finite*, and the players play *alternatively*.

Although this game is very simple, it has a position x° (namely, $x^\circ = (4, 0)$) such that *there does not exist a plain equilibrium point in* x°. Consequently, *$\mathscr{G}$ has no plain absolute equilibrium point*. But it is easy to see that *$\mathscr{G}$ has an absolute equilibrium* point.

(The idea of this example is similar to that used by Berge in the example in [3], ch. 6, sect. "Strategies".)

5.8.2. Example. Now, let the game $\mathscr{G}$ arise from the preceding game (§ 5.8.1) by omitting the positions (i, k) with $i \geqq 3$ (and by the corresponding restrictions of Γ, f_j etc.).

Again, the new game $\mathscr{G}$ has the properties mentioned in § 5.8.0, and it is *locally finite*, and the players play *alternatively*.

$\mathscr{G}$ has a plain equilibrium point in any position, but *$\mathscr{G}$ has not a plain absolute equilibrium point*. (Note that the analogical situation for general strategies is impossible: *a g.p.i. has an absolute equilibrium point if and only if it has an equilibrium point in each position*.)

5.8.3. Example. Let P and Γ be such that $0 < \operatorname{card} P < \aleph_0$, $\operatorname{card} \Gamma x \geqq 2$ for each $x \in P$. (Thus, $P_0 = \emptyset$.) Let the partition $(P(j);\, j \in J)$ be such that $\Gamma x \nsubseteq P(j)$ for each $j \in J$, $x \in P(j)$. (E.g., it is possible to choose $P = \{1, 2\}$, $P(1) = \{1\}$, $P(2) = \{2\}$, $\Gamma x = P$ for $x = 1, 2$. Or $P = \{1, 2\} \times \{1, 2\}$, $P(j) = \{1, 2\} \times \{j\}$, $\Gamma(i, j) = \{1, 2\} \times \{3 - j\}$ for $(i, j) \in P$; in such a case the players play *alternatively*.) Using the results of [4] (§ 4.24.2, § 3(6) etc.), we obtain immediately that there exist $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ such that: $\boldsymbol{A}_1 \cup \boldsymbol{A}_2 = \boldsymbol{X}_\Gamma$; $\boldsymbol{A}_1 \cap \boldsymbol{A}_2 = \emptyset$; for each $j \in J$, $x \in P$, and for any $\boldsymbol{\sigma} \in \bigtimes_{\boldsymbol{z} \in \boldsymbol{Z}(j)} \Gamma\varkappa(\boldsymbol{z})$ (cf. § 5.7.1) there exists $\boldsymbol{x} \in (\boldsymbol{\Gamma} x) \setminus \boldsymbol{A}_j$ complying with $\boldsymbol{\sigma}$. (The proof, contained in [4], is quite non-constructive.) Let $\boldsymbol{f}_j : \boldsymbol{X}_\Gamma \to \{-1, 1\}$ be such that $\boldsymbol{f}_j(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} \in \boldsymbol{A}_j$. The game $\mathcal{G}$ has the properties mentioned in § 5.8.0. But, for *any* $x^\circ \in P$, *$\mathcal{G}$ has no equilibrium point in* x°; therefore, *$\mathcal{G}$ has not an absolute equilibrium point.*

