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A CONTRIBUTION TO THE CARTAN'S METHOD OF SPECIALIZATION OF FRAMES

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A. Švec (see [1]) has given a precise form to the Cartan's method of specialization of frames, namely in terms of fibre bundles, connections and certain Grassmann manifolds. Here we show how to exploit some geometrical objects connected with the mentioned Grassmann manifolds for the specialization procedure. Some complementary results to the main theorem of [1] are also given.

PART I

A. Švec has admitted the equivalence problem as his start point. Here we shall limit ourselves to the specialization problem only. As for the equivalence problem, the reader is advised to consult the original Paper [1].

Following [1], *a space $\sigma(P, M, G, Q, H, \omega)$ is defined as a principal fibre bundle $P(M, G)$ with a given connection ω and a given reduction Q to a subgroup $H \subset G$.* The most important particular case is the following: Let G be a Lie group, H^0 its closed subgroup. Consider the homogeneous space G/H^0 and let G act on G/H^0 to the left. Form a trivial bundle $P = G/H^0 \times G$ over G/H^0 and denote by p the bundle projection $P \to G/H^0$. Further, denote by π the canonical projection $G \to G/H^0$. Let Q^0 be a reduction of P to the subgroup H^0 given by $Q^0 = \{(x, g) \in P \mid \pi(g) = x\}$. Then Q^0 is isomorphic to the principal fibre bundle $G(G/H^0, \pi, H^0)$ with π as bundle projection. (See [3], p. 137.) Consider the *natural Cartan connection* ω on P defined as follows: the horizontal subspace of a tangent space $T_{z_0}(P)$, $z_0 = (x_0, g_0) \in P$, is the tangent space of the submanifold $(G/H^0 \times g_0) \subset P$ at z_0. If X is a vertical vector of $T_{z_0}(P)$, then $\omega(X)$ is an element A of the Lie algebra $\mathfrak{g}$ representing that left-invariant vector field A^* which contains the vector $dqX \in T_{g_0}(G)$. Here q denotes the natural projection $P \to G$, dq its differential. Consequently, ω is a vector 1-form on P with values in the algebra $\mathfrak{g}$. Further, let $M \subset G/H^0$ be a sumanifold, $H \subset H^0$ a Lie subgroup, and $Q \subset Q^0$ a reduction of Q^0 to the group H. Let us denote by P_M, Q_M the restrictions of P, Q to the base M. We have obtained a special space $\sigma(P_M, M, G, Q_M, H, \omega)$. It will be called briefly a *h-induced space σ.*

In particular, if $G = GA(n)$ is the n-th affine group and $H^0 = GA_0(n)$ the isotropy group of a point $o \in A^n$, then A^n is a homogeneous space G/H^0 and we obtain the special bundle having been constructed in Chapter I of [1]. The elements of P are all couples (p, F) where $p \in A^n$ and F is an ordinary affine frame; a couple (p, F) belongs to the sub bundle Q^0 iff p is the initial point of F. Let Q be a reduction of Q^0, and Q_M a restriction of Q to a submanifold $M \subset G/H^0$. If we choose some basis of the algebra $\mathfrak{g} = \mathfrak{ga}(n)$, then the coordinates of the vector form ω on Q_M with respect to that basis are ordinary infinitesimal components ω_i, ω_i^j of the frame, in the sense of E. Cartan.

Now we shall announce a simple existence Theorem. As we shall limit ourselves to the local theory, we shall suppose all the fibre bundles under consideration to be trivial. All manifolds, maps, fibre bundles and their sections will be supposed to be differentiable of class C^∞.

Theorem 1. $\mathscr{A}$. *Denotations.*

1. *Let be given a space* $\sigma(P, M, G, Q, H, \omega)$ *where a)* $\dim M < \dim G/H$, *b) the* 1*-form* ω *is regular on the subbundle* Q, *i.e., of maximal rank.*

2. *Let* Q' *be a reduction of* Q *to a Lie subgroup* $H' \subset H$.

3. *Denote by* $\mathfrak{g}$, $(\mathfrak{h})$ *the Lie algebra of* $G(H)$. *Put* $h = \dim \mathfrak{h}$, $n = \dim M$, $k = h + n$; *consequently* $\dim Q = k$.

4. *Let* $\mathbf{Z}_k$ *be a Grassmann manifold of all subspaces* $\mathscr{P} \subset \mathfrak{g}$ *such that* $\mathscr{P} \supset \mathfrak{h}$ *and* $\dim \mathscr{P} = k$.

5. *Further, let* $\mathbf{Z} \subset \mathbf{Z}_k$ *be an invariant manifold with respect to the group* H' *acting on* $\mathfrak{g}$ *to the left as the adjoint group* Ad (H').

6. *Let* $\mathbf{Z}^* \subset \mathbf{Z}$ *be an invariant manifold with respect to a subgroup* $H^* \subset H'$.

$\mathfrak{B}$. *Assumptions.*

(i) *For any* $b \in Q'$ *we have* $\omega[T_b(Q)] \in \mathbf{Z}$.

(ii) *To any map* $\alpha : M \to \mathbf{Z}$ *there is a map* $\tilde{h} : M \to H'$ *such that* $\tilde{h}(x)\,\alpha(x) \in \mathbf{Z}^*$ *for each* $x \in M$.

(iii) *Whenever* $z \in \mathbf{Z}^*$, $h \in H'$ *and* $h \,.\, z \in \mathbf{Z}^*$ *we have* $h \in H^*$.

Under these assumptions there is exactly one reduction Q^* *of the bundle* Q' *to the group* H^* *such that* $\omega[T_b(Q)] \in \mathbf{Z}^*$ *for each* $b \in Q^*$.

Proof. Let us denote $Q^* = \{b \in Q' \mid \omega[T_b(Q)] \in \mathbf{Z}^*\}$. It suffices to prove that Q^* is a reduction of Q' to the group H^*. Let us choose some section $s : M \to Q'$ of the (trivial) bundle Q'. Then we have a map $\alpha : M \to \mathbf{Z}$ given by the formula $\alpha(x) = \omega[T_{s(x)}(Q)]$. According to (ii) there is a map $\tilde{h} : M \to H'$ such that $\tilde{h}(x)\,\alpha(x) \in \mathbf{Z}^*$. It follows from the elementary properties of connections that $\omega[T_{s(x)\tilde{h}(z)}(Q)] = \tilde{h}(x)\,\alpha(x) \in \mathbf{Z}^*$ for all $x \in M$.

