Felix M. Arscott; Graham P. Wright

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FLOQUET THEORY FOR DOUBLY-PERIODIC DIFFERENTIAL EQUATIONS

F. M. Arscott and G. P. Wright

To Professor Otakar Borůvka at his Seventieth Birthday

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1. INTRODUCTORY

The Floquet theory of ordinary differential equations with periodic coefficients is one of the basic elements in the study of such equations, not only in the abstract theory but in applications also. In essence, the Floquet theory states that if the coefficients of an ordinary linear differential equation

$$L_z(w) = 0 \tag{1.1}$$

are periodic functions of z with period π, then under very mild restrictions there always exists at least one *multiplicative solution*, that is, one solution $u(z)$ with the property that

$$u(z + \pi) \equiv su(z), \tag{1.2}$$

for an appropriately chosen constant s, in general complex. It follows immediately, if $s = e^{\mu\pi}$, that $u(z)$ can be put into the form

$$u(z) = e^{\mu z} P(z) \tag{1.3}$$

where $P(z)$ is periodic with period π.

A natural question is to ask how far this theory may be extended to differential equations whose coefficients are doubly-periodic functions, with periods ω, ω', say. As long ago as 1877 Hermite [3] established the remarkable result that, if the general solution of the differential equation is uniform in the complex plane, then there exists at least one doubly-multiplicative solution $u(z)$ such that

$$u(z + \omega) \equiv su(z), \qquad u(z + \omega') \equiv s'u(z) \tag{1.4}$$

for appropriate s, s'. Such a solution can be put into the form

$$u(z) = e^{\mu z} \frac{\Theta(z - a)}{\Theta(z)} P(z),$$

where $\Theta(z)$ is the Jacobian theta function, μ and a are suitably chosen constants, and $P(z)$ is doubly periodic with periods ω, ω'.

Up till the present, this appears to be the sole published result with a direct bearing on the problem of extending the Floquet theory, though some of the results in Arscott and Sleeman [2] were obtained with the application to such equations in mind.

The essential difficulty in extending the Floquet theory lies in the following fact: an equation with singly-periodic coefficients can normally be put into a form in which the equation has no singularities in a strip of the complex z-plane which includes all the real axis; consequently, there is no difficulty in continuing any solution analytically throughout this strip, and the complete analytic function so obtained is single valued there. Many equations of practical importance have, indeed, no finite singularities at all.

The situation is, however, quite different in the case of doubly-periodic equations. In the first place, a doubly-periodic function with no singularities is merely a constant, so that any doubly-periodic equation which is not completely trivial must have an infinite number of singularities in the finite part of the plane. Moreover, these singularities cannot in any sense be by-passed because double periodicity is essentially a property which involves the whole complex plane.

In order to make progress, therefore, the equations in this paper are restricted to have only one singularity in each fundamental period-parallelogram, and a fundamental part is played by a certain parameter, ν, which we call the "exponent" of the equation. The Hermite theory applies essentially to the case when ν is an integer, and this paper considers the extensions which can be made to certain rational values of ν. It is in the nature of a preliminary study, indicating the kind of results which can be expected to hold for general rational values of ν. The work is of a fairly intricate character, as might be expected from the fact that no advances in this direction have been made for nearly a century, and although this present study does not take its origin directly from the work of Professor Borůvka, it is offered as a tribute to the brilliance of his researches in the field of linear differential equations.

2. THE CHARACTERISTIC EXPONENT

It is convenient to use the notation of Jacobian elliptic functions. We consider the equation

$$\frac{d^2w}{dz^2} + \Phi(z)\, w = 0\,, \tag{2.1}$$

where $\Phi(z)$ is an even function of z, periodic with periods $2K$, $2\,iK'$, and analytic except at points congruent to iK' mod. $(2K, 2\,iK')$. It follows that $\Phi(z)$ is an integral function of $\mathrm{sn}^2 z$, say

$$\Phi(z) = \sum_{m=0}^{\infty} A_m \,\mathrm{sn}^{2m} z \tag{2.2}$$

the series being absolutely and uniformly convergent for all finite sn z. Special cases are the Lamé equation, which we here write as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (h - \tilde{\nu}(\tilde{\nu} + 1)\, k^2 \,\mathrm{sn}^2 z)\, w = 0 \tag{2.3}$$

and the ellipsoidal wave equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - (a + bk^2 \,\mathrm{sn}^2 z + qk^4 \,\mathrm{sn}^4 z)\, w = 0. \tag{2.4}$$

