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NOTES ON GAME THEORY EQUILIBRIA

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One approximation theorem on simplicial inclusive multivalued transformation, two versions of Brouwer's fixed-point theorem, following of B. Peleg's result [9] and L. S. Shapley's one [11] the independence of Nash equilibrium of polyhedral cones preferences, but dependence of stability in cooperative games and certain computational remark, are settled in § 1. § 2 follows L. S. Shapley's [10] results about non-existence of saddlepoints of special matrices and partially studies a structure of A's submatrices with saddlepoints if A has no such point.

First a word about denotations: a point $x \in \mathrm{E}^n$ is an n by 1 matrix (i.e. a column), ${}^TA$ means a transpose of A (i.e. $x, y \in \mathrm{E}^n$, ${}^Txy$ is an inner product of x and y), A_S or A^L means a submatrix of an m by n matrix A, indices of its columns or rows form the set $S \subset N = = : \{1, 2, \ldots, n\}$ or $L \subset M = : \{1, 2, \ldots, m\}$ respectively, $A_{\vartheta(S)} = = : A_{N-S}$, $A^{\vartheta(L)} = : A^{M-L}$ (i.e. $A = A_N^M$), for $X \subset \mathrm{E}^n$ CX is the convex hull of X, $A \leqq B$ means $a_{ij} \leqq b_{ij}$ for all i, j and $A \lessdot B$ means $A \leqq B$ but not $A = B$.

§ 1

By S_n one denotes an n-dimensional simplex in Euclidean space E^n, $\mathscr{C}(S_n)$ the set of all its nonvoid convex subsets and $\mathscr{S}(S_n)$ the set of all its nonvoid sides (i.e. all its vertices, edges, ..., $(n — 1)$-sides and S_n itself). A simplicial partition $\mathfrak{S}$ of S_n is such its partition on n-dimensional simplices that any two Δ's from $\mathfrak{S}$ are either disjoint or have only one side (of any dimension) in common. A point-set transformation Φ of S_k into $\mathscr{S}(S_l)$ is called *simplicial inclusive* according to $\mathfrak{S}$ if $\mathfrak{S}$ is a simplicial partition of S_k, any two points have the same transform if they belong to the interior of the same side of $\Delta \in \mathfrak{S}$ and have their transforms in the inclusive relation if the sides of $\Delta \in \mathfrak{S}$ to the interiors of which they belong are in the inclusive relation (not necessarily in the same sense; the interior of 0-side is the vertex itself). Evidently Φ is simplicial inclusive according to any $\mathfrak{S}'$ which is a refinement of $\mathfrak{S}$.

$\mathfrak{S}$ is called *primitive* if for any $\Delta \in \mathfrak{S}$ the set of images of all Δ's vertices forms the inclusive chain (i.e. any two transforms are in inclusive relation). Without loss of generality one can suppose Φ has primitive $\mathfrak{S}$.

(If $\mathfrak{S}$ is not primitive, choose for every $\Delta \in \mathfrak{S}$ its interior point and construct convex hulls of it with Δ's $(k-1)$-sides. The union of all such $k+1$ simplices forms the siplicial division $\mathfrak{S}^{(1)}$ of S_k. For every $\Delta^{(1)} \in \mathfrak{S}^{(1)}$ all points have the same transform except those which belong to certain "distinguished" $(k-1)$-side. Deviding $\Delta^{(1)}$ into k simplices (by means of a similar operation with distinguished $(k-1)$-side) one obtains $\mathfrak{S}^{(2)}$ etc. Evidently $\Delta^{(k)} \in \mathfrak{S}^{(k)}$ has for all its points (except a distinguished $(k-1)$-side) the same image, all points of the distinguished $(k-1)$-side have (except of a distinguished $(k-2)$-side) the same image, ..., all points of the distinguished edge have (except of a distinguished vertex) the same transform. Denote ${}^0x$, ${}^1x, \ldots, {}^kx$ the vertices of $\Delta^{(k)}$ in such a way that ${}^0x, {}^1x, \ldots, {}^sx$ $(0 \leqq s < k)$ is the distinguished s-side of $\Delta^{(k)}$ and let ${}^ix$, ${}^jx$, $i < j$ be any two vertices. Choose ${}^iy$, ${}^jy$ arbitrarily in the interiors of sides $C({}^0x, \ldots, {}^ix)$, $C({}^0x, \ldots, {}^jx)$. As the first one is a subset of the second and ${}^iy_\Phi = {}^ix_\Phi$, ${}^jy_\Phi = {}^jx_\Phi$ it must be either ${}^ix_\Phi \subset {}^jx_\Phi$ or ${}^ix_\Phi \supset {}^jx_\Phi$. Hence $\mathfrak{S}^{(k)}$ is primitive.)

We call a point-set transformation F continuous if F transforms S_k into $\mathscr{C}(S_l)$ and if $y \in x_F$ when ${}^nx \to x$, ${}^ny \to y$, ${}^ny \in {}^nx_F$, ${}^nx$, $x \in S_k$, where the convergence is in the sense of the usual metric topology (see [4]).

Remark 1. Let F or P be a continuous transformation of S_k into $\mathscr{C}(S_l)$ or S_l into $\mathscr{C}(S_k)$. Then F and P have a coincidence (i.e. $x \in S_k$, $y \in S_l$ exist such that $y \in x_F$, $x \in y_P$). Proof: The transformation R of cartesian product $S_k \otimes S_l$ into $\mathscr{C}(S_k \otimes S_l) : (x, y) \to y_P \otimes x_F$ is evidently continuous and hence a fixed point exists $(\bar{x}, \bar{y}) \in (\bar{x}, \bar{y})_R = \bar{y}_P \otimes \bar{x}_F$ (see [4]). Hence $\bar{x} \in \bar{y}_P$, $\bar{y} \in \bar{x}_F$; Q.E.D.

