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THE REFINEMENT OF TWO ISOMORPHIC GENERALIZED LEXICOGRAPHIC PRODUCTS

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INTRODUCTION

In a recent paper [3] M. Novotný has described a decomposition induced by an isomorphism of two Cartesian products. The purpose of this note is to introduce a generalization of the lexicographic product and to extend some of the results of [3] and [4] to this product.

1.

Without reference we shall use the terminology and notation of [3]. We start with a generalization of the lexicographic product. Let A, M be nonempty sets and let M^A denote the set of all mappings from A to M. The elements of M^A will be denoted also by $(m_a)_{a\in A}$. A subset ϱ of M^A will be called an *A-relation on M*. If card $A = h < \aleph_0$, we agree to identify the A-relations on M with the h-ary relations on M, i.e. with the subsets of the Cartesian power $M^h = M \times \ldots \times M$.

Let A, C, Q and $B_q (q \in Q)$ be nonempty sets. Let α_q be A-relations on B_q and let γ be a C-relation on Q. The Cartesian product of the sets B_q will be denoted by $B = \bigtimes_{q\in Q} B_q$. We shall consider the set Φ of all logical formulae which:

1° have exactly the free variables $(f_q^a)_{q\in Q} \in B$ and

2° are built up from the following atomic predicates:

(i) the equalities: $E_{a'a''q} \equiv_{df} f_q^{a'} = f_q^{a''}$ $(q \in Q,\ a',\ a'' \in A)$,

(ii) the predicates defined by the relations $\alpha_q : P\nu_q \equiv_{df} (f_k^{\nu a})_{a\in A} \in \alpha_q$ $(q \in Q,\ \nu : A \to A)$,

(iii) the predicates defined by the relation $\gamma : W_\mu \equiv_{df} (\mu c)_{c\in C} \in \gamma$ $(\mu : C \to Q)$. Hence the formulae from Φ are formed from the atomic predicates (i)—(iii) by means of the disjunction $\vee$, the conjunction &, the negation $\sim$, and the quantifiers $\exists$ and $\forall$ (whose bound variables range over Q) according to the laws of the predicate calculus.

Example. Let card A = card $C = 2$ and let $\leq_q$ and $\leqq$ be binary reflexive relations on $B_q (q \in Q)$ and Q respectively. We have the following example of a formula from Φ:

$$(1)\quad L(f^1, f^2) \equiv_{df} \forall_q \Big((f_q^1 = f_q^2) \vee (\exists_u \big((u \leq q) \,\&\, (f_u^1 \leq_u f_u^2) \,\&\, (\sim (f_u^1 = f_u^2)) \big) \Big).$$

It is easy to check that (1) is in fact the definition of the lexicographic product.

Let $L \in \Phi$. The A-relation corresponding to L will be denoted by λ_B. Hence $(f^a)_{a \in A} \in \lambda_B$ iff $L(f^a)_{a \in A}$ holds. Let $m = (m_q)_{q \in Q} \in B$ and let λ_B satisfy the following condition (C_m):

(C_m): If $f^a \in B (a \in A)$, $s \in Q$, and $f^a_q = m_q$ for all $a \in A$ and all $q \in Q \setminus \{s\}$, then

$$(f^a)_{a \in A} \in \lambda_B \text{ iff } (f^a_s)_{a \in A} \in \alpha_s .$$

We shall call the set B with the A-relation λ_B satisfying (C_m) the *L_m-product of (B_q, α_q) over (Q, γ)*, in symbols $\prod_{q \in Q} B_q$ (shortly L-product only). It is a simple matter to check that L defined by (1) satisfies (C_m) for any $m \in B$; hence the lexicographic product is an L_m-product, in particular the cardinal product is also an L_m-product. Thus the L_m-product is a generalization of the lexicographic product.

2.