We have proved the existence of a section $s(x)\,\tilde{h}(x) : M \to Q^*$. From (iii) we obtain that Q^* is a subbundle in Q', which is a principal fibre bundle with the structural group H^*, q.e.d.

Note. In the differential geometry, the bundle P or each of its subbundles is called a *mobile frame on the manifold* M. A reduction Q^* as above is called *a specialization of a mobile frame* Q' *to the group* H^* (in the sense of E. Cartan).

Definition. *We say that a principal bundle* $Q(M, H)$ *admits a local reduction to a group* $H^* \subset H$ *if for each point* $x \in M$ *there is a neighbourhood* $U(x)$ *in* M *such that the restricted bundle* $Q \mid U$ *admits a reduction to the group* $H^* \subset H$.

Corollary of Theorem 1. *Let us replace the assumption* (ii) *of Theorem* 1 *by the following requirement: the group* H' *acts transitively on* $\boldsymbol{Z}$ *and* $\boldsymbol{Z}^*$ *consists of a single point. Then* Q' *admits exactly one local reduction* Q^* *to the group* H^* *such that* $\omega[T_b(Q)] = \boldsymbol{Z}^*$ *for any* $b \in Q^*$.

Proof. According to (iii) H^* is the isotropy group of the point $\boldsymbol{Z}^* \in \boldsymbol{Z}$ with respect to H'. Let $\alpha : M \to \boldsymbol{Z}$ be a map. We can identify $\boldsymbol{Z}$ canonically with the homogeneous space H'/H^*. For any $x \in M$ there is a neighbourhood V of the element $\alpha(x)$ in H'/H^* and a local section $s : H'/H^* \to H'$, of the principal fibre bundle with the bundle projection $\pi' : H' \to H'/H^*$, defined on V. We obtain a map $s \circ \alpha : U \to H'$ where U is a neighbourhood of $x \in M$ such that $\alpha(U) \subset V$. Let $\sigma : H' \to H'$ be the map $h \to h^{-1}$ and put $\tilde{h} = \sigma \circ s \circ \alpha$. Our construction shows that $\tilde{h}(x)\, \alpha(x) \in \boldsymbol{Z}^*$ for each $x \in U$. Consequently, the condition (ii) is locally satisfied and this proves our assertion. In case that $Q' = Q$ we obtain essentially Theorem 3,8 from [1].

*

Now we shall show that the requirement 1*b*) of Theorem 1 is valid for any *h-induced space* $\sigma(P, M, G, Q, H, \omega)$. We shall preserve our de notations from the beginning of this Part.

Theorem 2. *Let* $P = G/H^0 \times G$ *be a principal fibre bundle with the natural Cartan connection* ω *over a homogeneous space* G/H^0. *Let* M *be a manifold of dimension n imbedded into* G/H^0. *Demote by* P_M, Q^0_M, ... etc. *the corresponding restrictions of the bundles* P, Q^0, ... etc. *to the base* M. *Then the* 1*-form* ω *is regular on any reduction* Q_M *of* Q^0_M *to a Lie group* $H \subset H^0$, i.e., *the map* $\omega_b : T_b(Q_M) \to \mathfrak{g}$ *is an injection for any* $b \in Q_M$. *Moreover, the subspace* $\omega[T_b(Q_M)]$ *is spanned by the subalgebra* $\mathfrak{h}$ *and by additional n vectors of* $\mathfrak{g}$ *that are linearly independent over* $\mathfrak{h}^0 \supset \mathfrak{h}$.

The proof of the Theorem is based upon the following Lemma:

Lemma 1. *Let be given* $b \in Q^\circ$, $X \in T_b(Q^\circ)$, $\mathrm{d}p(X) \neq 0$ $(p : P \to G/H^\circ$ *being the bundle projection*). *Denote by* $v'X$ *the vertical component of* X *in the connection* ω. *Then* $v'X \notin T_b(Q^\circ)$.

Proof. There is a neighbourhood U_b of the point $b = (x, g)$ in P and diffeomorphisms of the form $(p, v)\colon\ U_b \to W_x \times V_g$, $(\psi, \pi)\colon\ V_g \to$

$\to \tilde{U} \times W'_x$, where $V_g \subset G$, $\tilde{U} \subset H$, $W_x \subset G/H^\circ$, $W'_x \subset G/H^\circ$. We can restrict U_b so that $W_x = W'_x$. Now $\pi \circ v = p$ on the subspace $U_b \cap Q^\circ$: in particular $x = p(b) = \pi(g)$. Since $dp(X) \neq 0$ there is a function f on W_x such that $(dpX)f \neq 0$. Now, $F = f \circ \pi \circ v$ is a function on U_b whose restriction to $p^{-1}(x)/_Q = \{x\} \times \pi^{-1}(x)$ is constant. We have $Y(F) = 0$ for each vertical vector $Y \in T_b(Q^\circ)$. Finally,

$$(v'X)F = v'X(f \circ \pi \circ v) = (dv \circ v'X)(f \circ \pi) =$$
$$= (dvX)(f \circ \pi) = X(f \circ \pi \circ v) = X(f \circ p) = (dpX)f \neq 0.$$

Since $v'X$ is vertical, we obtain $v'X \notin T_b(Q^\circ)$, q.e.d.

Proof of Theorem 2. Let be given $b \in Q_M$, $b = (x, g)$. Choose a basis $\{f_1, \ldots, f_n\}$ of the tangent space $T_x(M)$ and n vectors $e_1, \ldots, e_n$ of $T_b(Q_M)$ such that $dp(e_i) = f_i$. Then the vectors $\omega(e_1), \ldots, \omega(e_n)$ form a linearly independent system together with any basis of the subalgebra $\mathfrak{h}^\circ$. Otherwise, there would be a vector $e = \sum \alpha^i e_i \neq 0$ such that $\omega(e) \in \mathfrak{h}^\circ$. Hence $e \in T_b(Q^\circ)$, $v'(e) \in T_b(Q^\circ)$ and at the same time $dp(e) = \sum \alpha^i f_i \neq 0$ — a contradiction to Lemma 1.

*

If we proceed to reduce a space $\sigma(P, M, G, Q, H, \omega)$ to a smaller group $H^* \subset H$ we have to find, first of all, the smallest Grassmann manifold $\boldsymbol{Z} \subset \boldsymbol{Z}_k$ such that $\omega[T_b(Q)] \in \boldsymbol{Z}$ for each $b \in Q$; $\boldsymbol{Z}$ being invariant under H. (See Theorem 1). In the classical differential geometry we employ the Cartan's lemma and the structural equations for this purpose. A very simple rule of this kind takes place if the space σ is h-induced.