It may be noted that (2.1) can be put into algebraic form, and indeed it seems possible that an alternative attack on this problem could be made by considering the equation in such a form and applying the methods of [2]. If we set sn $z = t$ we have

$$\begin{aligned}(1 - t^2)(1 - k^2t^2)\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - t(1 + k^2 - 2k^2t^2)\frac{\mathrm{d}w}{\mathrm{d}t} &+ \\ + \{\Sigma A_m t^{2m}\}\, w &= 0,\end{aligned} \tag{2.5}$$

which has four regular singularities, at $t = \pm 1$, $\pm k^{-1}$, and a singularity at infinity, generally irregular. If we set $u = t^2 = \mathrm{sn}^2 z$, we have an alternative form

$$\begin{aligned}4u(1 - u)(1 - k^2u)\frac{\mathrm{d}^2 w}{\mathrm{d}u^2} + 2(3k^2u^2 - 2(1 + k^2)\, u + 1)\frac{\mathrm{d}w}{\mathrm{d}u} &+ \\ + \{\Sigma A_m u^m\}\, w &= 0,\end{aligned} \tag{2.6}$$

which has three regular singularities at $u = 0, 1, k^{-2}$ and a singularity at infinity, generally irregular.

Returning now to (2.1) we apply a standard type of argument (e.g. [1], p. 162) to show that there exists at least one solution of the differential equation in the neighbourhood of iK' (more precisely, in $|z - iK'| < R = \min(2K, 2K')$) which on making a negative half-circuit about iK' is multiplied by an appropriate constant, say σ. Symbolically, this property may be expressed by the assertion that there is a solution $w(z)$ such that

$$w(iK' + (z - iK')\, e^{-i\pi}) = \sigma w(z). \tag{2.7}$$

Clearly, the solution $w(z)$ is determined only up to a multiplicative constant.

The constant σ is determined as a root of a quadratic equation of the form

$$\sigma^2 - 2A\sigma - 1 = 0, \tag{2.8}$$

so that $-\sigma^{-1}$ is also a root; that is to say, there is also a solution $\hat{w}(z)$ (say) such that

$$\hat{w}(\mathrm{i}K' + (z - \mathrm{i}K')\,\mathrm{e}^{-i\pi}) = -\sigma^{-1}\hat{w}(z). \tag{2.9}$$

Now we introduce the exponent ν by the relation

$$\sigma = \mathrm{e}^{\nu\pi i}, \tag{2.10}$$

with the consequence that

$$-\sigma^{-1} = \mathrm{e}^{-(\nu+1)\pi i}. \tag{2.11}$$

Then the solutions $w(z)$, $\hat{w}(z)$ of (2.7), (2.9) can be expressed in the circle $C : | z - \mathrm{i}K' | < R$ as Laurent series

$$w(z) = (z - \mathrm{i}K')^{-\nu} \sum_{-\infty}^{\infty} c_n(z - \mathrm{i}K')^{2n}, \tag{2.12a}$$

$$\hat{w}(z) = (z - \mathrm{i}K')^{\nu+1} \sum_{-\infty}^{\infty} \hat{c}^n(z - \mathrm{i}K')^{2n}, \tag{2.12b}$$

and these are clearly independent if 2ν is not an odd integer.

We have thus defined ν, in effect, as any number such that there exists a solution of (2.1) with the property of being multiplied by $\exp(\nu\pi\,\mathrm{i})$ on making a negative half-circuit about $\mathrm{i}K'$. This definition is imprecise, because from (2.8), (2.9), (2.10) it is clear that all the numbers

$$\nu + 2k, \qquad -\nu - 1 + 2k, \tag{2.13}$$

where k is any integer, also satisfy the definition. We therefore make our specification of ν precise by the condition

$$-\tfrac{1}{2} < \nu \leqq \tfrac{1}{2}; \tag{2.14}$$

it is easily seen that precisely one of the numbers in (2.13) satisfies this condition.