2.1. Definition. Let K, K', S, and $U_{kk'}$ $(k \in K,\ k' \in K')$ be sets. Let h be a mapping of $\mathsf{X}_{(k,k') \in K \times K'} U_{kk'}$ into S. We define $h^* \colon \mathsf{X}_{k \in K} (\mathsf{X}_{k' \in K'} U_{kk'}) \to S$ by

$$h^*\big((u_{kk'})_{k' \in K'}\big)_{k \in K} = h(u_{kk'})_{(k,k') \in K \times K'}$$

for any $u_{kk'} \in U_{kk'} (k \in K,\ k' \in K')$. Obviously $h \to h^*$ is an one-to-one correspondence between the set of all mappings of $\mathsf{X}_{(k,k') \in K \times K'} U_{kk'}$ into S and the set of all mappings of $\mathsf{X}_{k \in K} (\mathsf{X}_{k' \in K'} U_{kk'})$ into S. In the sequel we shall denote both mappings h and h^* by the same symbol.

2.2. Definition. Throughout this note S will be a nonempty set, σ an A-relation on S, (U, λ_U) will be a fixed L_m-product $\prod_{k \in K} U_k$ of the sets (U_k, ϱ_k) over $(K, \varkappa)$ and $(U', \lambda_{U'})$ will be a fixed $L_{m'}$-product $\prod_{k' \in K'} U'_{k'}$ of the sets $(U'_{k'}, \varrho'_{k'})$ over $(K', \varkappa')$ (where ϱ_k and $\varrho'_{k'}$ are A-relations on U_k and $U'_{k'}$ respectively and $\varkappa$ and $\varkappa'$ are C-relations on K and K', respectively) In the sequel we shall assume that f is an isomorphism of (U, λ_U) onto (S, σ) (that is f is a bijection such that $(g^a)_{a \in A} \in \lambda_U$ iff $(fg^a)_{a \in A} \in \sigma$), f' is an isomorphism of $(U', \lambda_{U'})$ onto (S, σ) and $fm = f'm' = n$.

If this holds then $(S, (U_k)_{k \in K}, f, n)$ and $\big(S, (U'_{k'})_{k' \in K'}, f', n\big)$ are admissible quadruples in the sense of [3] 1.1 and the mappings q_k and $q'_{k'}$ can be defined as in [3] 1.2. The following conditions are equivalent (see [3] 4 and [4] th. 1):

(α): $q_k q'_{k'} = q'_{k'} q_k$ for all $k \in K$ and $k' \in K'$,

(δ) There exist sets $U_{kk'}$ and bijections $f'_k\colon \mathop{\times}\limits_{k'\in K'} U_{kk'} \to U_k$ and bijections $f_{k'}\colon \mathop{\times}\limits_{k\in K} U_{kk'} \to U'_{k'}$ for every $k \in K$, $k' \in K'$ such that $f(f'_k)_{k\in K} = f'(f_{k'})_{k'\in K'}$,

where $f(f'_k)_{k\in K}$ and $f'(f_{k'})_{k'\in K'}$ are defined in [3] 3.5.

We shall prove now that if (α) holds, then there exist A-relations $\xi_{kk'}$ on $U_{kk'}$ such that the bijections $f(f'_k)_{k\in K}$ and $f'(f_{k'})_{k'\in K'}$ are isomorphisms of the L-product $X = \prod\limits_{k\in K} (\prod\limits_{k'\in K'} U_{kk'})$ and the L-product $Y = \prod\limits_{k'\in K'} (\prod\limits_{k\in K} U_{kk'})$ onto (S, σ), respectively. The proof is based on several lemmas. Throughout k and k' are elements of K and K', respectively. We shall assume that (α) holds. We define $U_{kk'}$ as in [3] 3.8 (proof part 5) by

(2) $$U_{kk'} = q_k q'_{k'} S = q'_{k'} q_k S (= q_k S \cap q'_{k'} S).$$

Further let

(3) $$X_k = \prod_{k'\in K'} U_{kk'}, \qquad Y_{k'} = \prod_{k\in K} U_{kk'}.$$

From [3] 1.7 and 1.5(i) it follows:

2.3. Lemma. *If* (α) *holds and* $u_{kk'} \in U_{kk'}$, *then*

(4) $$q_k u_{kk'} = u_{kk'},\ q_j u_{kk'} = n (j \in K,\ j \neq k),$$

(5) $$q_{k'}' u_{kk'} = u_{kk'},\ q'_{l'} u_{kk'} = n (l \in K',\ l' \neq k').$$

2.4. Lemma. *If* $t \in q_l S$, *then for every* $k \in K$, $k \neq l$, *we have*

(6) $$p_k f^{-1} t = p_k f^{-1} n = n_k.$$

Proof: By [3] 1.2 and 1.5 (i) $t = q_l t$, hence $f^{-1}t = f^{-1} q_l t = f^{-1} f o_l p_l f^{-1} t = o_l p_l f^{-1} t$ and the lemma follows.