Theorem 1. *Let Φ be a simplicial inclusive point-set transformation of S_k into $\mathscr{S}(S_l)$ according to primitive $\mathfrak{S}$. Then a continuous transformation F of S_k into $\mathscr{C}(S_l)$ exists so that $\Phi = F$ on the vertices of $\mathfrak{S}$.*

Proof: Let ${}^0x, {}^1x, \ldots, {}^kx$ be the vertices of $\Delta \in \sigma$. Let among $\{{}^ix_\Phi\}_{i=0}^k$ be r different ones $(1 \leqq r \leqq k+1)$. Without loss of generality one can suppose the existence of a sequence $\{i_s\}_{s=0}^r$, $-1 = i_0 < i_1 < i_2 < \ldots < i_r = k$ of integers such that $1 \leqq s' < s \leqq r$, $i_{s'-1} < i \leqq i_{s'}$, $i_{s-1} < j \leqq i_s$ implies ${}^ix_\Phi \subset {}^jx_\Phi$ and ${}^ix_\Phi = {}^jx_\Phi$ holds only if $s' = s$ (it follows from the primitivity of $\mathfrak{S}$ immediately). Construct x_F for arbitrary $x \in \Delta$, $x = \sum_{i=0}^{k} \lambda_i {}^ix$ as follows

$$x_F = \{y : y = \sum_{j=0}^{t_r} \nu_j {}^jy, \quad \sum_{j=0}^{t_r} \nu_j = 1, \quad \nu_j \geqq 0, \tag{1}$$

$$\sum_{j=0}^{t_1} \nu_j \geqq \mu_1, \quad \sum_{j=0}^{t_2} \nu_j \geqq \mu_1 + \mu_2, \quad \ldots, \quad \sum_{j=0}^{t_r} \nu_j \geqq \sum_{s=1}^{r} \mu_s\}$$

where ${}^0y, {}^1y, \ldots, {}^{t_s}y$ are vertices of ${}^{i_s}x_\Phi$ $(s = 1,2, \ldots, r)$ and $\mu_s =: \sum_{i_{s-1} < i \leqq i_s} \lambda_i$.

The last inequality is superfluous for $\sum_{s=1}^{r} \mu_s = 1$. Evidently x_F is a convex polyhedron. Even it is $x_F \neq \emptyset$ (For $0 \leqq t_1 < t_2 < \ldots < t_r$ it suffices $\mu_s (1 \leqq s \leqq r)$ to explain as a sum of $t_s - t_{s-1} (\geqq 1)$ non-negative numbers ($t_0 = : -1$). In this case in (1) equalities only hold.). We have ${}^i x_F = {}^i x_\Phi$ (For $x = {}^i x\ \lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$. Hence for $i_{s-1} < i \leqq i_s$ it is $\mu_s = 1$ and $\mu_{s'} = 0$ for $s' \neq s$. Thus $\sum_{j=0}^{t_s} \nu_j = 1$ and it results $\nu_j = 0$ for $j > t_s$. Since $\sum_{j=0}^{t_s'} \nu_j \geqq \sum_{l=1}^{s'} \mu_l$ with $s' < s$ is in (1) superfluous, we have $x_F = C({}^\circ y, \ldots, {}^{t_s} y)$, q.e.d.).

(2) x_F depends only on $\{{}^i x_\Phi\}_{{}^i x \in \Delta_v}$ by the rule (1) if x lies in the interior of v-dimmensional side Δ_v of Δ. (Let x_F depend on $\{{}^i x_\Phi\}_{{}^i x \in \Delta_z}$ by the rule (1) and suppose $x \in \Delta_{z-1} = C({}^{j_0} x, \ldots {}^{j_{u-1}} x, {}^{j_{u+1}} x, \ldots, {}^{j_z} x)$, (not necessarily in its interior), where ${}^{j_u} x \in \Delta_z$, $i_{s-1} < j_u \leqq i_s$. We have finished in the case $i_{s-1} < i_s - 1$ because there exists $j \neq j_u$, ${}^j x \in \Delta_z$ such that ${}^j x_\Phi = {}^{j_u} x_\Phi$ and hence all inequalities in (1) remain. Thus let $i_{s-1} + 1 = i_s = j_u$. Then ${}^{j_u} x_\Phi \notin \{{}^i x_\Phi\}_{{}^i x \in \Delta_{z-1}}$ and hence $\mu_s = 0$. It results $\sum_{=0}^{t_s} \nu_j \geqq \mu_1 + \ldots + \mu_s$ is superfluous and it follows x_F depends on $\{{}^i x_\Phi\}_{{}^i x \in \Delta_{z-1}}$ by the rule (1).).

Hence F defined according to (1) on Δ and on Δ', $\Delta, \Delta' \in \mathfrak{S}$ is the same on $\Delta \cap \Delta'$. Thus F transforms S_k into $\mathscr{C}(S_l)$. F is continuous (Let ${}^n u \to u\ {}^n v \to v\ {}^n v \in {}^n u_F$. Without loss of generality one can consider all ${}^n u$ lie in a certain $\Delta \in \mathfrak{S}$. For each j, $0 \leqq j \leqq t_r$, ${}^n \nu_j \to \nu_j$, where ${}^n v = \sum_{j=0}^{t_r} {}^n \nu_j {}^j y$ and $v = \sum_{j=0}^{t_r} \nu_j {}^j y$. As for each i, $0 \leqq i \leqq k$ it is ${}^n \lambda_i \to \lambda_i$ where ${}^n u = \sum_{i=0}^{k} {}^n \lambda_i {}^i x$, $u = \sum_{i=0}^{k} \lambda_i {}^i x$, we have ${}^n \mu_s \to \mu_s$ for each s, $1 \leqq s \leqq r$. $v \in u_F$ is now a consequence of (1) and ${}^n v \in {}^n u_F$.); Q.E.D.

Remark 2. Since x_F is a subset of the greatest simplex among $\{{}^i x_\Phi\}_{{}^i x \in \Delta_v}$, $x \in \Delta_v$ (it follows from (2) immediately), F has this property: if x lies in the interior of Δ_v and x_F contains an inner point of any side S of S_l, then for one ${}^i x \in \Delta_v$ it is ${}^i x_\Phi \supset S$.

Theorem 2. *Let Φ or Ψ be a simplicial inclusive transformation of S_k into $\mathscr{S}(S_l)$ or S_l into $\mathscr{S}(S_k)$. Then Φ and Ψ have a coincidence.*

Proof: Let $\mathfrak{S}_k$ or $\mathfrak{S}_l$ be a primitive simplicial partition of S_k or S_l belonging to Φ or Ψ respectively and F or P the corresponding continuous transformation mentioned in the Theorem 1. F and P have a coincidence, i.e. $x \in S_k$, $y \in S_l$ exist such that $y \in x_F$, $x \in y_P$. Evidently the sides

U, V, Δ_u, Δ_v (not necessarily all of the same dimension) exist with these properties: U is a side of S_k, V of S_l, Δ_u of a certain $\Delta^{(1)} \in \mathfrak{S}_k$, Δ_v of $\Delta^{(2)} \in \mathfrak{S}_l$ and x lies in the interiors of U and Δ_u, v in the interiors of V and Δ_v. As $\mathfrak{S}_k$ is a simplicial division of S_k, we have $U \supset \Delta_u$. For the same reason it is $V \supset \Delta_v$. According to Remark 2 a vertex ${}^i x \in \Delta_u$ exists such that ${}^i x_\Phi \supset V$ and a vertex ${}^j y \in \Delta_v$ with ${}^j y_\Psi \supset U$. Hence ${}^i x \in \Delta_u \subset U \subset {}^j y_\Psi$, ${}^j y \in \Delta_v \subset V \subset {}^i x_\Phi$ and Φ and Ψ have a coincidence; G.E.D.