Theorem 3. *Let $P = G/H^\circ \times G$ be a fibre bundle with the natural Cartan connection ω and $M \to G/H^\circ$ some embedded manifold. Let be given two reductions $Q''_M \subset Q'_M \subset \mathring{Q}_M$ of the bundle Q^0_M to the groups $H'' \subset H' \subset H^\circ$. Assume that there is a subspace $\mathscr{R} \subset \mathfrak{g}$ such that $\omega[T_h(Q'_M)] = \mathscr{R}$ for any $h \in Q''_M$. Put $\mathscr{P}_h = \omega[T_h(Q''_M)]$. Then the inclusion $[\mathscr{P}_h, \mathscr{P}_h] \subset \mathscr{R}$ holds for each $h \in Q''_M$; i.e., whenever $h \in Q''_M$, $A, B \in \mathscr{P}_h$, we have $[A, B] \in \mathscr{R}$.*

We state beforehand a lemma again.

Lemma 2. *Let V be a manifold, $N \subset M$ its submanifolds. Let be given two vector fields X, Y on V such that $X_q, Y_q \in T_q(M)$ for each $q \in N$ and $X_q, Y_p \in T_p(N)$ for a fixed point $p \in N$. Then $[X, Y]_p \in T_p(M)$.*

Proof. Let U be some neighbourhood of p in V. Let f be an arbitrary function on U such that $f/M \cap U$ is constant. Then the function Yf is zero on $N \cap U$ and $X_p(Yf) = 0$. Similarly $Y_p(Xf) = 0$ and hence $[X, Y]_p f = 0$, which proves our assertion.

Proof of Theorem 3. Since the connection ω in P is trivial, we have $v'[X, Y] = [v'X, v'Y]$ for any two vector fields on P or on Q°. Let us consider some $\mathscr{P}_h = \omega[T_h(Q''_M)]$, $h \in Q''_M$, and two vectors $A, B \in \mathscr{P}_h \subset \mathscr{R}$.

According to Theorem 2 the form ω is regular on the bundle Q°, hence $\omega_q: T_q(Q^\circ) \to \mathfrak{g}$ is an *isomorphism* for each $q \in Q^\circ$. Consequently, there are uniquely determined vector fields X, Y on Q° such that $\omega(X) = A$, $\omega(Y) = B$. The group G acts on P to the right and $v'X$ (or $v'Y$) is a fundamental vector field corresponding to A (or B). The bundle Q'_M is a submanifold of Q° and $\omega[T_q(Q'_M)] = \mathscr{R}$ for each $q \in Q''_M$. Hence $X_q, Y_q \in T_q(Q'_M)$ for $q \in Q''_M$, and in particular, $X_h, Y_h \in T_h(Q''_M)$. Lemma 2 implies $[X, Y]_h \in T_h(Q'_M)$ and $\omega_h[X, Y] \in \mathscr{R}$. On the other hand

$$\omega_h([X, Y]) = \omega_h(v'[X, Y]) = \omega_h([v'X, v'Y]) = \\ = \omega_h[A^*, B^*] = [A, B].$$

Consequently, $[A, B] \in \mathscr{R}$, q.e.d.

Now we shall introduce some new concepts, which will be usefull in applications.

Let a Lie group G act to the left on two manifolds $\boldsymbol{X}$ and $\boldsymbol{Y}$ simultaneously.

Definition. *An equivariant object on* $\boldsymbol{Y}$ *with values in* $\boldsymbol{X}$ *(with respect to the group G) is a map* $\mathsf{O}: \boldsymbol{Y} \to \exp(\boldsymbol{X})$ *such that* $\mathsf{O}(g \cdot y) = g \cdot \mathsf{O}(y)$ *for each* $y \in \boldsymbol{Y}$, $g \in G$. *Here* $\exp(\boldsymbol{X})$ *denotes the set of all subsets of* $\boldsymbol{X}$.

Denote by $\mathsf{O}(\boldsymbol{Y})$ the set of all values of the object O; then the group G acts on $\mathsf{O}(\boldsymbol{Y})$. If G acts on $\mathsf{O}(\boldsymbol{Y})$ transitively and some of its isotropy groups is a Lie group $G^* \subset G$, then $\mathsf{O}(\boldsymbol{Y})$ can be made a manifold, diffeomorphic to the homogeneous space G/G^*.

Theorem 4. *Let us preserve all the denotations and the first assumption of Theorem* 1. *Assume that H' acts on a manifold* $\boldsymbol{X}$, *and let* O *be an equivariant object on* $\boldsymbol{Z}$ *with values in* $\boldsymbol{X}$ *w(ith respect to H'). Further suppose that a) H' acts transitively on* $\mathsf{O}(\boldsymbol{Z})$, *b) there is an element* $t \in \mathsf{O}(\boldsymbol{Z})$ *such that H^* is its isotropy group and* $\mathsf{O}^{-1}(t) = \boldsymbol{Z}^*$. *Then the bundle Q' admits exactly one local reduction Q^* to the group H^* such that* $\omega[T_b(Q)] \in \boldsymbol{Z}^*$ *for each* $b \in Q^*$.

Proof is the same as that of Corollary of Theorem 1. We only have to consider the manifold $\mathsf{O}(\boldsymbol{Z})$ instead of $\boldsymbol{Z}$. On the other hand, the Corollary can be obtained from here putting $\mathsf{O}: \boldsymbol{Z} \to \boldsymbol{Z} =$ the identity map.

Let G be a subgroup of the full affine group $GA(n)$ acting on the affine space A^n. By a *coordinate G-typ* in A^n we mean the set of all affine coordinate systems of the form $R^\circ \circ g$, where $R^\circ: A^n \to \mathbf{R}^n$ is a fixed coordinate system and $g \in G$. The corresponding coordinate G-typ will be denoted by $R^\circ \circ G$. Let us remark that G also acts on the complex extension CA^n of A^n and each coordinate G-typ in A^n determines some coordinate G-typ in CA^n. (See [2] for some more details). Now, let $\boldsymbol{X}$ be the affine space A^n or its complex extension CA^n, $H \subset G$ two subgroups in $GA^{(n)}$ and $\boldsymbol{Y} \subset \mathbf{Z}_k$ some H-invariant Grassmann manifold of subspaces of the Lie algebra $\mathfrak{g}$. Then H acts simultaneously on $\boldsymbol{X}$

and on $\mathbf{Y}$ (on the latter space as the adjoint group $\mathrm{Ad}(H)$). The following Criterion is a very special case of Theorem VI from [2] and its very simple proof will be omitted.