From [3] 3.8 (proof part 5), 3.6 and 1.2 it follows that f'_k is given by

(7) $$f'_k(u_{kk'})_{k'\in K'} = p_k f^{-1} f' (p'_{k'} f'^{-1} u_{kk'})_{k'\in K'}.$$

Thus,

(8) $$f(f'_k)_{k\in K}(u_{kk'})_{(k,k')\in K\times K'} = f\big(p_k f^{-1}(f'(p'_{k'} f'^{-1} u_{kk'})_{k'\in K'}\big)_{k\in K}.$$

Further from [3] 3.2 and 1.2 we see that

(9) $$f(f'_k)_{k\in K} = g o g'.$$

In that what follows $u^a_{kk'}$ will be elements of $U_{kk'} (a \in A)$. We put

(10) $$t^a_k = (u^a_{kk'})_{k'\in K'},$$

(11) $$v^a_k = (p'_{k'} f'^{-1} u^a_{kk'})_{k'\in K'}.$$

Thus $t_k^a \in X_{k'}$ $v_k^a \in U'$, and

(12) $$f(f_k')_{k\in K}(u_{kk'}^a)_{(k,\, k')\in K\times K'} = f(p_k f^{-1} f' v_k^a)_{k\in K}.$$

2.5. Lemma. $f' v_k^a \in q_k S$, *i.e.*

(13) $$q_k f' v_k^a = f' v_k^a.$$

Proof: As $u_{kk'}^a \in U_{kk'} = q_k S \cap q_{k'}' S$, it follows that $gog'(u_{kk'}^a)_{(k,k')\in K\times K'}$ is defined ([3] 3.3, proof part 1). From [3] 3.3, proof part 1 we obtain also that $g'(u_{kk'}^a)_{k'\in K'} \in q_k S$. But $g'(u_{kk'}^a)_{k'\in K'} = f' v_k^a$ ((11) and [3] 1.2) and (13) now follows by [3] 1.5 (i).

We note that $U_{kk'} \subseteq S$ and therefore the A-relations $\xi_{kk'}$ on $U_{kk'}$ can be defined as the restrictions of σ to $U_{kk'}$ (i.e. $\xi_{kk'} = \{g \in \sigma \mid g(A) \in U_{kk'}\}$.

2.6. Lemma. *The following conditions are equivalent:*

(i) $(u_{kk'}^a)_{a\in A} \in \xi_{kk'}$,
(ii) $(p_k f^{-1} u_{kk'}^a)_{a\in A} \in \varrho_k$,
(iii) $(p_{k'}' f'^{-1} u_{kk'}^a)_{a\in A} \in \varrho_{k'}$.

Proof: It follows from 2.3, (C_m), $(C_{m'})$, $\xi_{kk'} \subseteq \sigma$, and from the fact that f and f' are isomorphisms.

2.7. Lemma. *If R is one of the atomic predicates and $k \in K$, then*

(14) $$R(t_k^a)_{a\in A} \Leftrightarrow R(v_k^a)_{a\in A}.$$

Proof: By the definition in the section 1 we have to consider the following three cases:

1) Let $R = E_{a'a''l'}(a', a'' \in A,\ l' \in K')$. Assume that $R(v_k^a)_{a\in A}$ holds. By (11) $R(v_k^a)_{a\in A}$ means $p_l' f'^{-1} u_{kl'}^{a'} = p_l' f'^{-1} u_{kl'}^{a''}$. In view of (5) we have $u_{kl'}^{a'} = q_{l'}' u_{kl'}^{a'} = f' o_l' p_l' f'^{-1} u_{kl'}^{a'} = f' o_l' p_l' f'^{-1} u_{kl'}^{a''} = q_l' u_{kl'}^{a''} = u_{kl'}^{l''}$; thus by (10) $R(t_k^a)_{a\in A}$ holds. Conversely $u_{kl'}^{a'} = u_{kl'}^{a''}$ obviously implies $p_l' f'^{-1} u_{kl'}^{a'} = p_l' f'^{-1} u_{kl'}^{a''}$ and (14) holds.