The exponent ν, thus defined, is determined uniquely by the function $\Phi(z)$ of equation (2.1) and is thus inherent in the equation; in general, however, it does not appear explicitly in the equation. The case of Lamé's equation (2.3) is exceptional, because there is a close connection between ν and $\hat{\nu}$; we have

$$\nu \equiv \hat{\nu} \ (\text{mod. } 2) \quad \text{or} \quad \nu \equiv -\hat{\nu} - 1 \ (\text{mod. } 2); \tag{2.15}$$

that is, ν is that number in the range $(-\frac{1}{2}, \frac{1}{2}]$ which differs from $\hat{\nu}$ or $-\hat{\nu} - 1$ by a multiple of 2.

It should be remarked also that ν, as we have defined it, is one of the exponents of the algebraic differential equation (2.5) at the singularity $t = \infty$, in the usual sense of that term (see, e.g. [3], p. 60).

The case $\nu = \frac{1}{2}$ is somewhat exceptional, since then (and only then) the solutions $w(z)$, $\hat{w}(z)$ may not be independent, and the complete solution near $z = iK'$ may involve a logarithm.

3. THE HERMITE THEORY: GENERAL SOLUTION UNIFORM [4]

It is easily seen that the general solution of (2.1) will be uniform in the entire z-plane if and only if $\nu = 0$. Hermite's argument, which we repeat here partly for completeness but more particularly because it can be applied with modifications in other cases, is simple and elegant.

By the ordinary Floquet theory for singly-periodic equations (regarding (2.1) for the moment as a singly-periodic equation with real period $2K$), there always exists a multiplicative solution $u(z)$ analytic throughout the strip $|\operatorname{Im} z| < K'$, i.e. such that

$$u(z + 2K) = su(z). \tag{3.1}$$

for some appropriately chosen constant s. Let $u^*(z)$ be defined by

$$u^*(z) = u(z + 2\,iK') \tag{3.2}$$

If $u^*(z)$ is a constant multiple of $u(z)$, say, $u^*(z) = s'u(z)$ then $u(z)$ is a doubly-multiplicative solution, and there is nothing more to prove. If, however, $u^*(z)$ is not a constant multiple of $u(z)$, it is then linearly independent of $u(z)$, and the general solution of (2.1) is of the form

$$v(z) = cu(z) + c^*u^*(z). \tag{3.3}$$

Now consider $v(z + 2K)$. We have

$$\begin{aligned} v(z + 2K) &= cu(z + 2K) + cu^*(z + 2K), \\ &= csu(z) + c^*u(z + 2K + 2\,iK'), \\ &= csu(z) + c^*u(z + 2\,iK' + 2K), \end{aligned}$$

(since $u(z)$ is uniform)

$$\begin{aligned} &= csu(z) + c^*su^*(z). \\ &= sv(z). \end{aligned} \tag{3.4}$$

Now since $\Phi(z + 2\,\mathrm{i}K') = \Phi(z)$, $u^*(z + 2\,\mathrm{i}K')$ is a solution of the differential equation, and hence expressible as a linear combination of $u(z)$, $u^*(z)$ say

$$u^*(z + 2\,\mathrm{i}K') = \alpha u(z) + \alpha^* u^*(z). \tag{3.5}$$

Then

$$\begin{aligned} v(z + 2\,\mathrm{i}K') &= cu(z + 2\,\mathrm{i}K') + c^* u^*(z + 2\,\mathrm{i}K'), \\ &= c^*\alpha u(z) + (c + c^*\alpha^*)\, u^*(z). \end{aligned} \tag{3.6}$$

Thus we shall have $v(z + 2\,\mathrm{i}K') = s'v(z)$ if and only if

$$s'c = c^*\alpha, \qquad s'c^* = c + c^*\alpha^*,$$

i.e.

$$s'c - \alpha c^* = 0, \tag{3.7a}$$

$$c + (\alpha^* - s')\, c^* = 0 \tag{3.7b}$$

and it will be possible to choose c, c^* non-trivially to satisfy these equations if and only if

$$s'(\alpha^* - s') + \alpha = 0$$

i.e.

$$s'^2 - \alpha^* s' - \alpha = 0. \tag{3.8}$$

If s' is chosen as a root of this equation, then the solution $v(z)$ can be constructed so that

$$v(z + 2\,\mathrm{i}K') = s'v(z)$$

and from this and (3.4) it is clear that $v(z)$ is doubly-multiplicative.

It is worth noting that since (2.1) is a special case of Hill's equation we can show that the equation to determine s is of the form $s^2 - 2As + 1 = 0$, so that if s is a root, so is s^{-1}. Similarly the equation to determine s' is of the form $s'^2 - 2A's' + 1 = 0$.