Criterion. *Let $R^\circ \circ H$ be a coordinate H-typ in A^n. Assume that to each coordinate systém $R^\alpha \in R^\circ \circ H$ there is assigned a global card S^α: $\mathbf{Y} \to \mathbf{R}^k$ ($k = \dim \mathbf{Y}$), the correspondence $R^\alpha \to S^\alpha$ being equivariant with respect to H. (It means, if $h \in H$, $R^\beta = R^\alpha \circ h$, we have $S^\beta = S^\alpha \circ h$). Let be given a map O from $\mathbf{Y}$ into* $\exp(A^n)$, *or* $\exp(CA^n)$. *Further assume that there are complex valued functions $F_i(x_1, \dots, x_n, \xi_1, \dots, \xi_k)$, $i = 1, 2, \dots, t$, defined on $\mathbf{R}^n \times \mathbf{R}^k$, or on $\mathbf{C}^n \times \mathbf{R}^k$, such that each set $\mathsf{O}(u)$, $u \in \mathbf{Y}$, is given by an equation system*

$$F_i(x_1, \dots, x_n, u_1, \dots, u_k) = 0, \ (i = 1\iota, 2, \dots, t),$$

with respect to each couple $\{R^\alpha, S^\alpha\}$ of mutually corresponding global cards, where $R^\alpha \in R^\circ \circ H$.

Then the map O is an equivariant object on $\mathbf{Y}$ with values in A^n, or in CA^n, with respect to the transformation group H.

PART II

In the classical differential geometry, if we want to specialize a frame bundle of a given surface to a smaller group, we can usually proceed as follows: at each point of the surface we select exactly those frames that are somehow related with some geometrical object at this point. For instance, we make some vectors of the frame to lie in the tangent space or in the asymptotic directions, and similarly, If we proceed in virtue of the Theorems 1—3, such a geometrical interpretation is not apparent at first sight. To make it apparent is the true purpose of this second Part. Roughly speaking, we are going to show that the same kind of geometrical objects can be joined with a point of the considered surface M and with the Grassmann manifolds representing the gradual steps of the specialization procedure.

There is only one difference here: at a point of the surface M we have to compare a variable frame with a fixed geometrical object. As for a Grassmann manifold, we have to compare its variable geometrical object with a fixed frame. So we have some kind of duality. We shall exhibit this idea in case that we are given a surface of a 3-dimensional equiaffine space.

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Let us consider the equiaffine space $A_e^{(3)}$ and a fixed coordinate frame $R^\circ = \langle o, \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$. (We shall denote by the same symbols the affine frames and the corresponding coordinate cards). Denote by G the equiaffine group $GA_e(3)$ and by H° the corresponding isotropy group of the point o. Let $\mathfrak{G}$, $\mathfrak{H}^\circ$ be, as usual, the Lie algebras of G, H°. Consider a surface M imbedded into the space A_e^3. The latter space will be allways identified with the homogeneous space G/H°. Now, let us consider the trivial bundle $P = M \times G$ with the natural Cartan connection ω and its reduction Q° to the group H°. We are to apply our new reduction procedure to the bundle Q°.

Note that only the case of a general hyperbolic surface M will be discussed completely; the other cases will be touched very briefly. The reader is given a possibility to compare our results with [4] and [5] for instance.

Put $\mathsf{R} = R^\circ H^\circ$. We shall join to each card $R^\alpha \in \mathsf{R}$, $R^\alpha = (x^1, x^2, x^3)$ a global card $\tilde{S}^\alpha$ of the algebra $\mathfrak{GA}(3)$ in such a way that $\tilde{S}^\alpha(X) = (s^i, s^i_j)$ iff $X = \sum_{i=1}^{3} S^i \frac{\partial}{\partial x^i} + \sum_{i,j=1}^{3} S^i_j x^j \frac{\partial}{\partial x^i}$. Further, we shall join to each $\tilde{S}^\alpha$ a card S^α in $\mathfrak{G}$ by restricting $\tilde{S}^\alpha$ to the subspace $\mathfrak{G} \subset \mathfrak{GA}(3)$. It is well-known that $X \in \mathfrak{G}$ if and only if $S^1_1 + S^2_2 + S^3_3 = 0$ in any card $\tilde{S}^\alpha$. *The assignement $R^\alpha \to S^\alpha$ is equivariant.*

Conventions:

a) We shall allways omit the index α at coordinates and we shall also write x, y, z instead of x^1, x^2, x^3.

b) In the following, the *round brackets* will designate a linear space spanned by some linear subspaces, or vectors.

c) The elements of any Grassmann manifold in question will be called briefly *blocks*.

Now, for any card $R^\alpha \in \mathsf{R}$, we have $\mathfrak{G} = \left(\mathfrak{H}^\circ, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Because $\dim Q^\circ = \dim H^\circ + \dim M = 10$, the values $\omega[T_q(Q^\circ)]$, $q \in Q^\circ$, belong to the manifold $\boldsymbol{Z}$ of all 10-dimensional subspaces of $\mathfrak{G}$ comprising $\mathfrak{H}^\circ$. (Theorem 2.) Any block $\mathscr{P} \in \boldsymbol{Z}$ is given by a relation

$$\alpha_1 S^1 + \alpha_2 S^2 + \alpha_3 S^3 = 0 \tag{1}$$

besides the usual condition $\sum_{i=1}^{3} S^i_i = 0$, hence $\dim \boldsymbol{Z} = 2$.

Every $\mathscr{P}$ comprises a 2-dimensional subspace $\mathfrak{T}_{\mathscr{P}}$ of $\mathfrak{T} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. The directions of the infinitesimal translations of $\mathfrak{T}_{\mathscr{P}}$ fill out an improper line $C^\infty(\mathscr{P})$ of A_e^3. Let us join to each $\mathscr{P} \in \boldsymbol{Z}$ a plane $\tau(\mathscr{P})$ determined by the origin o and by the improper line $C^\infty(\mathscr{P})$. We shall call $\tau(\mathscr{P})$ *the tangent plane joined to the block* $\mathscr{P}$. We can see that

$$\tau(\mathscr{P}) \equiv \alpha_1 x + \alpha_2 y + \alpha_3 z = 0 \tag{2}$$

with respect to each $R^\alpha \in \mathsf{R}$.