2) Let $R = P\nu_{l'}(l' \in K',\ \nu\colon A \to A)$. Then $R(t_k^a)_{a\in A}$ means $(u_{kl'}^a)_{a\in A} \in \xi_{kl'}$ and $R(v_k^a)_{a\in A}$ means $(p_l' f'^{-1} u_{kl'}^a)_{a\in A} \in \varrho_{l'}$ and the assertion follows from 2.6.

3) Let $R = W_\mu(\mu\colon C \to K')$. Since $t_k^a \in X_k$ and $v_k^a \in U'$ and both X_k and U' are products over the same set $(K', \varkappa')$, both sides of (14) mean simply $(\mu c)_{c\in C} \in \varkappa'$ and (14) is trivially satisfied.

2.8. Lemma. *If $k \in K$, then*

(15) $$(t_k)_{a\in A} \in \lambda_{Xk} \Leftrightarrow (v_k^a)_{a\in A} \in \lambda_{U'}.$$

Proof: L is a formula constructed from the atomic predicates. Since $\vee$, &, $\sim$, $\exists$, and $\forall$ preserve the equivalence $\Leftrightarrow$, it follows from (14), that $L(t_k^a)_{a\in A}$ holds iff $L(v_k^a)_{a\in A}$ holds, hence (15) holds.

2.9. Lemma. *If R is one of the atomic predicates then*

$$(16)\qquad R\big((t_k^a)_{k\in K}\big)_{a\in A} \Leftrightarrow R\big((p_k f^{-1} f' v_k^a)_{k\in K}\big)_{a\in A}.$$

Proof: According to the definition of R we have to consider the following three cases:

1. Let $R = E_{a'a''l}(a', a'' \in A,\ l \in K)$. Then the left and right side of (16) mean $t_l^{a'} = t_l^{a''}$ and $p_l f^{-1} f' v_l^{a'} = p_l f^{-1} f' v_l^{a''}$, respectively. Since f^{-1} and f' are bijections and t_l^a determines completely v_l^a, we have $t_l^{a'} = t_l^{a''} \Rightarrow$ $\Rightarrow p_l f^{-1} f' v_l^{a'} = p_l f^{-1} f' v_l^{a''}$. Conversely let $p_l f^{-1} f' v_l^{a'} = p_l f^{-1} f' v_l^{a''}$. Then by 2.5 and by the definition of q_l we have $f' v_l^{a'} = q_l f' v_l^{a'} = f o_l p_l f^{-1} f' v_l^{a'} =$ $= f o_l p_l f^{-1} f' v_l^{a''} = q_l f' v_l^{a''} = f' v_l^{a''}$. But f' is a bijection and therefore $v_l^{a'} = v_l^{a''}$, By (11) we have $p_{k'} f'^{-1} u_{lk'}^{a'} = p_{k'} f'^{-1} u_{lk'}^{a''}$ for any $k' \in K'$. Therefore by (5) and the definition of $q'_{k'}$ for each $k' \in K'$ we obtain $u_{lk'}^{a'} = q_{k'} u_{lk'}^{a'} = f' o_{k'} p_{k'} f'^{-1} u_{lk'}^{a'} = f' o_{k'} p'_{k'} f'^{-1} u_{lk'}^{a''} = q_{k'} u_{lk'}^{a''} = u_{lk'}^{a''}$, i.e. $t_l^{a'} =$ $= t_l^{a''}$.