4. OUTLINE OF RESULTS

If $\nu \neq 0$, the general solution of (2.1) will not be uniform throughout the z-plane. When ν is rational, however, say $\nu = l/m$ we cut the z-plane in an appropriate manner, making cuts which join chains of precisely m singularities. One might expect that the general solution would then be uniform in the cut plane, but such is not the case. However, we shall show that in certain circumstances the general solution is indeed uniform in the cut plane, and then the Hermite theory can be extended.

The simplest non-integral value of ν, namely $\frac{1}{2}$, is exceptional in tha logarithmic terms may then appear in the general solution; consideration of this case has therefore been excluded.

The values of ν which have been studied are: $\pm 1/3$, $\pm 1/4$, $\pm 1/5$, $\pm 2/5$, $\pm 1/6$. In each case, appropriate cuts are made in the z-plane; more precisely, if $\nu = l/m$, we make a cut from $\mathrm{i}K'$ to $(2m - 1)\,\mathrm{i}K'$ and similar cuts through all congruent points mod . $(2K, 2m\,\mathrm{i}K')$; each cut thus covers a chain of precisely m singularities parallel to the imaginary axis. Then we ask: in what circumstances is the general solution uniform in the cut plane? The answer is expressible in terms of a certain matrix T, introduced in § 6; we find that for the general solution to be uniform, T must have one of a small number of particular forms. The details vary slightly according as m is even or odd.

When m is odd, every matrix T is found to be diagonal; this carries the implication that each of the solutions w, $\hat{w}$ is multiplicative for the period $2\,\mathrm{i}K'$, but the periodicity factors (i.e. the constants by which they are multiplied) are different, being $\pm\sigma^{-1}$, $\pm\sigma$ respectively. The solutions are, of course, also multiplicative for the period $2m\,\mathrm{i}K'$, and the periodicity factors are the same, being ± 1. Consequently, the general solution has period $4m\,\mathrm{i}K'$ (it may have period $2m\,\mathrm{i}K'$). In this case, either w or $\hat{w}$ is a doubly-multiplicative solution (with pseudo-periods $2K$, $2\,\mathrm{i}K'$); it is even or odd in z, and has the real period $4K$ (possibly $2K$).

When m is even, the same diagonal forms for T appear, giving corresponding results, but there is also one non-diagonal form, yielding quite different results which will be described in § 8.

5. NOTATION

Throughout the remainder of this paper, we use the following notation:

$$(5.1) \qquad \nu = l/m(-\tfrac{1}{2} < \nu \leqq \tfrac{1}{2}), \qquad \sigma = \mathrm{e}^{\nu\pi i}.$$

Defining $w(z)$, $\hat{w}(z)$ as in (2.12) we write further

$$(5.2) \qquad w(z - 2m\,\mathrm{i}K') = w_m(z), \qquad \hat{w}(z - 2m\,\mathrm{i}K') = \hat{w}_m(z),$$

so that $w_0(z) = w(z)$, $\hat{w}_0(z) = \hat{w}(z)$; thus $w_m(z)$, $\hat{w}_m(z)$ are valid in $|\, z - (2m + 1)\,\mathrm{i}K' \,| < R = \min\,(2K, 2K')$. We further write $W(z)$ for the solution column vector

$$(5.3) \qquad W(z) = \{w(z), \hat{w}(z)\}$$

and similarly for $W_m(z)$.

It should be noted that $w_m(z)$, $\hat{w}_m(z)$ are each indeterminate to the extent of an arbitrary constant multiple, so that $W(z)$ is indeterminate to the extent of pre-multiplication by an arbitrary diagonal matrix.

Finally, we write M for the matrix

$$(5.4)\qquad M = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma^{-1} \end{pmatrix} = \begin{pmatrix} e^{\nu\pi i} & 0 \\ 0 & e^{-(\nu+1)\pi i} \end{pmatrix}.$$

6. THE SHIFT MATRIX T

The solution vector $W_m(z)$, valid near $(2m + 1)\,iK'$, is generally not periodic and so is not the same as $W_{m+1}(z)$, valid near $(2m + 3)\,iK'$. Nevertheless, these solution vectors have a common region of validity, so there will be a constant "shift matrix" T, such that

$$(6.1)\qquad W_{m+1} = TW_m;$$

because of the periodic character of $\Phi(z)$, the matrix T is clearly independent of m. To a large extent, our investigation depends on the nature of the matrix T.

