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# DIRECT PRODUCTS OF HOMOMORPHIC MAPPINGS

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It is well-known in the theory of abstract algebras that for arbitrary class of algebras $\mathfrak{A}$ the direct product $\prod_{\tau \in T} h_\tau$ of homomorphic mappings $h_\tau$ of algebras $A_\tau \in \mathfrak{A}$ onto $B_\tau \in \mathfrak{A}$ is a homomorphic mapping of the direct product $\prod_{\tau \in T} A_\tau$ onto $\prod_{\tau \in T} B_\tau$. The prupose of this paper is to give sufficient conditions for the converse of this theorem. It will be shown that the class of algebras, for which the converse of the theorem is valid, is enough extensive. It contains for example atomic Boolean algebras, discrete direct products of completely ordered groups or rings and lattices which are direct products of chains with the least or greatest element.

## 1.

## BASIC CONCEPTS

Let $\mathfrak{A}$ be a class of algebras with the zero element 0 and the binary operation $\oplus$ and a set $\Omega$ of n-ary operations ($n \geqq 1$) fulfilling for each algebra $A \in \mathfrak{A}$ and each element $a \in A$ identities:

(i) $a \oplus 0 = a = 0 \oplus a$

(ii) for each $\omega \in \Omega$ is $00 \dots 0\, \omega = 0$

The operations in all algebras of $\mathfrak{A}$ will be denoted by the same symbols.

**Definition 1.** An algebra $A \in \mathfrak{A}$ is said to be *without zero-divisors* iff there exists $\Omega' \subseteq \Omega$, $\Omega' \neq \varnothing$ with following properties:

(a) the arity of each $\omega \in \Omega'$ is greater than 1
(b) for each $\omega \in \Omega'$ the identity $a_1 a_2 \dots a_n \omega = 0$ holds
iff $a_i = 0$ for at least one $i$ ($1 \leqq i \leqq n$).

The set $\Omega'$ is called *the set of regular operations.*

**Definition 2.** An algebra $A \in \mathfrak{A}$ is called *N-algebra* iff there exist algebras $A_\tau \in \mathfrak{A}$, $\tau \in T$ without zero-divisors such that $A$ is equal to the direct product of $A_\tau$, i.e. $A = \prod_{\tau \in T} A_\tau$ and at least one of the following conditions is satisfied:

(iii) for each $\tau \in T$ in $A_\tau$ there exists "sum" (in the sense of $\oplus$) of arbitrary set $\{a^\gamma;\, a^\gamma \in A_\tau,\ \gamma \in G,\ a^\gamma = 0$ for $\gamma \neq \gamma_0 \in G$, *card* $G \leqq$ *card* $T\}$ and it is equal to $a^{\gamma_0}$.

(iv) $\prod_{\tau \in T} A_\tau$ is the discrete direct product.

**Remark.** If T is a finite index set, then the conditions (iii) and (iv) of the definition 2 can be omitted because for a finite index set is each direct product discrete one and (iii) follows directly from (i). From (i) it follows that the "sum" of the set from the condition (iii) does not depend on a bracketing.

For some algebras the conditions for direct decomposition to algebras without zero-divisors are known. For example an atomic Boolean algebra is direct decomposable to two-element Boolean algebras (they do not contain zero-divisors), see [10] and [3]. The conditions for $\Omega$-algebras and $\Omega$-groups are given recently in [8] and [9]. These algebras are N-algebras.

**Notation.** Let $\mathfrak{A}$ be a fixed class of algebras with 0, operation $\oplus$ and a set $\Omega$ of operations fulfilling (i), (ii), let $T$ be an index set. The direct product of algebras $A_\tau \in \mathfrak{A}$ for $\tau \in T$ will be denoted by $A = \prod_{\tau\in T} A_\tau$, the zero of $A$ is denoted by $O_A$. Let $a \in A$, the projection of $a$ into $A_\tau$ is denoted by $pr_\tau(a) = a(\tau)$. It is easy to show that $pr_\tau(O_A) = 0$ for each $\tau \in T$. For $T' \subseteq T$ there is $\overline{\prod_{\tau\in T'} A_\tau} = \{a;\ a \in A,\ pr_\tau(a) = 0$ for $\tau \in T - T'\}$. Specially for $T' = \{\tau_0\}$ is $\overline{\prod_{\tau_0} A_\tau}$ denoted by $\overline{A}_{\tau_0}$. An element of $\overline{A}_\tau$ is denoted by $\overline{a}_\tau$. Let $A_\tau$, $B_\sigma \in \mathfrak{A}$. By the symbol $H(A_\tau, B_\sigma)$ we denote the set of all homomorphic mappings of $A_\tau$ into $B_\sigma$, by $\overline{H}(A_\tau, B_\sigma)$ the set of all homomorphic mappings of $A_\tau$ onto $B_\sigma$.