2. Let $R = P\nu_l (l \in K,\ \nu : A \to A)$. Then the left and right side of (16) mean $(t_l^a)_{a\in A} \in \lambda_{Xl}$ and $(p_l f^{-1} f' v_l^a)_{a\ni A} \in \varrho_l$. According to 2.8 and in view of the fact that f' and f^{-1} are isomorphisms we have: $(t_l^{\nu a})_{a\in A} \in \lambda_{Xl} \Leftrightarrow$ $\Leftrightarrow (v_l^{\nu a})_{a\in A} \in \lambda_{U'} \Leftrightarrow (f^{-1} f' v_l^a)_{a\in A} \in \lambda_U$. By 2.5, 2.4, and (C_m) we have $(f^{-1} f' v_l^{\nu a})_{a\in A} \in \lambda_U \Leftrightarrow (p_l f^{-1} f' v_l^{\nu a})_{a\in A} \in \varrho_l$ and (16) holds.

3. Let $R = W_\mu(\mu : C \to K)$. Since both X and U are products over $(K, \varkappa)$, both sides of (16) mean simply $(\mu c)_{c\in C} \in \varkappa$ and (16) is trivially satisfied.

2.10. Lemma. *$f(f_k)_{k\in K}$ is an isomorphism of X onto S.*

Proof: $f(f_k)$ is a bijection of X onto S. f is an isomorphism, hence in view of (8), (10), and (11) we have to prove only that

$$(17)\qquad L\big((t_k^a)_{k\in K}\big)_{a\in A} \Leftrightarrow L\big((p_k f^{-1} f' v_k^a)_{k\in K}\big)_{a\in A}.$$

L is a formula constructed from atomic predicates. Since $\vee$, & $\sim$, $\exists$, and $\forall$ preserve the equivalence $\Leftrightarrow$, (17) is a consequence of (16).

By symmetry we have a similar statement for $f'(f_{k'})_{k'\in K'}$. Thus we have

2.11. Theorem. *Let S be a set, σ an A-relation on S, (U, λ_U) and $(U', \lambda_{U'})$ be an L_m-product of the sets (U_k, ϱ_k) over $(K, \varkappa)$ and an $L_{m'}$-product of the sets $(U_{k'}, \varrho_{k'})$ over $(K', \varkappa')$ respectively (where ϱ_k and $\varrho_{k'}$ are A-relations on U_k and $U'_{k'}$ respectively and $\varkappa$ and $\varkappa'$ are C-relations on K and K' respectively). Further let f and f' be isomorphisms of (U, λ_U) and $(U', \lambda_{U'})$ onto (S, σ) respectively such that $fm = f'm' = n$. Then the following assertions are equivalent:*

(α) *For every $k \in K$ and $k' \in K'$ the mappings q_k and $q'_{k'}$ (determined by n) satisfy*

$$q_k q_{k'} = q_{k'} q_k.$$

δ^*) *For every $k \in K$ and $k' \in K'$ there exists a set $U_{kk'}$ and an A-relation $\xi_{kk'}$ on $U_{kk'}$; for every $k \in K$ there exists a bijection $f_k' : \bigtimes_{k' \in K'} U_{kk'} \to U_k$, and for every $k' \in K'$ there exists a bijection $f_{k'}' : \bigtimes_{k \in K} U_{kk'} \to U'_{k'}$ such that $f(f_k')_{k \in K}$ and $f'(f_{k'})_{k' \in K'}$ are isomorphisms of the L-product $\Pi_{k \in K}(\Pi_{k' \in K'} U_{kk'})$ and of the L-product $\Pi_{k' \in K'}(\Pi_{k \in K} U_{kk'})$ onto (S, σ) respectively.*

3.

Let S be a set. Further let Ω be a set and let $\mathscr{A} = \{A_\omega \mid \omega \in \Omega\}$ be a system of nonempty sets. If σ^ω are any A_ω-relatios on S, then the system $\{\sigma^\omega \mid \omega \in \Omega\}$ will be said to be an *$\mathscr{A}$-relational structure on S*. Let $\{\varrho_k^\omega \mid \omega \in \Omega\}$ be an $\mathscr{A}$-relational structure on U_k for each $k \in K$. If the L-product of (U_k, ϱ_k^ω) over $(K, \varkappa)$ is denoted by (U, λ_U^ω) for each $\omega \in \Omega$, then $\{\lambda_U^\omega \mid \omega \in \Omega\}$ is obviously an $\mathscr{A}$-relational structure on U. We say that f is an *isomorphism* of the $\mathscr{A}$-relational structure $\{\lambda_U^\omega \mid \omega \in \Omega\}$ onto the $\mathscr{A}$-relational structure $\{\sigma^\omega \mid \omega \in \Omega\}$ iff f is an isomorphism of the relation λ_U^ω onto the relation σ^ω for each $\omega \in \Omega$.