Now, because of the symmetry of the points $\pm iK'$ with respect to the origin, and the fact that $\Phi(z)$ is even, $W(-z) \equiv \{w(-z), \hat{w}(-z)\}$ is a solution column vector valid near $-iK'$. But from (2.12)

$$\begin{aligned} w(-z) &= (-z - iK')^{-\nu} \sum_{-\infty}^{\infty} c_n(-z - iK')^{2n} \\ &= e^{-\nu\pi i}(z + iK')^{-\nu} \sum_{-\infty}^{\infty} c_n(z + iK')^{2n} \\ &= e^{-\nu\pi i} w_{-1}(z) \end{aligned}$$

and similarly

$$\hat{w}(-z) = e^{(\nu+1)\pi i}\hat{w}_{-1}(z),$$

and consequently (see (5.4))

$$W(-z) = M^{-1}W_{-1}(z).$$

But by the definition of T,

$$(6.2)\qquad W(z) = TW_{-1}(z) = TMW(-z).$$

Writing $-z$ for z we have, immediately,

$$W(-z) = TMW(z) = (TM)^2W(-z),$$

so

$$(6.3)\qquad (TM)^2 = I,$$

I being the unit matrix.

From this, we can deduce that T must have, basically, one of two possible forms.

Write

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{6.4}$$

then a brief calculation shows that (6.3) holds if and only if

$$\sigma^2\alpha^2 - \beta\gamma = \sigma^{-2}\delta^2 - \beta\gamma = 1, \tag{6.5a}$$

$$\beta(\sigma^2\alpha - \delta) = \gamma(\sigma^2\alpha - \delta) = 0. \tag{6.5b}$$

We must now distinguish two cases.
Case I: $\sigma^2\alpha - \delta \neq 0$. Then from (6.5b), $\beta = \gamma = 0$ and the only possible forms for T are found to be (but see note on p. 124)

$$T = \pm M^{-1} = \pm \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & -\sigma \end{pmatrix}. \tag{6.6}$$

Case II: $\sigma^2\alpha - \delta = 0$. In this case T has the form

$$T = \begin{pmatrix} \alpha & \beta \\ \dfrac{\sigma^2\alpha^2 - 1}{\beta} & \sigma^2\alpha \end{pmatrix}. \tag{6.7}$$

This apparently involves two parameters, α and β, but one of these may be removed. We recall that the solution vector $W(z)$ is indeterminate to the extent of pre-multiplication by an arbitrary non-singular diagonal matrix D. If we put

$$W^*(z) = DW(z),\ W^*_m(z) = DW_m(z), \tag{6.8}$$

then (6.1) becomes $W^*_{m+1} = T^*W_m$, where $T^* = DTD^{-1}$, and, as before, $(T^*M)^2 = I$. Thus without loss of generality T can be replaced by DTD^{-1} with an arbitrary diagonal matrix D. To put T into a standard form we choose D so that DTD^{-1} is symmetric; this yields the form

$$T = \begin{pmatrix} \alpha & (\sigma^2\alpha^2 - 1)^{\frac{1}{2}} \\ (\sigma^2\alpha^2 - 1)^{\frac{1}{2}} & \sigma^2\alpha \end{pmatrix} \tag{6.9}$$

which is unique except for the two possible values of $(\sigma^2\alpha^2 - 1)^{\frac{1}{2}}$, and depends on the single parameter α.

Clearly, the form of T obtained in Case I, being diagonal, is unaffected by a transformation such as $DTD^{-1} = T^*$.

However, the form obtained in Case I turns out to be inadmissible. For, if $T = M^{-1}$, then $TM = I$, and then, by (6.2), $W(z) = W(-z)$, so that $w(z) = w(-z)$, $\hat{w}(z) = \hat{w}(-z)$. That is to say, both w and $\hat{w}$ are even functions of z; similarly, if $T = -M^{-1}$, w and $\hat{w}$ are both odd. But since $\Phi(z)$ is even in z, and $z = 0$ is an ordinary point of equation (2.1), the usual theory of linear differential equations shows that there cannot be two even or two odd independent non-trivial solutions. Thus $T = \pm M^{-1}$, while a formal solution of (6.3), is not admissible as a shift matrix.

7. UNIFORM SOLUTIONS IN THE CUT PLANE

Now we determine the possible forms which the shift matrix T can have if the general solution is uniform in the cut plane.