Remark 3. Note that the coincidence takes place even for the vertices ${}^i x$, ${}^j y$ of $\mathfrak{S}_k$, $\mathfrak{S}_l$.

Remark 4. Theorem 2 can be characterized as a simplicial-inclusive-coincidence version of Brouwer's fixed-point theorem. Other coincidence versions see [3], [6].

Remark 5. It can be settled also two-sphere-collision version of Brouwer's theorem: Let S^1, S^2 be two disjoint $(n-1)$-spheres of E^n, F a continuous transformation of S^1 onto S^2. Let during a unit time interval S^1 be in quiet (i.e. ${}^t S^1 \equiv S^1$ for all $t \in [0,1]$) but S^2 continuously changes (i.e. it moves and deforms; the set of points of S^2 in the time t we denote ${}^t S^2$) as far as ${}^1 S^2 \subset S^1$ (hence the continuous transformation F^t of ${}^t S^1$ onto ${}^t S^2$ is defined: $x \in S^1$, ${}^t x = x$, $y =: x_F$, ${}^t x_{F^t} =: {}^t y \in {}^t S^2$, i.e. a homotopy $\{F^t\}_{t \in [0,1]}$ is defined). Then $\bar{t} \in [0,1]$ and $\bar{x} \in S^1$ exist such that ${}^{\bar{t}}\bar{x} = {}^{\bar{t}}\bar{x}_{F^{\bar{t}}}$.

Proof: Let S be an $(n-1)$-sphere having the same center o as S^1 and containing S^2. Denote $\bar{S}$ the union of S with its interior. Let $\{P^t\}_{t \in [0,1]}$ be the set of homotheties with the center o representing the continuous change of S^1 into S and such that $S^1_{P^t} \cap S^1_{P^{t'}} = \emptyset$ for $t \neq t'$. Since $\{F^t\}_{t \in [0,1]}$ is a continuous set of continuous transformations (i.e. ${}^t S^1_{F^t} = {}^t S^2$), one can choose S such great that ${}^t S^2_{P^t} \subset \bar{S}$ for all $t \in [0,1]$. For $x \in S^1$ define $x_f = x_F$ and prolong f on the whole $\bar{S}^1$ that f may continuously transform $\bar{S}^1$ onto $\bar{S}^2$. For $x \in \bar{S} - \bar{S}^1$ let us choose $t \in (0,1]$, $x \in S^1_{P^t}$ (such t exists unique) and define $x_f = x_{(P^t)^{-1} F^t P^t}$. Since f continuously transforms $\bar{S}$ into itself (it is in fact a continuous transformation between two flows), f has a fixed point x, $x_f = x$, i.e. $x = x_{(P^{\bar{t}})^{-1} F^{\bar{t}} P^{\bar{t}}}$, $x \in S^1_{P^{\bar{t}}}$. Hence for $\bar{x} =: x_{(P^{\bar{t}})^{-1}}$ we have $\bar{x} \in S^1 = {}^{\bar{t}} S^1$, $\bar{x} = x_{(P^{\bar{t}})^{-1} F^{\bar{t}} P^{\bar{t}} (P^{\bar{t}})^{-1}} = x_{(P^{\bar{t}})^{-1} F^{\bar{t}}} = \bar{x}_{F^{\bar{t}}}$, i.e. ${}^{\bar{t}}\bar{x} = {}^{\bar{t}}\bar{x}_{F^{\bar{t}}}$; Q.E.D.

Let us consider a set $N = \{1,2, \ldots, n\}$ of players, each player i has a finite set $\mathscr{S}_i$ of strategy possibilities and a sharp polyhedral convex cone $K_i \subset E^p$ as its preference relation (i.e. i finds a' better than a if $a' \neq a$ and $a' \in a + K_i$ ($\equiv a' - a \in K_i$). Let n transformations f_i be given of $\mathscr{S}_1 \otimes \ldots \otimes \mathscr{S}_n$ into E^p such that $f_i(x)$ means a payoff to the player i if x are players' choices. By a natural way let us define f_i as transformations of $S = S_{s_1} \otimes \ldots \otimes S_{s_n}$($s_i =:$ card $\mathscr{S}_i - 1$ and S_{s_i}

is the i^{th} probability simplex) into E^p prolonging f_i' s given above on the vertices of S. Such a situation is called an n-person non-cooperative game $\Gamma(N, S, f_i, K_i)$. For $x \in S$ x^{S^i} let us denote such a point of $S^i =: S_{s_1} \otimes \dots \otimes S_{s_{i-1}} \otimes S_{s_{i+1}} \otimes \dots \otimes S_{s_n}$ which is the orthrogonal projection of x on S^i. For a convex polyhedron P in E^p and a sharp convex polyhedral cone K $P^{\max K} = \{x : x \in P$ and it does not exist y in P so that $y \neq x, y \in x + K\}$. Evidently $P_i(z) =: C\,(f_i[(1,0, \dots, 0), z], \dots, f_i[(0, \dots, 0,1), z])$ is the set of possible i' s payoffs for other players' choices $z \in S^i$. Hence the optimal i' s play in this case is to have his payoff in $P_i(z)^{\max K_i}$. We call $\bar{x} \in S$ a Nash equilibrium if for each $i \in N$ $f_i(\bar{x}) \in P_i^{\max K_i}(\bar{x}^{S_i})$.

Theorem 3. *For each $\Gamma(N, S, f_i, Ki)$ a Nash equilibrium exists.*

Remark 6. For $p = 1$ it is the well known Nash's theorem (see [7]). For general p but K_1 positive and K_2 negative cones we have L. S. Shapley's result published in [11]. For different than our preference relations (but in a very general form) the theorem is proved by B. Peleg in [9]. We shall give two proofs. First as a trivial corollary of the Nash's theorem, the second (independent of the Nash's one but for $n = 2$) by means of our Theorem 2.