**Definition 3.** Let $A$, $B \in \mathfrak{B}$, $A = \prod_{\tau\in T} A_\tau$, $B = \prod_{\tau\in T} B_\tau$, $\varphi_\tau : A_\tau \to B_\tau$ for each $\tau \in T$, $\mathfrak{B}$ beeing an arbitrary class of algebras. The mapping $\varphi : A \to B$ defined by the rule:

$$pr_\tau \varphi(a) = \varphi_\tau(pr_\tau(a)) \text{ for each } \tau \in T,\ a \in A$$

is called the *direct product of mappings* $\varphi_\tau$ and it is denoted by $\varphi = \prod_{\tau\in T} \varphi_\tau$ (see [12], p. 127, Lemma 3).

**Lemma.** *Let $A$, $B$ be $N$-algebras, $\varphi \in \overline{H}(A, B)$ and $O_B$ be a zero of the algebra $B$. Then $\varphi(O_A) = O_B$.*

Proof. Let $\omega$ be a direct product of $n$-ary regular operations ($n \geqq 2$), let $\varphi^{-1}(O_B) = V$. Then for each $v \in V$ it holds

$$\varphi(O_A) = \varphi(vO_A \dots O_A\omega) = \varphi(v)\ \varphi(O_A) \dots \varphi(O_A)\ \omega = O_B\varphi(O_A) \dots \varphi(O_A)\ \omega = O_B.$$

**Theorem 1.** Let $\mathfrak{L}$ be a class of algebras with a set of operations $\Omega$, let $A_\tau$, $B_\tau \in \mathfrak{L}$ for $\tau \in T$ and $\varphi_\tau \in \overline{H}(A_\tau, B_\tau)$. Then $\prod_{\tau\in T} \varphi_\tau \in \overline{H}(\prod_{\tau\in T} A_\tau, \prod_{\tau\in T} B_\tau)$ (see [12]).

Proof. For each $\tau \in T$ and arbitrary $n$-ary operation $\omega \in \Omega$ there is $pr_\tau\varphi(a_1a_2 \dots a_n\omega) = \varphi_\tau(pr_\tau(a_1a_2 \dots a_n\omega)) = \varphi_\tau(a_1(\tau)\, a_2(\tau) \dots a_n(\tau)\, \omega) = \varphi_\tau(a_1(\tau))\ \varphi_\tau(a_2(\tau)) \dots \varphi_\tau(a_n(\tau))\ \omega$, i.e. $\varphi(a_1a_2 \dots a_n\omega) = \varphi(a_1)\ \varphi(a_2) \dots \varphi(a_n)\ \omega$. This implies that $\varphi = \prod_{\tau\in T} \varphi_\tau \in \overline{H}(\prod_{\tau\in T} A_\tau, \prod_{\tau\in T} B_\tau)$.

**Definition 4.** A mapping $\varphi$ of an $N$-algebra $A$ into an $N$-algebra $B$ is said to be *trivial* iff card $\varphi(A) = 1$. If $\varphi(A) = \{O_B\}$ and $\varphi \in H(A, B)$, $\varphi$ is called a *zero-homomorphism* and it is denoted by $o$.

## 2.

## THEOREMS ON HOMOMORPHIC MAPPINGS OF $N$-ALGEBRAS

**Theorem 2.** Let $A_\tau$, $B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors for $\tau \in T$, $\sigma \in S$ and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$ be $N$-algebras. Let $\varphi$ be a homomorphic mapping of $A$ onto $B$ which is not trivial. Then for each $\sigma \in S$ there exists just one $\tau_\sigma \in T$ such that $\overline{B}_\sigma \subseteq \varphi(\overline{A}_{\tau_\sigma})$.

Proof. Let the assumption of the theorem be valid and let there exists $\sigma_0 \in S$ such that the assertion of the theorem is not true. Let $T'$ be an arbitrary subset of $T$ such that $\varphi\ (\overline{\prod_{\tau \in T'} A_\tau}) \supseteq \overline{B}_{\sigma_0}$ (such $T'$ exists, for example $T' = T$). Evidently $card\ T' \geqq 1$. Denote $A' = \prod_{\tau \in T'} A_\tau$. As the assertation of the theorem is not valid, it is $card\ T' > 1$, so that there exist $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$.