The map $\mathscr{P} \to \tau(\mathscr{P})$ is one-to-one and it is an equivariant object on $\boldsymbol{Z}$ with values in A^3_e with respect to the group H° (See: [2], Theorem VII). The manifold $\tau(\boldsymbol{Z})$ is clearly an orbit with respect to H° and there is exactly one block $\mathscr{P}^1 \in \boldsymbol{Z}$ such that $\tau(\mathscr{P}^1) = (\mathrm{o}, \vec{e}_1, \vec{e}_2)$. According to Theorem 4 there is a local reduction Q^1 of the bundle Q° to the isotropy group $H^1 \subset H^\circ$ of $\tau(\mathscr{P}^1)$ such that $\omega[T_q(Q)] = \mathscr{P}^1$ for all $q \in Q^1$. The elements of the bundle Q^1 will be called *the tangent frames of the surface* M. We shall restrict M if necessary so that the new bundle Q^1 may have M for its base again. Put $\mathsf{R}^1 = R^\circ \circ H^1$. With respect to each couple of corresponding cards $\{R^\alpha \in \mathsf{R}^1, S^\alpha\}$ the plane $\tau(\mathscr{P}^1)$ is given by $z = 0$ and consequently, the block $\mathscr{P}^1$ is given by $S^3 = 0$ [we use (1), (2)].

As for the Lie algebra $\mathfrak{H}^1$ of the group H^1, we have

$$\mathfrak{H}^1 = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},\ x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ z\frac{\partial}{\partial y}\right)$$

in any coordinate system $R^\alpha \in \mathsf{R}^1$.

The values $\mathscr{P} = \omega[T_q(Q^1)]$ for $q \in Q^1$ belong to the manifold $\tilde{\boldsymbol{Z}}^1$ of all 8-dimensional subspaces of $\mathscr{P}^1$ such that $\mathscr{P} \cap \mathfrak{H}^\circ = \mathfrak{H}^1$. Any $\mathscr{P} \in \tilde{\boldsymbol{Z}}^1$ is of the form $\mathscr{P} = (\mathfrak{H}^1, U_1, U_2)$, where $U_1, U_2 \in \mathscr{P}^1$ are linearly independent vectors over $\mathfrak{H}^0$ (Theorem 2). With respect to any card $R^\alpha \in \mathsf{R}^1$ we can put $U_1 = U_1^\alpha$, $U_2 = U_2^\alpha$, where

$$U_1^\alpha = \frac{\partial}{\partial x} + \alpha_1 x\frac{\partial}{\partial z} + \beta_1 y\frac{\partial}{\partial z}$$

$$U_2^\alpha = \frac{\partial}{\partial y} + \alpha_2 x\frac{\partial}{\partial z} + \beta_2 y\frac{\partial}{\partial z}. \tag{3}$$

The map $\mathscr{P} \to (\alpha_1, \beta_1, \alpha_2, \beta_2)$ is a global card $\tilde{C}_1^\alpha$ on $\tilde{\boldsymbol{Z}}^1$ and moreover, the correspondence $R^\alpha \to \tilde{C}_1^\alpha$ is equivariant. $\tilde{\boldsymbol{Z}}^1$ can be made into a linear space .According to Theorem 3 we have $[\mathscr{P}, \mathscr{P}] \subset \mathscr{P}^1$ for each „ω-value" $\mathscr{P}$ on the bundle Q^1. This requirement is equivalent with $[U_1^\alpha, U_2^\alpha] \in \mathscr{P}^1$ and hence $\alpha_2 = \beta_1$. Consequently the blocks $\omega[T_q(Q^1)]$, $q \in Q^1$, belong to a 3-dimensional linear subspace $\boldsymbol{Z}^1$ of $\tilde{\boldsymbol{Z}}^1$, whose global cards are of the form C_1^α: $\mathscr{P} \to (\alpha_1, \alpha_2, \beta_2)$. Every $\mathscr{P} \in \boldsymbol{Z}^1$ is given by an equation system of the form

$$\begin{aligned} &S^3 = 0, &&S_1^3 - \alpha_1 S^1 - \alpha_2 S^2 = 0,\\ &S_1^1 + S_2^2 + S_3^3 = 0, &&S_2^3 - \alpha_2 S^1 - \beta_2 S^2 = 0. \end{aligned} \tag{4}$$

In the following, if a Lie transformation group G preserves some set M, we shall say that its *Lie algebra* $\mathfrak{G}$ *preserves* M. If, in particular, M is a point, it will be called *a singularity of* $\mathfrak{G}$.

The algebra $\mathfrak{H}^1$ preserves the plane $\tau(\mathscr{P}^1) = (o, \vec{e}_1, \vec{e}_2)$. Denote by $\mathfrak{G}_1 \subset \mathfrak{H}^1$ the greatest subalgebra preserving each of the parallel planes, too. Clearly

$$\mathfrak{G}^1 = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \; x\frac{\partial}{\partial y}, \; y\frac{\partial}{\partial x}, \; z\frac{\partial}{\partial x}, \; z\frac{\partial}{\partial y}\right).$$

For $X \in \mathfrak{G}^1$ let us denote by hX the restriction of the vector field X to the plane $\tau(\mathscr{P}^1) = A^2_e$. The map h is a Lie algebra homomorphism h: $\mathfrak{G}^1 \to \mathfrak{G}\mathfrak{A}_e(2)$. Given $\mathscr{P} \in \boldsymbol{Z}^1$ we denote by $\xi(\mathscr{P})$ the subspace of all vectors $Y \in \mathfrak{G}\mathfrak{A}_e(2)$ such that $[h^{-1}Y, \mathscr{P}] \subset \mathscr{P}$. Put

$$Y = a\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + bx\frac{\partial}{\partial y} + cy\frac{\partial}{\partial x}$$

with respect to any card $R^\alpha \in \mathsf{R}^1$. Then we obtain the following equations determining $\xi(\mathscr{P})$:

$$\begin{aligned} \beta_2 b + \alpha_1 c &= 0 \\ -\beta_2 a \qquad\quad + \alpha_2 c &= 0 \\ \alpha_1 a + \alpha_2 b \qquad\quad &= 0 \end{aligned} \tag{5}$$