Proof 1. Let c_i's be vectors of unit lengths lying in the interiors of corresponding K_i^{dual}'s. Define new payoffs $\varphi_i(x) =: {}^T c_i f_i(x)$. Hence we have $p = 1$ and a Nash equilibrium $\bar{x}$ exists for payoffs φ_i. But $\bar{x}$ is the Nash equilibrium for f_i' s, too, for c_i lies in the interior of K_i^{dual} for all i and K_i is sharp; Q.E.D.

Proof 2. Let us consider $n = 2$. Change the denotations $a_{ij} =: f_1(i,j)$, $b_{ij} =: f_2(i, j)$ for $i \in \mathscr{S}_1$, $j \in \mathscr{S}_2$ and $k =: s_1 - 1$, $l =: s_2 - 1$. $\mathscr{A} =: =: (a_{ij})$, $\mathscr{B} =: (b_{ij})$ are vector matrices, $\mathscr{A}_j = (a_{1j}, \dots, a_{k+1\,j})$ a $(k + 1)$ by p and $\mathscr{A}^i = (a_{i1}, \dots, a_{i\,l+1})$ a p by $(l + 1)$ matrices of real numbers ($\mathscr{B}_j, \mathscr{B}^i$ are defined analogously). For $x \in S_k$ define $x_\Phi = \{y : y \in S_l, ({}^T c_2 \mathscr{B}_1 x, \dots, {}^T c_2 \mathscr{B}_{l+1} x)\, y = \max\limits_{1 \leqq j \leqq l+1} \{{}^T c_2 \mathscr{B}_j x\}\}$. Evidently x_Φ is not a nullset and it is a convex hull of all such vertices ${}^j y$ of S_l for which ${}^T c_2 \mathscr{B}_j x$ is maximal. Hence Φ transforms S_k into $\mathscr{S}(S_l)$. Φ is simplicial inclusive (In E^{k+2} denote first $k + 1$ coordinates of a point as $x^1, \dots, x^{k+1}$ and the last one as t. Consider $l + 1$ closed halfspaces $t \geqq {}^T c_2 \mathscr{B}_j x$ and orthogonally project the boundary of their intersection into the space E^{k+1} of x-axis. The projection is a polyhedral partition $\mathfrak{S}$ (with some sides being unbounded) of E^{k+1} because each halfspace has an inner normal with positive t^{th} coordinate (the dimension of any boundary side and its projection is the same). If one corresponds to the interior of any side λ of $\mathfrak{S}$ such a side of S_l vertices ${}^j y$ of which are all ${}^j y$'s with j having this property: the boundary of the above considered halfspace j contains the side its projection being λ, then this correspondence is inclusive. Evidently $\mathfrak{S}$ defines a polyhedral partition on S_k which we refine on simplicial one. The above considered inclusive correspondence between interiors of sides of $\mathfrak{S}$ and $\mathscr{S}(S_l)$ defines our Φ.). Analogously one obtains a simplicial inclusive transformation Ψ of S_l into $\mathscr{S}(S_k)$: $y_\rho = \{x : x \in S_k, ({}^Tc_1\mathscr{A}^1y, {}^Tc_1\mathscr{A}^2y, \ldots, {}^Tc_1\mathscr{A}^{k+1}y)\ x = = \max_{1 \leq i \leq k+1} \{{}^Tc_1\mathscr{A}^iy\}\}$. According to Theorem 2 Φ and Ψ have a coincidence $(\bar{x}, \bar{y})$. Since y_Ψ (or x_Φ) is a part of best replies of the first (second) player to the strategy y (x) of the second (first) one (because K_1^{dual}, K_2^{dual} are sharp and c_1, c_2 lie in their interiors), $(\bar{x}, \bar{y})$ is our required equilibrium; Q.E.D.

Remark 7. Evidently the set of the best i's replies x, $z_\Phi = \{x : f_i(x, z) \in \in P_i^{\max K_i}(z)\}$, to other players' choices z is the union of some S_{s_i}'s sides. Φ need not to be inclusive: Let $n = 2$, $p = 2$, $s_1 = 1$, $s_2 = 2$, K_2 a positive cone (i.e. the set of points in E^2 with all coordinates non-negative), denote $S_1 = C(X_1, X_2)$, $S_2 = C(Y_1, Y_2, Y_3)$, $f_2(X_1, Y_1) = \left\{\left(\frac{1}{2}, 0\right)\right\}$,

$f_2(X_1, Y_2) = \left\{\left(0, \frac{1}{2}\right)\right\}$, $f_2(X_1, Y_3) = \{(0,0)\}$, $f_2(X_2, Y_1) = \left\{\left(-\frac{1}{2}, 0\right)\right\}$,

$f_2(X_2, Y_2) = \left\{\left(0, \frac{1}{2}\right)\right\}$)}, $f_2(X_2, Y_3) = \{(0,1)\}$ and for $Z = : \frac{1}{2} X_1 + \frac{1}{2} X_2$ define $\mathfrak{S}_1 = \{\Delta_1, \Delta_2\}$, $\Delta_1 = C(X_1, Z)$, $\Delta_2 = C(X_2, Z)$. Evidently $x_\Phi = = C(Y_1, Y_2)$ for $x = X_1$ and x in the interior of Δ_1, $x_\Phi = C(Y_2, Y_3)$ for $x = Z$ and $x_\Phi = \{Y_3\}$ for x in the interior of Δ_2 and $x = X_2$. Hence Φ is not inclusive according to $\mathfrak{S}_1$ and even it cannot be inclusive according to any other $\mathfrak{S}_1$; Q.E.D.