(a) Let there exist $\overline{a}_{\tau_1} \in \overline{A}_{\tau_1}$, $\overline{a}_{\tau_2} \in \overline{A}_\tau$, $\overline{a}_{\tau_1}, \overline{a}_{\tau_2} \in A'$ such that $\varphi(\overline{a}_{\tau_1}) \neq O_B \neq \varphi(\overline{a}_{\tau_2})$. For each n-ary operation $\omega \in \Omega'$ which is the direct product of regular operations the relation

$$\varphi(\overline{a}_{\tau_1}\overline{a}_{\tau_2} \ldots \overline{a}_{\tau_2}\omega) = \varphi(O_A) = O_B$$

holds by the lemma, but by the assumption (a):

$$\varphi(\overline{a}_{\tau_1})\ \varphi(\overline{a}_{\tau_2}) \ldots \varphi(\overline{a}_{\tau_2})\ \omega \neq O_B$$

which is a contradiction.

(b) Let (a) does not hold. Thus there exists $\tau_0 \in T'$ such that $\varphi(\overline{a}_\tau) = O_B$ for each $\overline{a}_\tau \in \overline{A}_\tau$, $\tau \in T'$ $\tau \neq \tau_0$. Let $\overline{b}_{\sigma_0} \in \overline{B}_{\sigma_0}$, $\overline{b}_{\sigma_0} \neq O_B$. Choose $a \in \varphi^{-1}(\overline{b}_{\sigma_0})$, $a \in A'$ arbitrary; according to lemma we have $a \neq O_A$. We can write $a = \overline{a(\tau_0)} \oplus c$, where $c(\tau_0) = 0$. Then $\varphi(a) = \varphi(\overline{a(\tau_0)}) \oplus \varphi(c) = \varphi(\overline{a(\tau_0)})$, $\varphi(c)$ being $O_B$ according to the assumption (b) and (iii) of the definition 2. Thus $\overline{B}_{\sigma_0} \subseteq \varphi(\overline{A}_{\tau_0})$, in contradiction with the assumption of the proof.

The proof of the theorem 2 is complete.

**Definition 5.** An algebra $A \in \mathfrak{A}$ without zero-divisors is said to be *pseudo-ordered*, if there exists a set $\Omega'' \subseteq \Omega'$, $\Omega'' \neq \varnothing$ such that for each $n$-ary $\omega \in \Omega''$ there is $a_1 a_2 \ldots a_n \omega = a_i \alpha$ where $i \in \{1, 2, \ldots, n\}$ and $\alpha$ is the identity operation (i.e. $a\alpha = a$) or $\alpha \in \Omega$ is a unary operation with $a\alpha = 0$ iff $a = 0$.

From the inclusion $\Omega'' \subseteq \Omega'$ it holds that the arity of $\omega \in \Omega''$ is greater than 1. Let us denote $T^* = \{\tau_\sigma ; \sigma \in S\}$ where $\tau_\sigma$ is corresponding to $\sigma \in S$ by the theorem 2, evidently $T^* \subseteq T$.

**Theorem 3.** Let $A_\tau, B_\sigma \in \mathfrak{A}$ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$ and $\varphi$ be a non trivial homomorphic mapping of $A$ onto $B$. Then there exists an algebra $C = \prod_{\tau \in T} C_\tau$ (isomorphic with $B$), where $C_\tau = B_\sigma$ for $\tau = \tau_\sigma \in T^*$ and $C_\tau = \{0\}$ for $\tau \in T - T^*$ such that $i.\varphi = \prod_{\tau \in T} \varphi_\tau$ where $\varphi_\tau$ is a homomorphic mapping of $A_\tau$ onto $C_\tau$ and $i$ is a natural isomorphism of $B$ onto $C$.

Proof. It is clear that $C$ is isomorphic with $B$. By the theorem 2 for each $\sigma \in S$ there exist just one $\tau_\sigma \in T$ for which $\overline{C}_{\tau_\sigma} = \overline{B}_\sigma \subseteq \varphi(\overline{A}_{\tau_\sigma})$.

(a) Let $\overline{C}_{\tau_\sigma} = \overline{B}_\sigma = \varphi(\overline{A}_{\tau_\sigma})$ for each $\sigma \in S$, then $pr_{\tau_\sigma}\ \varphi \in \overline{H}(A_{\tau_\sigma}, B_\sigma) = \overline{H}(A_{\tau_\sigma}, C_{\tau_\sigma})$. Let $\varphi_\tau = pr_{\tau_\sigma}\ \varphi$ for $\tau = \tau_\sigma \in T^*$ and $\varphi_\tau = o$ for $\tau \in T - T^*$, then $i\ .\ \varphi = \prod_{\tau \in T} \varphi_\tau$ and $\varphi_\tau \in \overline{H}(A_\tau, C_\tau)$.