7.1 Case $\nu = \pm 1/3$

We make cuts in the z-plane from iK' to $5iK'$ and congruent points mod. $(2K, 6iK)'$, and determine the conditions on T for the general solution to be uniform in the cut plane. This will be the case if and only if the solution vector $W(z)$, when continued analytically along a circuit about the cut from iK' to $5iK'$, returns to its starting-point unchanged. We therefore continue $W(z)$ analytically along the path ABCDEFGA shown, which has been deformed to consist of shifts of $2iK'$ and simple circuits about singularities. We observe first that the solution vector $W_m(z)$, on making a circuit about $(2m+1)\, iK'$, is multiplied by

$$(7.1.1) \qquad \begin{pmatrix} e^{-2\nu\pi i} & 0 \\ 0 & e^{2\nu\pi i} \end{pmatrix} = \begin{pmatrix} \sigma^{-2} & 0 \\ 0 & \sigma^{2} \end{pmatrix} = M^{-2}\,.$$

Starting with $W(z) = W_0(z)$ at A, its analytic continuation at B is $T^{-1}W_1$ (using (6.1)), and at C is $T^{-2}W_2$. By the remark above the same solution vector becomes $T^{-2}M^{-2}W_2$ at D, $T^{-2}M^{-2}TW_1$ at E, $T^{-2}M^{-2}TM^{-2}W_1$ at F, $T^{-2}M^{-2}TM^{-2}TW_0$ at G, and finally returns to A with the value $T^{-2}M^{-2}TM^{-2}TM^{-2}W_0 = T^{-2}(M^{-2}T)^2M^{-2}W_0$. Thus the general solution will be uniform in the cut plane if and only if $T^{-2}\,(M^{-2}T)^2\, M^{-2} = \mathrm{I}$, that is

$$(7.1.2) \qquad T^2M^2 = (M^{-2}T)^2.$$

We now ask whether T, as given by (6.9), can satisfy (7.1.2) and find that this is so only for particular values of α. After tedious working, we obtain

$$(7.1.3)\quad T^2M^2 = \begin{pmatrix} \sigma^4\alpha^2 + \sigma^2\alpha^2 - \sigma^2 & (\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^{-2}+1)\,\alpha \\ (\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^2+\sigma^4)\,\alpha & \sigma^2\alpha^2 - \sigma^{-2} + \alpha^2 \end{pmatrix}$$

and

$$(7.1.4)\quad (M^{-2}T)^2 = \begin{pmatrix} \sigma^{-4}\alpha^2 + \sigma^2\alpha^2-1 & (\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^{-4}+\sigma^2)\,\alpha \\ (\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^6+1)\,\alpha & \sigma^2\alpha^2-1+\sigma^8\alpha^2 \end{pmatrix}$$

so the four equations

$$(7.1.5a)\qquad \sigma^4\alpha^2 + \sigma^2\alpha^2 - \sigma^2 = \sigma^{-4} + \sigma^2\alpha^2 - 1,$$

$$(7.1.5b)\qquad \alpha(\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^{-2}+1) = \alpha(\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^{-4}+\sigma^2),$$

$$(7.1.5c)\qquad \sigma^2\alpha(\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^2+1) = \alpha(\sigma^2\alpha^2-1)^{\frac{1}{2}}(\sigma^6+1),$$

$$(7.1.5d)\qquad \sigma^2\alpha^2 - \sigma^{-2} + \alpha^2 = \sigma^2\alpha^2 - 1 + \sigma^8\alpha^2$$

must all hold.

It is easy to see that (7.1.5b) holds if and only if $\alpha = 0$ or $\alpha = \pm\sigma^{-1}$, but if $\alpha = 0$ then (7.1.5a) is not satisfied (since $\sigma^2 \neq 1$). But if $\alpha = \pm\sigma^{-1}$, then all the equations (7.1.5) hold, and we have thus obtained all the possible forms of T, namely

$$(7.1.6)\qquad T = \pm\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix}.$$

If we write

$$(7.1.7)\qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then these two forms of T can conveniently be expressed as

$$(7.1.8)\qquad T = \pm M^{-1}J;$$

it is simple to verify that these satisfy (7.1.2), but the above calculation shows the more important and less obvious fact that these are the *only* forms of T satisfying (7.1.2) (but see note on p. 124).