Remark 8. The independence of the game theory on cones preferences fails in this question of an n-person cooperative game Γ with a characteristic vector-function $v(S) \in E^p$, $S \subset N$, where $N = \{1,2, \ldots, n\}$ is a set of players and $v(S) \geqq o$, $v(\{i\}) = o$, $i \in N$: Such an $(X, \mathbf{B})$ (where X is a p by n matrix, $X \geqq 0$, $\mathbf{B} = \{B_1, \ldots, B_l\}$ is a partition of N and $\sum_{i \in B_j} X_i = v(B_j)$) is called stable (see [2], [8] where the stability for $p = 1$ is defined) if for each $\mu \in N$ μ is not weaker than any other player of B_j, $\mu \in B_j$, i.e. each objection Y_C against $\mu(C \subset N - \{\mu\}$, Y_C is a p by card C matrix, $\sum_{k \in C} Y_k = v(C)$, $Y_C \geqq X_C$ and such $\nu \in C \cap B_j$ exists that $Y_\nu \geqslant X_\nu$) can be countered (i.e. there exist such D and Z_D that $\mu \in D \subset N - \{\nu\}$, Z_D a p by card D matrix, $Z_D \geqq X_D$, $\sum_{k \in D} Z_k = v(D)$ and $Z_{D \cap C} \geqq Y_{D \cap C}$). The following example shows a game Γ (with $p = 2$, $n = 3$) for which for a given $\mathbf{B}$ no stable $(X, \mathbf{B})$ exists (compare the result of B. Peleg, M. Davis, and M. Maschler, see [8], [2], for $p = 1$):

Example 1. $N = \{1,2,3\}$, $\mathbf{B} = \{(1,2), 3\}$, $v(1,2) = \begin{pmatrix}1\\3\end{pmatrix}$, $v(1,3) = $
$= \begin{pmatrix}2\\4\end{pmatrix}$, $v(2,3) = \begin{pmatrix}5\\2\end{pmatrix}$.

It is $X = \begin{pmatrix}x_{11}x_{12}o\\x_{21}x_{22}o\end{pmatrix}$, $x_{ij} \geqq 0$, $x_{11} + x_{12} = 1$, $x_{21} + x_{22} = 3$. If $0 \leqq x_{21} < 1$ it is $2 < x_{22} \leqq 3$. There exists an objection $Y_{(1,3)} = \begin{pmatrix}x_{11} + \varepsilon_1 & 2 - x_{11} - \varepsilon_1\\x_{21} + \varepsilon_2 & 4 - x_{21} - \varepsilon_2\end{pmatrix}$ where $\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix} \gg o$ of the player 1 against 2. It cannot be countered by 2 because 2 cannot get $\begin{pmatrix}\bullet\\z_{22} \geqq x_{22} > 2\end{pmatrix}$ in the coalition (2,3) due to $v(2,3) = \begin{pmatrix}\bullet\\2\end{pmatrix}$. If $x_{21} \geqq 1$ it is $x_{22} \leqq 2$ and an objection $Y_{(2,3)} =$
$= \begin{pmatrix}x_{12} + \varepsilon_1 & 5 - x_{12} - \varepsilon_1\\x_{22} + \varepsilon_2 & 2 - x_{22} - \varepsilon_2\end{pmatrix}$, $\varepsilon_1 > 0$, $\varepsilon_2 \geqq 0$ of 2 against 1 exists. Since $\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix} \gg 0$ it cannot be countered by 1 if, say, $\varepsilon_1 = \frac{1}{10}$ because 3 will get at least $\begin{pmatrix}39/10\\\bullet\end{pmatrix}$ in the coalition (2,3) whereas in (1,3) he will get at most $\begin{pmatrix}2\\\bullet\end{pmatrix}$.

Two other added examples may have an interest, too.

Example 2. $N = \{1, 2, 3, 4\}$, $\mathbf{B} = \{(1, 2), (3, 4)\}$, $X = \begin{pmatrix}0\,1\,1\,0\\1\,0\,1\,0\end{pmatrix}$, $v(2, 3) = \begin{pmatrix}4\\1\end{pmatrix}$, $v(1, 3) = \begin{pmatrix}0\\3\end{pmatrix}$, $v(1, 2) = \begin{pmatrix}1\\1\end{pmatrix}$, $v(3, 4) = \begin{pmatrix}1\\1\end{pmatrix}$, $v(S) = \begin{pmatrix}0\\0\end{pmatrix}$ otherwise. This game has the property that $(X, \mathbf{B})$ is not stable but $(x_{11}, x_{12}, x_{13}, x_{14};\ (1, 2), (3, 4))$, $(x_{21}, x_{22}, x_{23}, x_{24};\ (1, 2), (3, 4))$ are stable.

Example 3. N, $\mathbf{B}$, X as in the Example 2. $v(2, 3) = \begin{pmatrix}0\\2\end{pmatrix}$, $v(1, 2) =$
$= \begin{pmatrix}1\\1\end{pmatrix}$, $v(1, 3) = \begin{pmatrix}3\\0\end{pmatrix}$, $v(3, 4) = \begin{pmatrix}1\\1\end{pmatrix}$, $v(S) = \begin{pmatrix}0\\0\end{pmatrix}$ for other S. It shows, on the other hand, that $(X, \mathbf{B})$ is stable whereas the single parts are not stable.

Remark 9. Let $C(A)$ mean the convex hull of columns of a matrix A, $P = \{x : x \in \mathrm{E}^n, Bx \geqq b\}$ a convex polyhedron lying in $C(A)$ ($b \in \mathrm{E}^k$, B is a k by n matrix, A an n by l matrix). Evidently $Y = \{y : y \in S_{l-1},$

$BAy \geqq b\}$ is a convex polyhedron. Denote for $y \in Y$ $K(y)$ (or $L(y)$) a set of all indices j such that $B^j x = b^j$ (or $y^j > 0$) and write $k(y) =: \,$ $=:\text{card } K(y)$ ($l(y) =: \text{card } L(y)$). Being inspired by an explicite formulas for basic optimal strategies solving a two-person zero-sum matrix game (see [5]) one can settle this necessary and sufficient condition for $y \in Y$ to be a vertex of Y (for a free eligibility of A one may find it useful from numerical point of view):

(3) $C(B^{K(y)}A_{L(y)})$ is an $(l(y) - 1)$-dimensional simplex.

Proof: I. Let y be a vertex of Y and (3) be not true i.e. either two columns of $B^{K(y)}A_{L(y)}$ are equal or all are different but in (3) mentioned polyhedron is at most $l(y)$—2 dimensional. According to Radon's theorem (see[1]) two disjoint sets L_1, L_2 of indices exist such that $C[(B^{K(y)}A_{L(y)})_{L_1}] \cap C[(B^{K(y)}A_{L(y)})_{L_2}] \neq \emptyset$ i.e. a vector $z \in E^l$ exists such that ${}^Tez = 0$, $z^{L(y)} \neq o$, $z^i = 0$ for $i \notin L(y)$ and $B^{K(y)}A_{L(y)}z^{L(y)} = o$. As $y^{L(y)} > o$ two points $y_{1,2} = y \pm \varepsilon z$ (for a suitable small $\varepsilon > 0$) lie in S_{l-1} and (for $i = 1, 2$) $B^{K(y)}Ay_i = b^{K(y)}$ and, evidently, ε can be chosen such small that $B^jAy_i > b^j$ for $j \notin Ky$, $i = 1, 2$. Hence $y \neq y_{1,2} \in$ $\in Y$ and y is not a vertex. II. Let $y \in Y$ and (3) is true. Suppose y is not a vertex of Y i.e. $y_1, y_2 \in Y$ exist, $y_1 \neq y_2$ such that $y = \frac{1}{2} y_1 + \frac{1}{2} y_2$. It results for $i = 1, 2$ $j \notin L(y)$, $y_i^j = 0$ and $B^{K(y)}Ay_i = b^{K(y)}$. As ${}^Tey_i = 1$ for $i = 1, 2$ we have $B^{K(y)}A_{L(y)}(y_1 - y_2)^{L(y)} = o$, ${}^Te(y_1 - y_2)^{L(y)} = 0$, $(y_1 - y_2)^{L(y)} \neq o$ which contradicts to (3); Q.E.D.