5.8.4. Example. Let $P = (\{0, 2\} \times \{0\} \cup \{1\} \times \{1, 2\}) \times \{1, 2\}$, let $\Gamma(i_1, i_2, i_3) = P \cap (\{i_1 - 1\} \times \{0, 1, 2\} \times \{i_3\})$ for each $(i_1, i_2, i_3) \in P$ with $i_1 \neq 0$, let $\Gamma(0, 0, i_3) = \{(2, 0, 3 - i_3)\}$ (this defines Γ uniquely). Let $P(1) = \{0, 2\} \times \{0\} \times \{1\} \cup \{1\} \times \{1, 2\} \times \{2\}$ $(\subseteq P)$, $P(2) = P \setminus P(1)$. Then the players play *alternatively*, and any play is infinite (i.e. $P_0 = \emptyset$). We say that $\boldsymbol{x} = (x_k;\, k \in W_\infty) \in \boldsymbol{X}_\Gamma$ is *almost cyclic* iff there exist $m > 0$ and n_0 such that $x_{n+m} = x_n$ for each $n \geqq n_0$. Let $\boldsymbol{f}_j : \boldsymbol{X}_\Gamma \to \{-1, 1\}$ be such that $\boldsymbol{f}_j(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} (\in \boldsymbol{X}_\Gamma)$ is almost cyclic. Let $\boldsymbol{f}_1 = \boldsymbol{f}$, let $\boldsymbol{f}_2 = -\boldsymbol{f}$. Then $\mathcal{G}$ has the properties mentioned in § 5.8.0. Evidently, *any* $\sigma \in \mathrm{T}_F(\Gamma)$ *is a weak plain absolute equilibrium point of* $\mathcal{G}$, while *$\mathcal{G}$ has no plain absolute equilibrium point*, as *$\mathcal{G}$ has not a plain equilibrium point in position* $x^\circ = (2, 0, 2)$.

5.9. Remark. The meaning of the main theorem and its corollaries (§§ 3.0; 3.2–3.5, 3.7–3.9, 5.4, 5.5) is commented in §§ 0, 3.1, 3.3, 3.5,3.6, 3.8, 3.9, 5.5; further, cf. §§ 2.3–2.5, 2.8, 2.11, 5.7.6, 5.8.0-4. Of course, other corollaries of the main theorem can be obtained, e.g., by particular choices of pseudolengths and quasiqualitative pay-off functions. The auxiliary propositions in § 4, and mainly the used construction itself, have a certain meaning, too; e.g., if P is finite, then the construction describes an algorithm giving a plain absolute equilibrium point; if the construction is applied in the case which is considered in § 3.4, then it gives the usual method of the construction of (weak, cf. § 3.6) plain absolute equilibrium point (cf., e.g., [2], Ch. 6, or [3] Ch. I, § 7; but instead of indexes 0, 1, 2, ... we have used 1, 2, 3, ...). Naturally the latter method cannot be used at games with infinite plays (if Γ is a graph which is not locally finite, it may happen that $P_0 \neq \emptyset$ but $\{x;\, x \in Z, \Gamma x \subseteq P_0\} = \emptyset$).

5.10. Remark. The author had examined several other ways of proving the main theorem (e.g., it is possible to choose the functions (§ 4.2) in a somewhat different manner, pay-off functions can be replaced preliminarily by certain auxiliary "finer" ones, etc.; the proof can be based on Zorn's lemma), but it turned out that they give no essential advantage.

5.11. Remark. As we have mentioned in § 0, the method of the construction used in § 4 is somewhat similar to that used in the proof of theorem 3.13 in [6]; but the latter proof (it concerns another kind of games!) needed only the usual mathematical induction (the considered games were finite), while in the present paper the fact that infinite games are admitted brought about the complications connected with the transfinite induction (cf. the auxiliary conditions (A), (B) in § 3.0, the proof in § 4.5.3 (cf. with case 5.3° in the proof of theorem 3.13 in [6]), etc.); on the other hand the games considered in [6] are simultaneous and nondeterministic, and thus the proof in [6] not only needed (among others) somewhat "richer" families of the functions χ and F ($\chi^r_{i,x}$, $h^r_{i,x}$ instead of χ^ξ_x, F^ξ_x), but also it had to be based on special "local" equilibrium point result (theorem 2.3 in [6]). Another difference consists in the definitions of the values of the functions χ (cf. § 4.2 with part 2° of the proof 3.13 in [6]), and in the corresponding "technical" details of both the proofs.

ACKNOWLEDGEMENTS

I want to express my gratitude to V. Novák for calling my attention (when writing § 5) to papers [15] and [16] which has been very useful at generalizing the results presented in § 3 to the case of poset-valued pay-off functions (cf. §§ 5.0 – 5.5).

J. Hanák
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia