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Robert Karpe

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ON A CERTAIN ANALOGY OF STIRLING'S NUMBERS OF THE 2ND KIND

Robert Karpe, Brno.

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Stirling's number of the second kind (which is usually noted as $S(n, k)$) gives the number of ways in which it is possible to carry out the partition of a set of n mutually different elements into k nonempty sets. To compare with the results that will follow, let us return to the recurrent formula, see [1], p. 168:

$$S(n, k) = k \,.\, S(n-1, k) + S(n-1, k-1) \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} S(n, k) &= 1 \text{ for } n = k,\ k = 0, 1, 2, \dots \\ &= 0 \text{ for } k = 0,\ n = 1, 2, 3, \dots \\ &= 0 \text{ for } n < k \end{aligned} \tag{2}$$

This formula was the base for the deduction of a generating function, see [1], p. 170:

$$y_k(x) = \prod_{j=1}^{k} \frac{1}{1 - jx}, \qquad k = 1, 2, 3, \dots \tag{3}$$

whose expansion into a power series runs as follows:

$$y_k(x) = S(k, k) + S(k+1, k) \,.\, x + S(k+2, k) \,.\, x^2 + \dots \tag{4}$$

Definition 1. A finite set will be called *pair* (*odd*) if it contains a pair (odd) number of elements. A *pair* (*odd*) partition on a finite set is a partition whose every set contains a pair (odd) number of elements.

Convention 1. We are going to use the following symbols: $N = \{x_1, x_2, \dots, x_n\}$ is a finite set, containing $n = 2\nu$ mutually different elements, where ν is a natural number. $L \subset N$; $L = \{x_1, x_2, \dots, x_{n-2}\}$. Let $p \geqq q$ be natural numbers. By the symbol $S_2(2p, q)$, resp. $S_2[2p, q]$ we shall denote the numbers, resp. the family of all pair partitions of the set of $2p$ different elements into q nonempty sets.

Theorem 1. For the numbers $S_2(n, k)$ the recurrent formula holds good:

$$S_2(n, k) = k^2 \,.\, S_2(n-2, k) + (2k-1) \,.\, S_2(n-2, k-1) \tag{5}$$

with boundary conditions:

$$\begin{aligned} S_2(n, k) &= \frac{(2k)!}{(2!)^k \,.\, k!} \text{ for } n = 2k,\ k = 0, 1, 2, \dots \\ &= 0 \text{ for } k = 0,\ n = 2, 4, 6, \dots \\ &= 0 \text{ for } n < 2k \end{aligned} \tag{6}$$

Proof. Let $\mathfrak{R}(X, s)$, resp. $R(X, s)$ denote a pair, resp. non-pair partition of the set X into s nonempty sets. Let us choose a certain partition $\mathfrak{R}(L, k)$. If we let the elements x_{n-1}, x_n belong to the sets of this partition (there are k^2 ways of doing so), two cases may occur:

The elements x_{n-1}, x_n belong: 1) both to the same set of the partition, 2) each to a different set of the partition.

In the first case the resulting partition is pair. In the second case it is possible to form a (1,1) correspondence between the resulting and its respective pair partition, through the following transformation:

Let $x_{n-1} \in M_i$, $x_n \in M_j$, so that M_i, M_j (the sets of the resulting partition $R(N, k)$) are odd. Let x_p be the element with the lowest index in the set $M_i \cup M_j$. Then if $x_p \in M_i$, resp. $x_p \in M_j$, we shall class this element into the set M_j, resp. M_i.

As all the partitions $\mathfrak{R}(L, k)$ form the group $S_2[n - 2, k]$, we shall derive from this group by means of the mentioned proceeding (i.e. either directly or by means of the transformation) on the whole $k^2 . S_2(n - 2, k)$ mutually different partitions $\mathfrak{R}(N, k)$.

Let us consider further that none of the partitions $\mathfrak{R}(N, k)$ we have formed doesn't turn through the transformation to a partition $R(N, k)$ having the following quality: one of its odd sets is formed by the element x_{n-1} or x_n itself. Besides none from the partitions $\mathfrak{R}(N, k)$ we have formed as yet contains $\{x_{n-1}, x_n\}$ as an independent set of the partition. That's why to form the remaining partitions $\mathfrak{R}(N, k) \in S_2[n, k]$ we shall take into consideration individual partitions $\mathfrak{R}(L, k - 1) \in S_2[n - 2, k - 1]$:

a) To each of these partitions we shall add one of the elements x_{n-1}, x_n as a k-th independent set of the partition, while the other of these elements will be included in one of the original sets of the partition. Thus we shall form $2(k - 1) . S_2(n - 2, k - 1)$ different partitions $R(N, k)$, each of which contains two odd and $k - 2$ even sets. We shall then form a correspondence between these partitions and the respective even partitions by means of the quoted transformation.

b) We shall join to each partition $\mathfrak{R}(L, k - 1) \in S_2[n - 2, k - 1]$ a set $\{x_{n-1}, x_n\}$ as a k-th set of the partition. Thus we shall form the remaining partitions of the family $S_2[n, k]$.