§ 2

If $A = (a_{ij})$ is an m by n matrix and r, s integers, $1 \leqq r \leqq m - 1$, $1 \leqq s \leqq n - 1$, then the sets of saddlepoints of matrices A, $A_{\vartheta(q_1 \ldots q_s)}$, $A^{\vartheta(p_1 \ldots p_r)}$, $A^{\vartheta(p_1 \ldots p_r)}_{\vartheta(q_1 \ldots q_s)}$ are denoted by $\mathcal{S}, \mathcal{S}_{q_1 \ldots q_s}, \mathcal{S}^{p_1 \ldots p_r}, \mathcal{S}^{p_1 \ldots p_r}_{q_1 \ldots q_s}$. ($a_{i_0 j_0}$ is the saddlepoint of A if for all i, j, $a_{ij_0} \leqq a_{i_0 j_0} \leqq a_{i_0 j}$).The row p or the column q means the p^{th} row or the q^{th} column in A.

Lemma 1. *Let $A = (a_{ij})$ be an m by n matrix, $m \geqq 1$, $n \geqq 3$ and let every m by $(n - 1)$ submatrix has a saddlepoint. Then $\mathcal{S} = \emptyset$ iff there exists a column, q, with two maximal elements x, y, $x \in \mathcal{S}_{q_1}$, $y \in \mathcal{S}_{q_2}$ for $q_1 \neq q_2$ and $\mathcal{S}_{q_1}$, $\mathcal{S}_{q_2}$ have no common element in q.*

Proof: I. Necessity. At least two columns q_1, q_2, $q_1 \neq q_2$ exist so that $\mathcal{S}_{q_1}, \mathcal{S}_{q_2}$ have a common column. (Otherwise $\mathcal{S}_r$ for $r = 1, 2, \ldots, n$ is a k_r by 1 matrix. Let $a = a_{kl}$ be the maximal element in A. There exists exactly one column, l', such that $a \in \mathcal{S}_{l'}$. It is $a_{ki} = a$ for all $i = 1, \ldots, n$, $i \neq l'$ and $a_{kl'} < a$. Then for every $s \neq l'$ it is $a_{ks} \in \mathcal{S}_{l'}$ which contradicts to the above result.) Further $\mathcal{S}_{q_1}$, $\mathcal{S}_{q_2}$ have disjoint sets of rows (in another case any common element of $\mathcal{S}_{q_1}$, $\mathcal{S}_{q_2}$ would be a saddlepoint of A). From this it follows the rest of the assertion.

II. Sufficiency. Let the assumptions be satisfied and $\mathcal{S} \neq \emptyset$. Let $a_{11} \in \mathcal{S}_{q_1}$, $a_{21} \in \mathcal{S}_{q_2}$, $a = a_{i2} \in \mathcal{S}$. Evidently $a_{i2} = a_{11} = a_{21}$ (since $a \in \mathcal{S}_{q_1} \cup \mathcal{S}_{q_2}$ and $a_{11} = a_{21}$). As $a_{1q_1} \leqq a_{11}$ (since $a_{11} \notin \mathcal{S}_{q_2}$) and $a_{2q_2} < a_{21}$ we have $i > 2$. At least one of integers q_1, q_2 is > 2; let $q_1 > 2$. Then $a \in \mathcal{S}_{q_1}$ and $a_{i1} = a = a_{11} = a_{21}$. Evidently $a_{i1} \in \mathcal{S}$ and from this it follows $a_{i1} \in \mathcal{S}_{q_1} \cap \mathcal{S}_{q_2}$—a contradiction; Q.E.D.

Remark 10. The assumption of $\mathcal{S}_{q_1}$, $\mathcal{S}_{q_2}$ in q is substantial as this example shows: For the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

it is: every $\mathcal{S}_q \neq \emptyset$, $a_{32} \in \mathcal{S}$, $x = a_{12}$, $y = a_{22}$ and $a_{32} \in \mathcal{S}_{q_1} \cap \mathcal{S}_{q_2}$ for $q_1 = 1$, $q_2 = 3$.

Remark 11. The similar assertion holds for $(m - 1)$ by n submatrices but the word "column" and "maximal" must be substituted by "row" and "minimal".

Theorem 4. *Let $A = (a_{ij})$ be an m by n matrix, $m \geqq 3$, $n \geqq 3$. For given integers r, s, $1 \leqq r \leqq m - 3$, $1 \leqq s \leqq n - 3$ let every $m - r$ by $n - s$ submatrix of A have a saddlepoint. Then $\mathcal{S} = \emptyset$ iff there exist integers $1 \leqq r_0 \leqq r$, $1 \leqq s_0 \leqq s$, $1 \leqq p_1 < p_2 < \ldots < p_{r_0} \leqq m$, $1 \leqq$ $\leqq q_1 < \ldots < q_{s_0} \leqq n$ such that* 1° *A has no saddlepoint in rows $p_1, \ldots, p_{r_0}$ and columns $q_1, \ldots, q_{s_0}$ and either* 2° a) *there exists a column, q, with two equal elements x, y, $x \in \mathcal{S}^{p_1 \ldots p_{r_0}}_{q_1 \ldots q_{s_0 - 1}}$, $y \in \mathcal{S}^{p_1 \ldots p_{r_0}}_{q_1 \ldots q_{s_0-2} q_{s_0}}$ and $\mathcal{S}^{p_1 \ldots p_{r_0}}_{q_1 \ldots q_{s_0-1}}$, $\mathcal{S}^{p_1 \ldots p_{r_0}}_{q_1 \ldots p_{s_0-2} q_{s_0}}$ are disjoint in q, or* 2° b) *there exists a row, p, with two minimal elements u, v, $u \in \mathcal{S}^{p_1 \ldots p_{r_0-1}}$, $v \in \mathcal{S}^{p_1 \ldots p_{r_0-2} p_{r_0}}$ and $\mathcal{S}^{p_1 \ldots p_{r_0-1}}$ $\mathcal{S}^{p_1 \ldots p_{r_0-2} p_{r_0}}$ are disjoint in p.*