Clearly, w and $\hat{w}$ are both multipliative solutions for the period $2\,iK'$ the periodicity factors being $\pm\sigma^{-1}$, $\pm\sigma$ respectively. For the period $6\,iK'$, they are also, of course, multiplicative, but their periodicity factors are the same, being ∓ 1 according as $T = \pm M^{-1}J$. Consequently, the *general* solution is multiplicative for the period $6iK'$, and indeed is periodic with period $12\,iK'$, possibly with period $6\,iK'$.

The doubly-multiplicative solutions will be considered in § 8.

7.2 Case $\nu = \pm 1/4$

In this case the cuts are made from iK' to $7\,iK'$ and congruently mod. $(2K,\ 8\,iK')$, each cut covering 4 singularities. By a similar process to that of §7.1 we find the condition for the general solution to be uniform to be

$$T^3M^2 = (M^{-2}T)^3. \tag{7.2.1}$$

An analysis similar to that of §7.1, but longer because of the extra multiplication, shows that if T has the form (6.9), then $\alpha = \pm\sigma^{-1}$ or $\alpha = 0$. Consequently, as in the case $\nu = 1/3$ we have the two possible forms of T

$$T = \pm M^{-1}J, \tag{7.2.2}$$

but we have also the possibility

$$T = \pm K, \text{ where } K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{7.2.3}$$

Again, it is quite easy to verify that each of the values in (7.2.2) and (7.2.3) satisfy (7.2.1); the interesting feature of the analysis is that these are the only possible values of T (but see note on p. 124).

Analogously to the case $\nu = 1/3$, if T has one of the diagonal forms (7.2.2), then w, $\hat{w}$ are both multiplicative for $2\,iK'$ with different periodicity factors; they are also multiplicative for period $8\,iK'$ with the same periodicity factor which has the value ± 1, so the general solution is also multiplicative with the same factor, and is periodic with period $16\,iK'$ (possibly with period $8\,iK'$).

If $T = \pm K$, however, w and $\hat{w}$ are not multiplicative for period $2\,iK'$, but they are both multiplicative for period $4iK'$, with the same factor -1 (since $K^2 = -I$). Consequently, the general solution is multiplicative for $4\,iK'$ with factor -1, and is periodic with period $8\,iK'$.

7.3 $\nu = l/5$, $\nu = l/6$ and further cases

For a general value $\nu = l/m$, $m > 2$, we easily establish that the condition for the general solution to be uniform in the cut plane is

$$T^{m-1}M^2 = (M^{-2}T)^{m-1}, \tag{7.3.1}$$

and can verify without difficulty that $T = \pm M^{-1}J$ satisfies this for all m, while $T = \pm K$ satisfies it only when m is even. For the cases $m = 5$, $m = 6$ it has been verified, by long and tedious analysis involving the multiplication of the various matrices, that these are in fact the only solutions, and it is reasonable to conjecture that this is true in general, but no way has yet been found of confirming this fact.

The periodic properties of w, $\hat{w}$ with respect to the periods $2\,\mathrm{i}K'$, $2\,\mathrm{m}\,\mathrm{i}K'$ are entirely analogous to those described in §§ 7.1, 7.2, and will not be detailed here.

8. DOUBLY-MULTIPLICATIVE SOLUTIONS IN THE CUT PLANE

Let ν be rational, $\nu = l/m$, let the z-plane be cut as described in § 4, and let T be such that the general solution is uniform in the cut plane. It then becomes relevant to ask whether the Hermite theory of doubly-multiplicative solutions (§ 3) can be extended to the cut plane.

In considering this, it is necessary to stress the meaning attached to the idea of addition of a period. If $u(z)$ is a solution, valid in a certain region including the point z, then $u(z + 2K)$, for instance, must be interpreted as the analytic continuation of $u(z)$ to a region including the point $z + 2K$, by a path which avoids the cuts. Because of the uniformity of the general solution, which we have imposed, this analytic continuation is unique and does not depend on the particular path followed, so long as it does not cross a cut.