We can see easily that the system (5) is of rank 2 if at least one of the numbers $\alpha_1, \alpha_2, \beta_2$ is non-zero; otherwise it is of rank 0. We have an invariant decomposition according to the rank of (5): $\boldsymbol{Z}^1 = \boldsymbol{Z}^1_2 \cup \boldsymbol{Z}^1_0$. $\boldsymbol{Z}^1_0$ is an invariant point of $\boldsymbol{Z}^1$ given by

$$S^3 = 0, \qquad S^1_1 + S^2_2 + S^3_3 = 0, \qquad S^3_1 = 0, \qquad S^3_2 = 0. \tag{6}$$

It will be called *the planar block* of $\boldsymbol{Z}^1$. The surface M is called *planar* if all ω-values on the bundle Q^1 are equal to $\boldsymbol{Z}^1_0$. We receive the usual conditions for planar surfaces in A^3_e if we replace S^i, S^i_j by ω^i, ω^i_j in the relations (6).

In case that $\mathscr{P} \in \boldsymbol{Z}^1_2$ the subspace $\xi(\mathscr{P})$ is of dimension 1, $\xi(\mathscr{P}) = \left(\alpha_2\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - \alpha_1 x\frac{\partial}{\partial y} + \beta_2 y\frac{\partial}{\partial x}\right)$. Put $D = \begin{vmatrix}\alpha_1 & \alpha_2 \\ \alpha_2 & \beta_2\end{vmatrix}$. According to [6] we have the following possibilities:

a) $D > 0$, then the corresponding 1-dimensional group $G[\xi]$ is a group of elliptic rotations around the origin in A^2_e.

b) $D < 0$, then $G[\xi]$ is a group of hyperbolic rotations around the origin in A^2_e.

c) $D = 0$, then $G[\xi]$ is a group of shear transformations with a directional line passing through the origin.

In any case, we shall call the set of all trajectories with respect to the group $G[\xi]$ *the indicatrix joined* to $\mathscr{P}$. In the case *a)* or *b)* the curves of the indicatrix have common asymptotics which are imaginary conjugate or real and different. In both cases we have the equation

$$(7) \qquad k(\mathscr{P}) = \alpha_1 x^2 + 2\alpha_2 xy + \beta_2 y^2 = 0$$

for the corresponding pair of asymptotics.

As for the case *c)*, the equation (7) expresses the double directional line of $G[\xi]$.

According to Criterion. $k(\mathscr{P})$ is an equivariant object on $\boldsymbol{Z}_2^1$ with values in CA_e^2 with respect to the group H^1. The lines of $k(\mathscr{P})$ will be called *the asymptotic directions joined to the block* $\mathscr{P}$. A block $\mathscr{P} \in \boldsymbol{Z}_2^1$ will be called *elliptic* or *hyperbolic* or *parabolic* according to the figure of the indicatrix of $\mathscr{P}$. It is clear that all ω-values along a fibre of Q^1 are of the same type. We have an invariant decomposition $\boldsymbol{Z}_2^1 = \boldsymbol{Z}_e^1 \cup \cup \boldsymbol{Z}_h^1 \cup \boldsymbol{Z}_p^1$. A point $x \in M$ will be called *elliptic* or *hyperbolic* or *parabolic* if any ω-value $\mathscr{P} = \omega[T_b(Q^1)]$, $p(b) = x$, is elliptic or hyperbolic or parabolic. *Assume M to be composed of hyperbolic points only.* Then the manifold $k(\boldsymbol{Z}_h^1)$ is formed by all pairs of real mutually different lines of A_e^2 crossing at the origin. The group H^1 acts transitively on $k(\boldsymbol{Z}_h^1)$. Denote $\boldsymbol{Z}^2$ the submanifold of $\boldsymbol{Z}_h^1$ consisting of all blocks $\mathscr{P}$ such that the corresponding asymptotic directions coincide with the lines e_1, e_2 of the frame R^0. Let H^2 be the maximal subgroup of H^1 preserving the set $\{e_1\} \cup \{e_2\}$. According to Theorem 4 there is a local reduction Q^2 of the bundle Q^1 to the group H^2 such that $\omega[T_q(Q^1)] \in \boldsymbol{Z}^2$ for any $q \in Q^2$. The elements of Q^2 will be called *the asymptotic frames of M.*

Let us restrict M if necessary so that M is the base for Q^2 again. We can see easily that for any $\mathscr{P} \in \boldsymbol{Z}^2$ and for any card $R^\alpha \in \mathsf{R}^2 = R^0 \circ H^2$ we have $k(\mathscr{P}) \equiv xy = 0$. Hence $\beta_2 = \alpha_1 = 0$ and from (4) follows

$$(8) \qquad \mathscr{P} \equiv \begin{cases} S_1^3 - \alpha_2 S^2 = 0, & S^3 = 0, \\ S_2^3 - \alpha_2 S^1 = 0, & S_1^1 + S_2^2 + S_3^3 = 0, \end{cases} \quad \alpha_2 \neq 0.$$

Now, in the card R^0, it is easy to verify that the subgroup $G^* \subset H^2$ of all matrizes of the form $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \pm \dfrac{1}{\lambda^2} \end{pmatrix}$, $\lambda > 0$, acts transitively on $\boldsymbol{Z}^2$.

Denote by $\mathscr{P}^3$ the block of $\boldsymbol{Z}^2$ such that $\alpha_2 = 1$ in the coordinate system R^0. Let H^3 be the isotropy group of $\mathscr{P}^3$. Then there is a local reduction

Q^3 of the bundle Q^2 to the group H^3 such that $\omega[T_q(Q^1)] = \mathscr{P}^3$ for each $q \in Q^3$. In any card $R^\alpha \in \mathsf{R}^3 = R^0 \circ H^3$ we have

$$(9) \qquad \mathscr{P}^3 \equiv \begin{cases} S_1^3 - S^2 = 0, & S^3 = 0, \\ S_2^3 - S^1 = 0, & S_1^1 + S_2^2 + S_3^3 = 0. \end{cases}$$

The elements of Q^3 will be called *the normalised asymptotic frames of* M. H^3 consists of 8 connected components; its Lie algebra $\mathfrak{H}^3$ is given by the condition $[\mathfrak{H}^3, \mathscr{P}^3] \subset \mathscr{P}^3$. Hence we obtain $\mathfrak{H}^3 = \left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ with respect to any card $R^\alpha \in \mathsf{R}^3$. According to (8) we obtain

$$\mathscr{P}^3 = \left(\mathfrak{H}^3, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \right.$$
$$\left. \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right).$$