Proof. I. Necessity. Since every $(m - r)$ by $(n - s)$ submatrix has a saddlepoint and $\mathcal{S} = \emptyset$ one of two cases will appear:

1. There exists k, $1 \leqq k \leqq s$ so that every $(m - r)$ by $(n - k)$ submatrix has a saddlepoint but at least one $(m - r)$ by $(n - k + 1)$ submatrix, $B = A^{\vartheta(p_1 \ldots p_r)}_{\vartheta(q_1 \ldots q_{k-1})}$ has no saddlepoint (let k be maximal with this property). According to Lemma 1 there exists a column, q, of B with two maximal elements x, y, $x \in \mathcal{S}^{p_1 \ldots p_r}_{q_1 \ldots q_k}$, $y \in \mathcal{S}^{p_1 \ldots p_r}_{q_1 \ldots q_{k-1}\, k+1}$ and $q \cap \mathcal{S}^{p_1 \ldots p_r}_{q_1 \ldots q_k} \cap \mathcal{S}^{p_1 \ldots p_r}_{q_1 \ldots q_{k-1} q_{k+1}} = \emptyset$ i.e. 2° a) holds where $r_0 = r$, $s_0 = k + 1$.

If 1. doesn't work then there exists l, $1 \leqq l \leqq r$ (maximal one) with the following property: every $(m - l)$ by n submatrix has a saddlepoint but there exists an $(m - l + 1)$ by n submatrix, $C = A^{\vartheta(p_1 \ldots p_{l-1})}$

with no saddlepoint. According to remark 11 there exists a row, p, of C with two minimal elements u, v, $u \in \mathcal{S}^{p_1 \dots p_l}$, $v \in \mathcal{S}^{p_1 \dots p_{l-1} p_{l+1}}$ and $\mathcal{S}^{p_1 \dots p_l}$, $\mathcal{S}^{p_1 \dots p_{l-1} p_{l+1}}$ are disjoint in the row p, i.e. it holds 2° b) for $r_0 = l + 1$. The property 1° is clear (as $\mathcal{S} = \emptyset$).

II. Sufficiency. Suppose, on the contrary, $\mathcal{S} \neq \emptyset$, $a_{ij} \in \mathcal{S}$. Then $i \neq p_1, \dots, p_{r_0}$, $j \neq q_1, \dots, q_{s_0}$. As $a_{ij} \in \mathcal{S}^{p_1 \dots p_{r_0}}_{q_1 \dots q_{s_0 - 1}} \cup \mathcal{S}^{p_1 \dots p_{r_0}}_{q_1 \dots q_{s_0 - 2} q_{s_0}}$ in 2° a) or $a_{ij} \in \mathcal{S}^{p_1 \dots p_{r_0 - 1}} \cup \mathcal{S}^{p_1 \dots p_{r_0 - 2} p_{r_0}}$ in 2° b) it is $a_{ij} = x = y$ or $a_{ij} = u = v$ resp. From this it follows $a_{iq} \in \mathcal{S}^{p_1 \dots p_{r_0}}_{q_1 \dots q_{s_0 - 1}} \cap \mathcal{S}^{p_1 \dots p_{r_0}}_{q_1 \dots q_{s_0 - 2} q_{s_0}}$ or $a_{pj} \in \mathcal{S}^{p_1 \dots p_{r_0 - 1}} \cap \mathcal{S}^{p_2 \dots p_{r_0 - 2} p_{r_0}}$ resp.—a contradiction with 2°; Q.E.D.

Theorem 5. *Let $A = (a_{ij})$ be an m by n matrix, $\mathcal{S} = \emptyset$ and*

(4) *no column have two maximal elements.*

The maximal number of m by $(n - 1)$ submatrices of A with saddlepoints equals two.

Proof. Let there exist three such submatrices, e.g. A_1, A_2, A_3. Then some saddlepoints of A_1, A_2, A_3 lie (after suitable denotation) in their turn also in the column 2, 3, 1; denote s_i these points, i.e. $s_i \in \mathcal{S}_i$ for $i = 1, 2, 3$, $s_i = a_{j_i k_i}$, $k_1 = 2$, $k_2 = 3$, $k_3 = 1$. (Let it be not the case. Thus there exists $i \in \{1, 2, 3\}$ such that $k_i \notin \{1, 2, 3\} - \{i\}$. Let, for example, it be $i = 1$; then $k_1 > 3$. For at least one $l \in \{2, 3\}$ it is $k_l \neq 1$ [due to (4)]. We can assume $l = 2$. Then $j_1 \neq j_2$ (in another case it would be $s_1 = s_2$ and $s_1 \in \mathcal{S}$ — a contradiction). From this it follows $s_i \in \mathcal{S}_{12}$ for $i = 1, 2$ and also $a_{j_2 k_1} \in \mathcal{S}_{12}$. Hence $s_1 = s_2 = a_{j_2 k_1}$, which contradicts to (4). Thus $s_3 \leqq a_{j_3 k_1} \leqq s_1 \leqq a_{j_1 k_2} \leqq s_2 \leqq a_{j_3 k_3} \leqq s_3$, i.e. only equality holds. It follows [from (4)] $j_1 = j_2 = j_3$ and we have a contradiction with $\mathcal{S} = \emptyset$; Q.E.D.

Theorem 6. *Let $A = (a_{ij})$ be an m by n matrix, $m \geqq 1$, $n \geqq 3$ $\mathcal{S} = \emptyset$ and (4) hold. Then the maximal number of m by $(n - 2)$ submatrices with saddlepoints is equal to $2n - 3$.*

The assertion follows immediately from the following lemmas.

Lemma 2. Let for a matrix $A = (a_{ij})$ (4) hold and $\mathcal{S} = \emptyset$. Then there doesn't exist four distinct elements being saddlepoints in their turn of four distinct submatrices of type $A_{\vartheta(pq)}$ such that none of them is a saddlepoint of any two submatrices $A_{\vartheta(pq_1)}$, $A_{\vartheta(pq_2)}$, $q_1 \neq q_2$.