The analysis of § 3 applies virtually unchanged to show the existence of at least one doubly-multiplicative solution with pseudo-periods $2K$, $2\,\mathrm{i}K'$—that is to say, a solution $u(z)$ and constants s, s' such that

$$u(z + 2K) = su(z),\ u(z + 2\,\mathrm{i}K') = s'u(z). \tag{8.1}$$

Consider now the case when $T = \pm M^{-1}J$, where a very interesting feature occurs. We have seen that w, $\hat{w}$ are both multiplicative for the period $2\,\mathrm{i}K'$ but with different periodicity factors $\pm\sigma^{-1}$, $\pm\sigma$, so no linear combination of them can be multiplicative for this period. The only possible doubly-multiplicative solutions are, therefore, w and $\hat{w}$. Now, if $T = M^{-1}J$, then (see (6.2)) $W(z) = JW(-z)$, so that w is even and $\hat{w}$ is odd; similarly, if $T = -M^{-1}J$ then w is odd and $\hat{w}$ even. But we now show that w, $\hat{w}$ can not both bemultiplicative for period $2K$, without being identically zero. For, let us suppose that $T = M^{-1}J$ for definiteness, and

$$w(z + 2K) = sw(z),\ \hat{w}(z + 2K) = \hat{s}\hat{w}(z); \tag{8.2a, b}$$

then, putting $z = -2K$ in (8.2a) we have $w(0) = sw(-2K) = sw(2K) = s^2w(0)$. Similarly, differentiating (8.2b) and putting $z = -2K$ gives $\hat{w}'(0) = \hat{s}^2\hat{w}'(0)$. Now $z = 0$ is an ordinary point of (2.1), so no non-trivial solution can have a double zero; hence

$$w(0) \neq 0,\ \hat{w}'(0) \neq 0,\ s^2 = 1,\ \hat{s}^2 = 1.$$

The function $w(z)$, therefore, has the properties

$w(z) = w(-z)$, $w(z + 2K) = sw(z)$ $(s = \pm 1)$, $w(z + 2\mathrm{i}K') = s'w(z)$, $(s' = \sigma^{-1})$. Then

$$w(K + \mathrm{i}K') = sw(-K + \mathrm{i}K') = ss'w(-K - \mathrm{i}K') = ss'w(K + \mathrm{i}K'),$$
$$\text{so } w(K + \mathrm{i}K') = 0.$$

Similar working shows that $\hat{w}(K + \mathrm{i}K') = 0$, but this is impossible since $K + \mathrm{i}K'$ is an ordinary point of (2.1) and the general solution cannot vanish there. The same holds if $T = -M^{-1}J$.

Thus one, but only one, of w, $\hat{w}$ is doubly-multiplicative for pseudo-periods $2K$, $2\mathrm{i}K'$, it is even or odd, with $s = \pm 1$, $s' = \pm\sigma^{\pm 1}$.

It is interesting to consider also the possibility of doubly-multiplicative solutions with pseudo-periods $2K$, $2m\mathrm{i}K'$; the reasoning of § 3 applies again, to show the existence of at least one doubly-multiplicative solution. When $T = \pm M^{-1}J$, the *general* solution is multiplicative for $2m\mathrm{i}K'$, with periodicity factor ± 1, so that the doubly-multiplicative solution is not necessarily w or $\hat{w}$ but may be a linear combination of these and the restriction $s = \pm 1$ no longer holds.

Finally, we consider the case when m is even and $T = \pm K$. If $T = K$, then we easily find that the multiplicative solutions for $2\mathrm{i}K'$ are $w \pm \hat{w}$, with periodicity factors $\pm\mathrm{i}$, while if $T = -K$, the periodicity factors are $\mp\mathrm{i}$. There seems no reason, in this case, for the solutions to have any particular properties of being even or odd, nor to have any special values of s; there could, indeed, be two doubly-multiplicative solutions.

Note (added in proof). Closer examination shows that for all m, even or odd, there exist two further forms of T, namely the unsymmetric matrices

$$T = \begin{pmatrix} \pm\sigma^{-1} & 0 \\ \gamma & \pm\sigma \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \pm\sigma^{-1} & \beta \\ 0 & \pm\sigma \end{pmatrix}$$

giving

$$TM = \begin{pmatrix} \pm 1 & 0 \\ \gamma\sigma & \mp 1 \end{pmatrix} \quad \text{and} \quad TM = \begin{pmatrix} \pm 1 & -\beta\sigma^{-1} \\ 0 & \mp 1 \end{pmatrix}$$

In the first case $w(z)$, and in the second place $\hat{w}(z)$ is doubly-multiplicative and either even or odd, while the other solution in each case has no special properties.

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University of Surrey
Guildford, Surrey
England