It is easy to see that 2.11 remains valid if we replace the relations σ, ϱ_k, ϱ_k'' by $\mathscr{A}$-relational structures.

Let ϱ be an A-relation on M and let $a_0 \in A$. We say that ϱ is an *operation* on M iff for any $(f_a)_{a \in A} \in \varrho$ and $(g_a)_{a \in A} \in \varrho$ we have : $f_a = g_a$ for every $a \in A$, $a \neq a_0$ implies $f_{a_0} = g_{a_0}$. Hence a finitary n-ary operation is a special case of $(n+1)$-ary relation. Our definition includes also partial and infinitary operations. From this it follows that universal algebras may be regarded as a special case of relational structures.

In the sequel we shall restrict ourselves to the ase of (full) direct product of algebras. A subalgebra with a single element $\{n\}$ is termed a *trivial subalgebra* (for reference see e.g. [1]). Obviously an isomorphism carries a trivial subalgebra onto a trivial subalgebra and (C_n) holds. Hence we have (see also [4] Theorem 2):

3.1. Theorem. *Let Ω be an operator domain. Let S be an Ω-algebra with the trivial subalgebra $\{n\}$ and let $U_k (k \in K)$ and $U'_{k'} (k' \in K')$ be Ω-algebras. Further let f and f' be isomorphisms of the direct products $\Pi_{k \in K} U_k$ and $\Pi_{k' \in K'} U_{k'}$ onto S respectively. Then the following assertions are equivalent:*

(α) *For every $k \in K$ and $k' \in K'$ the mappings q_k and q_k' (determined by n) satisfy*

$$q_k q'_{k'} = q'_{k'} q_k .$$

(δ') *For every $k \in K$ and $k' \in K'$ there exists an Ω-algebra $U_{kk'}$; for every $k \in K$ there exists an isomorphism $f_k' : \Pi_{k' \in K'} U_{kk'} \to U_k$ and for every*

$k' \in K'$ there exists an isomorphism $f_{k'} : \prod_{k \in K} U_{kk'} \to U_{k'}$ such that $f(f_k)_{k \in K}$ and $f'(f_{k'})_{k' \in K'}$ are isomorphisms of $\prod_{(k,k') \in K \times K'} U_{kk'}$ onto S respectively.

3.2. Remark. Hashimoto has proved in [2] that if σ is a binary, reflexive, antisymmetric and connected relation on S, then the condition (α) is satisfied for eardinal products. This is not true for lexicographic products as the following very simple example shows.

Let $h > 0$ be an integer. By **h** we understand the chain $0 < 1 < \ldots < h - 1$. Let $K = K' = \mathbf{2}$, $U_0 = U'_1 = \mathbf{2}$, $U_1 = U'_0 = \mathbf{3}$. Then $U = \prod_{k \in K} U_k$ and $U' = \prod_{k' \in K'} U_k'$ are chains with 6 elements and hence both are isomorphic to $S = \mathbf{6}$. Let $n = 0$, $s = \mathbf{3}$ $(\in S)$. Then $q_0\mathbf{3} = fo_0p_0(1,0) = f(1,0) = \mathbf{3}$ and similarly $q'_1\mathbf{3} = f'o'_1p'_1(1,1) = f'(0,1) = 1$. Thus $q'_1q_0\mathbf{3} = q'_1\mathbf{3} = 1$ and $q_0q'_1\mathbf{3} = q_0 1 = fo_0p_0(0,1) = f(0,0) = 0$, i.e. $q_0q'_1\mathbf{3} \neq q'_1q_0\mathbf{3}$.

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