Proof. Let four such saddlepoints $s_1, \dots, s_4$ exist and $A_{p_i q_i}$ for $i = 1, \dots, 4$ be the corresponding submatrices. At most two saddlepoints of $\{s_1, \dots, s_4\}$ can lie in the same row. (Let s_1, s_2, s_3 be three such points in a row i. Let $s_1 \leqq \min \{s_2, s_3\}$. Then there exists p_1, $1 \leqq p_1 \leqq n$ such that $a_{ip_1} < s_1$. From this it follows that the corresponding submatrices of points s_1, s_2, s_3 are of type $A_{p_1 q_1}, A_{p_1 q_2}, A_{p_1 q_3}$ where $q_1 \neq q_2 \neq q_3 \neq q_1$. Then there exist j, $k \in \{1, 2, 3\}$ so that $s_1 \in \mathcal{S}_{p_1 q_j} \cap \mathcal{S}_{p_1 q_k}$—a contradiction.) We can assume $s_i = a_{u_i i}$, $i = 1, 2, 3, 4$. For s_1 let k be the smallest integer, $1 \leqq k \leqq n$ with the property $k \neq 1, p_1, q_1$. Thus it is $k \leqq 4$, $s_1 \leqq a_{u_1k} \leqq s_k$. Further for s_k let l, $1 \leqq l \leqq n$ be the minimal index with the property $l \neq k, p_k, q_k$. Evidently $l \leqq 4$ and $s_k \leqq a_{u_kl} \leqq s_l$. If we continue this process, then after at most four steps we get some s_i previously had been obtained, say for instance s_1, i.e. $s_r \leqq a_{u_r1} \leqq s_1$, $1 \neq r, p_r, q_r$ and hence $s_1 = a_{u_1k} = s_k = a_{u_kl} = s_l = \ldots = s_r = a_{u_r1}$. If $u_1 = u_k$ then $l \neq 1$ (in another case it would be $s_1 \in \mathscr{S}_{pq} \cap \mathscr{S}_{pq'}$ for $q \neq q'$ or $s_1 \in \mathscr{S}$) and thus it must be (from the above result) $u_l \neq u_1$, $a_{u_1l} = a_{u_ll}$ which contradicts to (4). If $u_1 \neq u_k$ then $a_{u_1k} = a_{u_kk}$ and it is the contradiction, too; Q.E.D.

Remark 12. Three such saddlepoints can exist; see the following example:

For $A = \begin{pmatrix} 4 & 0 & 0 & 5 & 6 \\ 3 & 2 & 1 & 0 & 3 \\ 2 & 1+2\varepsilon & 1+\varepsilon & 1 & 0 \end{pmatrix}$, $\varepsilon > 0$ sufficiently small, it is $\mathscr{S} = \emptyset$, $a_{11} \in \mathscr{S}_{23}$, $a_{22} \in \mathscr{S}_{34}$ and $a_{33} \in \mathscr{S}_{45}$.

Lemma 3. Let an m by n matrix $A = (a_{ij})$ have no saddlepoint and (4) hold. Then at most two distinct columns p_1, p_2 of A and columns $p_i' \neq p_i$, $i = 1, 2$, $p_1' \neq p_2$ exist so that for all q, r, $1 \leqq q, r \leqq n$, $q \neq p_1$, p_1', $r \neq p_2$, p_2' the submatrices $A_{\vartheta(p_1q)}$ and/or $A_{\vartheta(p_2q)}$ have the common saddlepoint in the column p_1' or p_2' respectively.

Proof. Assume the existence of three such columns p_1, p_2, p_3. Denote s_1, s_2, s_3 the corresponding saddlepoints; $s_i = a_{k_ip_i}$ for $i = 1, 2, 3$. Without loss of generality we can suppose $p_1 = 1$, $p_1' = 2$ and $p_3 = 3$. Then $p_3' = 1$ (in another case (4) or $\mathscr{S} = \emptyset$ would be failed). From the same reason it must be $p_2 = 2$, $p_2' = 3$. Then it is $s_1 \leqq a_{k_13} < s_2 \leqq$ $\leqq a_{k_21} < s_3 \leqq a_{k_32} < s_1$—a contradiction; Q.E.D.

Lemma 4. Let $\mathscr{S} = \emptyset$ and (4) hold. Let there exist columns p_1', p_2', $p_1' \neq 1$, $p_2' \neq 2$, $p_1' \neq p_2'$ of A such that all submatrices of type $A_{\vartheta(1q)}$, $q \neq 1$, p_1' and/or $A_{\vartheta(2r)}$, $r \neq 2$, p_2' have the common saddlepoint s_1 or s_2 in the column p_1' or p_2' respectively. Then $\mathscr{S}_{uv} = \emptyset$ for every $\{u, v\} \neq \{p_1', p_2'\}, \{1, q\}, \{2, r\}$.

Proof. First of all it is $p_2' = 1$, $p_1' = 2$ or $p_2' = 1$, $p_1' \neq 1, 2$ (the case $p_1' = 2$, $p_1' \neq 1, 2$ is the same as the last one). In another case either (4) or $\mathscr{S} = \emptyset$ would be failed. Let it be $\mathscr{S}_{uv} \neq \emptyset$, i.e. there exists $s = a_{kl}$, $s \in \mathscr{S}_{uv}$ for at least one couple $\{u, v\}$ satisfying the condition of the Lemma. Then it is $u, v \neq 1, 2$. Let $s_i = a_{k_ip_i'}$. If $k = k_2$ it is $s = a_{k_22}$ and $s \in \mathscr{S}$—a contradiction. If $k \neq k_2$ then for the column 1 (4) doesn't hold, because $s_2 = a_{k_21} = a_{k1}$; Q.E.D.

Lemma 5. Let A be an m by n matrix, $n \geqq 4$. If there exist s, p, q_1, q_2, $q_1 \neq q_2$ so that $s \in \mathscr{S}_{pq_1} \cap \mathscr{S}_{pq_2}$ then $s \in \mathscr{S}_{pq}$ for each $q \neq q_0$. (It is evident.)

Remark 13. The maximal number $2n - 3$ of submatrices appears when there exist columns $p_1, p_2, p_1', p_2', p_1 \neq p_2$ such that for every $p \neq p_1$ A_{p_1p} has a saddlepoint in p_1', for every $p \neq p_2$ A_{p_2p} has a saddlepoint in p_2' and $\mathcal{S}_{p_1'p_2'} \neq \emptyset$.

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