Let us restrict M again if necessary. Let $\mathscr{P}$ be a ω-value on Q^3, i.e. $\mathscr{P} = \omega[T_q(Q_3)]$, $q \in Q^3$. Then $\mathfrak{H}^3 \subset \mathscr{P} \subset \mathscr{P}^3$ and according to Theorem 2. $\mathscr{P} = (\mathfrak{H}^3, V_1, V_2)$, where V_1, V_2 are two linearly independent vectors over $\mathfrak{H}^0$. With respect to any card $R^\alpha \in \mathsf{R}^3$, we can assume that V_1 and V_2 are of the form

$$V_1^\alpha = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + \alpha_1\left(x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right) + \alpha_2 x\frac{\partial}{\partial y} + \alpha_3 y\frac{\partial}{\partial x}$$

$$V_2^\alpha = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + \beta_1\left(x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right) + \beta_2 x\frac{\partial}{\partial y} + \beta_3 y\frac{\partial}{\partial x}.$$

According to Theorem 3 $[\mathscr{P}, \mathscr{P}] \subset \mathscr{P}^3$, i.e. $[V_1^\alpha, V_2^\alpha] \subset \mathscr{P}^3$ and hence $\alpha_1 = \beta_2$, $\beta_1 = \alpha_3$. Consequently; in any coordinate system $R^\alpha \in \mathsf{R}^3$

$$(10) \quad \mathscr{P} \equiv \begin{cases} S^3 = 0, \quad S^1 - S_2^3 = 0, \quad S^2 - S_1^3 = 0, \quad S_1^1 + S_2^2 + S_3^3 = 0 \\ S_3^3 + \beta_2 S^1 + \alpha_3 S^2 = 0 \\ S_1^2 - \alpha_2 S^1 - \beta_2 S^2 = 0 \\ S_2^1 - \alpha_3 S^1 - \beta_3 S^2 = 0. \end{cases}$$

The map $\mathscr{P} \to (\alpha_2, \beta_2, \alpha_3, \beta_3)$ is a global card C_3^α on the manifold $\boldsymbol{Z}^3 = \{\mathscr{P} \subset \mathscr{P}^3 \mid \dim \mathscr{P} = 5, \mathscr{P} \cap \mathfrak{H}^0 = \mathfrak{H}^3, [\mathscr{P}, \mathscr{P}] \subset \mathscr{P}^3\}$. We have $\boldsymbol{R^0 \circ h} \to \to C_3^0 \circ \boldsymbol{h}$ for any $\boldsymbol{h} \in H^3$ (the requirement of the Criterion). Denote $\mathfrak{G}_3 \supset \mathfrak{H}^3$ the isotropy subalgebra of the block $\mathscr{P}^3$ with respect to the action of *the full affine group* $GA(3)$ on $\boldsymbol{Z}^3$. The we obtain easily

$$\mathfrak{G}^3 = \left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

Denote by $\mathcal{Q}_1$ or $\mathcal{Q}_2$ the subspace of all infinitesimal transformations from $\mathcal{P}$ preserving the axis e_1 or e_2. We want to find out the subspaces $\eta_1(\mathcal{P}) \subset \mathfrak{G}^3$, $\eta_2(\mathcal{P}) \subset \mathfrak{G}^3$ such that $[n_1(\mathcal{P}), \mathcal{Q}_1] \subset \mathcal{Q}_1$, $[\eta_2(\mathcal{P}), \mathcal{Q}_2] \subset \mathcal{Q}_2$. It is immediate that $\{\mathcal{Q}_1, \mathcal{Q}_2\} = \{(\mathfrak{H}^3, V_1), (\mathfrak{H}^3, V_2)\}$ exact up to the order. That last depends on the connected component of R^3 including the card R^α. Let be $X \in \mathfrak{G}^3$ and put

$$X = az\frac{\partial}{\partial x} + bz\frac{\partial}{\partial y} + c\left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right) + d\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right).$$

After an easy computation we state that $\eta_1(\mathcal{P})$, $\eta_2(\mathcal{P})$ are determined (exact up to the order) by the equation systems ;

$$(11) \qquad \begin{aligned} &(a)\ \alpha_3 d - a = 0, \quad c\beta_2 - b = 0, \quad \beta_3(2d - c) = 0 \\ &(b)\ \alpha_3 d - a = 0, \quad c\beta_2 - b = 0, \quad \alpha_2(2c - d) = 0. \end{aligned}$$

The rank of each of these systems is 3 or 2. We have an invariant decomposition

$$(12) \qquad \boldsymbol{Z}^3 = \boldsymbol{Z}^3_0 \cup \boldsymbol{Z}^3_1 \cup \boldsymbol{Z}^3_2.$$

Here we put $\mathcal{P} \in \boldsymbol{Z}^3_0$ if (a) and (b) are both of rank 3, $\mathcal{P} \in \boldsymbol{Z}^3_2$ if these are both of rank 2, and $\mathcal{P} \in \boldsymbol{Z}^3_1$ otherwise. The subspace $\mathcal{L} = (\eta_1(\mathcal{P}), \eta_2(\mathcal{P})) \subset \mathfrak{G}^3$ is given by the system

$$(13) \qquad \alpha_3 d - a = 0, \quad \beta_2 c - d = 0$$

for any $\mathcal{P} \in \boldsymbol{Z}^3$ and any $R^\alpha \in \mathsf{R}^3$. Hence $X \in \mathcal{L}$ if and only if

$$X = c\left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + \beta_2 z\frac{\partial}{\partial y}\right) + d\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \alpha_3 z\frac{\partial}{\partial x}\right).$$

The singularities of X in the space A^3_e are solutions of the system

$$(14) \qquad cx + \alpha_3\, dz = 0, \quad dy + \beta_2 cz = 0, \quad (c + d)\, z = 0.$$

A vector $X \in L$ admits a *line of singularities* if and only if $c + d = 0$, $c \neq 0$. That line is given by

$$(15) \qquad \varkappa(\mathcal{P}): \quad x - \alpha_3 z = 0, \quad y - \beta_2 z = 0$$

and it will be called *the affine normal joined to the block* $\mathcal{P} \in \boldsymbol{Z}^3$. The map $\varkappa$ is an equivariant object on $\boldsymbol{Z}^3$. The manifold $\varkappa(\boldsymbol{Z}^3)$ is the set of all lines of A^3_e passing through the origin and not belonging to the plane A^2_e. The group H^3 acts transitively on it.