(b) Let there be $\overline{B}_{\sigma_0} \neq \varphi(\overline{A}_{\tau_{\sigma_0}})$, $\overline{B}_{\sigma_0} \subseteq \varphi(\overline{A}_{\tau_{\sigma_0}})$ for some $\sigma_0 \in S$. Because $\varphi$ is the mapping of the type "onto", there exists a that set $S' \subseteq S$, card $S' > 1$, such $\varphi(\overline{A}_{\tau_{\sigma_0}}) \supseteq \overline{B}_\sigma$ for $\sigma \in S'$. Let $\sigma_1 \neq \sigma_2$, $\sigma_1, \sigma_2 \in S'$ and $b_1 \in \overline{B}_{\sigma_1}$, $b_2 \in \overline{B}_{\sigma_2}$, $b_1 \neq O_B \neq b_2$. Let $a_1, a_2 \in \overline{A}_{\tau_{\sigma_0}}$ and $\varphi(a_1) = b_1$ $\varphi(a_2) = b_2$. Then for each $\omega$ which is the direct product of operations from $\Omega''$ we have:

$$O_B = b_1 b_2 \dots b_2 \omega = \varphi(a_1)\,\varphi(a_2) \dots \varphi(a_2)\,\omega = \varphi(a_1 a_2 \dots a_2 \omega) =$$
$$= \varphi(a_i \alpha) = \varphi(a_i)\,\alpha = b_i \alpha \neq O_B, \text{ where } i = 1 \text{ or } 2,$$

which is a contradiction. The proof is complete.

**Theorem 4.** Each chain with the least element 0 or the greatest element 1 is a pseudo-ordered algebra. Each completely ordered group is a pseudo-ordered algebra.

Proof. Let $A$ be a chain with the least element 0. Put: $a \oplus b = \max\{a, b\}$, $a\ .\ b = \min\{a, b\}$, $0 = \{0\}$, $\Omega' = \Omega'' = \{.\}$. Dually for a chain with the greatest element.

Let $A$ be a completely ordered group. Then $\oplus$ be the group composition, 0 the unit element of A and $\Omega' = \Omega'' = \{.\}$, where $a\ .\ b = \min(\max(a, a^{-1}), \max(b, b^{-1}))$.

**Corollary 5.** Let $A_\tau$, $B_\sigma \in \mathfrak{A}$ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in S} B_\sigma$ be N-algebras and $\varphi$ be a non trivial homomorphic mapping of $A$ onto $B$. Then card $S \leqq$ card $T$.

It follows directly from the theorems 2 and 3.

**Corollary 6.** Let $A_\tau$, $A_\gamma^*$ be pseudo-ordered algebras and $A = \prod_{\tau \in T} A_\tau$ and $A = \prod_{\gamma \in G} A_\gamma^*$ be N-algebras. Then card $G$ = card $T$ and $A_\gamma^* = A_{\pi(\tau)}$, where $\pi$ is a permutation of the set $T$.

It follows directly from the theorem 3 and corollary 5.

**Theorem 7.** Let $A_\tau$, $B_\tau \in \mathfrak{A}$ be pseudo-ordered algebras and $\varphi$ be a homomorphic mapping of an N-algebra $A = \prod_{\tau \in T} A_\tau$ onto N-algebra $B = \prod_{\tau \in T} B_\tau$. Then there exists a permutation $\pi$ of the set $T$ and the natural isomorphism $p$ of $\prod_{\tau \in T} B_\tau$ onto $\prod_{\tau \in T} B_{\pi(\tau)}$ such that

$$p \cdot \varphi = \prod_{\tau \in T} \varphi_\tau$$

where $\varphi_\tau$ is a homomorphic mapping of $A_\tau$ onto $B_{\pi(\tau)}$.

It follows directly from the theorem 3 and corollary 6. The theorem 7 is the converse of the theorem 1 for pseudo-ordered algebras. From theorems 7 and 4 we obtain:

**Corollary 8.** For atomic Boolean algebras, for l-groups discretely directly decomposable into completely ordered groups, for lattices which are direct products of chains with the least (or the greatest) element and for ordered rings which are discrete direct products of completely ordered rings is the converse of the theorem 1 valid.

**Remark.** For l-groups and lattice-ordered rings is by a "homomorphism" in the sense of this paper understood the homomorphic mapping preserving lattice operation (because it must preserve the direct product of operations introduced in the proof of theorem 4). It is easy to show that this homomorphism is also o-homomorphism in the sense of [11].

The conditions for discreie direct decompositions of l-groups and ordered rings into completely ordered groups and rings are given in [11].

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