The validity of the recurrent formula (5) is now evident from what has been said.

Let us go on and consider how many ways there are of performing the partition of a set containing $2k$ different elements into k even sets: Evidently each set of the partition must have just two elements. Let us choose one of those partitions and let us choose arbitrarily the order of its sets as well as the order of the elements in individual sets: We shall get a certain sequence of $2k$ different elements. Forming successively all possible permutations of the chosen order of sets and elements belonging to each set, we shall make the chosen partition correspond to $(2!)^k . k!$ different sequences containing $2k$ elements each. But the number of all sequences, formed from $2k$ different elements, is $(2k)!$. Thus we have $S_2(2k, k) = \dfrac{(2k)!}{(2!)^k . k!}$; let us notice that this expression is different from zero even for $k = 0$, i.e. $S_2(0, 0) = 1$.

The validity of the remaining boundary conditions in (6) is natural. That was to be proved.

Remark 1. From the given formula for $S_2(2k, k)$ there follows evidently the following recurrent formula:

(7) $$S_2(2k, k) = (2k - 1) . S_2(2k - 2, k - 1); \qquad k = 1, 2, 3, \ldots$$

Further we are going to treat the generating function for the numbers $S_2(n, k)$.

Definition 2. By the symbol $y_k(x)$ we shall denote the generating function for the numbers $S_2(n, k)$, that is the function which in the form of a power series runs as follows:

$$(8) \qquad y_k(x) = S_2(2k, k) + S_2(2k + 2, k) \,.\, x^2 + S_2(2k + 4, k) \,.\, x^4 + \dots$$

Theorem 2. The generating function of the numbers $S_2(n, k)$ runs in its final form as follows:

$$(9) \qquad y_k(x) = \prod_{j=1}^{k} \frac{2j - 1}{1 - (jx)^2}\,; \qquad k = 1, 2, 3, \dots$$

Proof. If we subtract from the equation (8) its multiple by the factor k^2x^2, then if we note briefly $S_2(a, b) = S_a^b$ we get: $(1 - k^2x^2) \,.\, y_k(x) = S_{2k}^k + (S_{2k+2}^k - k^2S_{2k}^k) \,.\, x^2 + (S_{2k+4}^k - k^2S_{2k+2}^k) \,.\, x^4 + \dots$, so that considering (5) and (7) we have: $(1 - k^2x^2) \,.\, y_k(x) = (2k - 1) \,.\, (S_{2k-2}^{k-1} + S_{2k}^{k-1} \,.\, x^2 + S_{2k+2}^{k-1} \,.\, x^4 + \dots) = (2k - 1) \,.\, y_{k-1}(x)$. At the same time we have: $S_2(2n, 1) = 1, n = 1, 2, 3, \dots$ so that $y_1(x) = 1 + x^2 + x^4 + \dots = \dfrac{1}{1 - x^2}$ for $|x| < 1$, which was to be proved.

Convention 2. Analogously to [2], p. 27 let us denote by the Symbol $P_2(r)$, resp. $P_2[r]$ the number, resp. the family of all even partitions on a set of r different elements ($r = 0, 2, 4, \dots$) independently on the number of the sets forming the partition.

Evidently the following relation holds good:

$$(10) \qquad P_2(r) = \sum_{j=1}^{r/2} S_2(r, j); \qquad r = 2, 4, 6, \dots$$

Theorem 3. For the numbers $P_2(r)$ this recurrent formula holds good:

$$(11) \qquad P_2(n) = \sum_{j=1}^{n/2} \binom{n-1}{2j-1} \,.\, P_2(n - 2j); \qquad P_2(0) = 1.$$

Proof. Let us choose an arbitrary but constant element of the set N. Let us denote it x_i. Let us consider that it is possible to form just $\binom{n-1}{k-1}$ sets by k elements containing the element x_i each. From the remaining $(n - k)$ elements of the set N it is always possible to form just $P_2(n - k)$ even partitions. Let us then divide the family $P_2[n]$ into partial families according to the number of elements which are contained by that set of partitions in which the chosen element is contained. If we denote $k = 2j$, then $j = 1, 2, \dots, \dfrac{n}{2}$, for n is an even number, see the def. 1.

Remark 2. See also the author's article [3].

Finally I give a few lines of the system of numbers $S_2(n, k)$, factors included by which the slipped numbers are supposed to be multiplied: see the numbers in the middle of the arrows.

n	$k = 0$	1	2	3	4	5	$P_2(n)$
0	**1**						**1**
2	**0**	**1**					**1**
4	**0**	**1**	**3**				**4**
6	**0**	**1**	**15**	**15**			**31**
8	**0**	**1**	**63**	**210**	**105**		**379**
10	**0**	**1**	**255**	**2205**	**3150**	**945**	**6556**

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R. Karpe
Department of Mathematics, Technical University
Brno, Gorkého 13
Czechoslovakia