Let $\boldsymbol{Z}^4$ be the submanifold of all blocks $\mathcal{P} \in \boldsymbol{Z}^3$ such that the affine normal $\varkappa(\mathscr{P})$ coincides with the axis e_3 of the frame R^0. We obtain a decomposition

$$(16) \qquad \boldsymbol{Z}^4 = \boldsymbol{Z}^4_0 \cup \boldsymbol{Z}^4_1 \cup \boldsymbol{Z}^4_2$$

similarly to (12).

According to Theorem 4 there is a local reduction Q^4 of the bundle Q^3 to the group H^4, the isotropy group of the line e_3 with respect to H^3, such that $\omega[T_b(Q^3)] \in \boldsymbol{Z}^4$ for $b \in Q^4$. The elements of the bundle Q^4 will be called *the Darboux frames of the surface M*. For $\mathscr{P} \in \boldsymbol{Z}^4$ we have $\varkappa(\mathscr{P})$: $x = 0$, $y = 0$ with respect to any card $R^\alpha \in \mathsf{R}^4 = R^0 \circ H^4$; hence $\alpha_3 = 0$, $\beta_2 = 0$.

Let us assume in the following that M is the base for Q^4 and that all ω-values $\omega[T_b(Q^3)]$, $b \in Q^4$, belong to $\boldsymbol{Z}^4_0$. Such a surface M is called *general. Given a general hyperbolic surface M and a* frame $b \in Q^4$, we obtain the following expression for the block $\mathscr{P} = \omega[T_b(Q^3)]$:

$$
(17) \quad
\begin{aligned}
&S^3 = 0, \quad S^1_1 + S^2_2 + S^3_3 = 0, \quad S^3_1 - S^2 = 0, \quad S^3_2 - S^1 = 0\\
&S^3_3 = 0\\
&S^2_1 - \alpha_2 S^1 = 0, \quad S^1_2 - \beta_3 S^2 = 0, \quad \alpha_2\beta_3 \neq 0.
\end{aligned}
$$

Or else, the block $\mathscr{P}$ is spanned by the subalgebra $\mathfrak{H}^3$ and by two vectors of the form

$$V^\alpha_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + \alpha_2 x\frac{\partial}{\partial y}, \quad V^\alpha_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + \beta_3 y\frac{\partial}{\partial x}, \alpha_2\beta_3 \neq 0.$$

We can see easily that the subspace $\mathscr{L}' = (V^\alpha_1, V^\alpha_2)$ does not depend on the coordinate system $R^\alpha \in \mathsf{R}^4$. Now we shall find out all the vectors $X \in \mathscr{L}'$ admitting a singularity in A^3_e. Put $X = cV^\alpha_1 + dV^\alpha_2$, then X has a singularity if and only if the following equation system is solvable:

$$(18) \qquad c + \beta_3 \mathrm{d}y = 0, \; cy + \mathrm{d}x = 0, \; \mathrm{d} + c\alpha_2 x = 0.$$

It requires that $c/d = -\sqrt[3]{\beta_3/\alpha_2}$, and the wanted singularity is given by

$$(19) \qquad R(\mathscr{P}) = [\bar{x}, \bar{y}], \; \bar{x} = -\sqrt[3]{1/\beta_3(\alpha_2)^2}, \; y = -\sqrt[3]{1/\alpha_2(\beta_3)^2}$$

$R(\mathscr{P})$ is an equivariant object on $\boldsymbol{Z}^4_0$ with values in A^2_e. (See [5], pp. 29,45 for the classical meaning of this point). The ray $s(\mathscr{P}) = \overrightarrow{(0, R(\mathscr{P}))}$ will be called *the real direction of Segre joined to* $\mathscr{P}$. (Cfr. [5], p. 45]. The manifold $s(Z^4_0)$ consists of all rays passing out of the origin in the plane A^2_e and not belonging to e_1 or e_2. It is composed of 4 connected components. Now $\mathfrak{H}^4 = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$ and H^4 consists of 8 connected components.

We can see easily that H^4 acts transitively on $s(\boldsymbol{Z}_0^4)$. Let $\boldsymbol{Z}^5 \subset \boldsymbol{Z}_0^4$ be the submanifold defined as follows: $\mathscr{P} \in \boldsymbol{Z}^5$ if and only if the real direction of Segre $s(\mathscr{P})$ coincides with the ray $\lambda(\vec{e}_1 + \vec{e}_2)$, $\lambda < 0$. There is a local reduction Q^5 of the bundle Q^4 to the group $\{e, e^*\}$, where e is the identity transformation and e^* is the reflection of the frame R^0 which permutes $\vec{e}_1$, $\vec{e}_2$. Moreover, we have $\omega[T_b(Q^4)] \in \boldsymbol{Z}^5$ for any $b \in Q^5$. The bundle Q^5 consists of two sections of the bundle Q^0, which are called *the canonical sections of Darboux*. These sections determine two opposite orientations of the surface M.

Let $\mathscr{P} = \omega[T_b(Q^4)]$, $b \in Q^5$. We can see easily that $\bar{x} = \bar{y} < 0$ and hence $\alpha_2 = \beta_3 < 0$. The number $K = \alpha_2 = \beta_3$ is called the *affine curvature* of $\mathscr{P}$. The space $\mathscr{P}$ is given by

$$S^3 = 0,\ S_1^1 + S_2^2 = 0,\ S_1^3 - S^2 = 0,\ S_2^3 - S^1 = 0$$

$$S_3^3 = 0,\ S_1^2 - KS^1 = 0,\ S_2^1 - KS^2 = 0,\ K \neq 0 \tag{20}$$

in both coordinate systems R^0 and $R^0 \,.\, e^*$. Our specialization procedure is finished. If we replace S_i, S_i^j by ω_i, ω_i^j in (20), we obtain a well-known canonical equation system for general hyperbolic surfaces in A_e^3 (See [4]).

Note. Similarly we can learn the case that all ω-values $\omega[T_b(Q^3)]$, $b \in Q^4$, belong to $\boldsymbol{Z}_1^4$ or to $\boldsymbol{Z}_2^4$. [See (16)]. As usual, we obtain two classes of non-developable ruled surfaces. The second class is formed by hyperboloids of one sheet.

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