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Erich Barvínek

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ON SOME CONVERGENCE TESTS FOR SERIES AND INTEGRALS

E. BARVÍNEK (BRNO)

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1. For determining convergence of a series $\sum_{n=0}^{\infty} a_n b_n$ the four following theorems usually are applied, see [1].

[AB_1] Abel's test. The series $\Sigma\, a_n b_n$ converges if $\Sigma\, a_n$ converges and if (b_n) is a monotonic convergent sequence.

[DR_1] Dirichlet's test. Let $\Sigma\, a_n$ be a series whose partial sums form a bounded sequence. Let (b_n) be a monotonic sequence which converges to 0. Then $\Sigma\, a_n b_n$ converges.

[DBR_1] Du Bois-Reymond's test. If both series $\Sigma\, a_n$ and $\Sigma\, | b_n - b_{n+1} |$ converge then $\Sigma\, a_n b_n$ conveges too.

[DD_1] Dedekind's test. Let $\Sigma\, a_n$ be a series whose partial sums form a bounded sequence. Let $\Sigma\, | b_n - b_{n+1} |$ converge and (b_n) converges to O. Then $\Sigma\, a_n b_n$ also converges.

Let $f(x)$ and $g(x)$ be functions defined in an interval $[a, b[$ on the set of real numbers. Let the proper Riemann integrals $\int_a^{b'} f(x)\,\mathrm{d}x$ and $\int_a^{b'} g(x)\,\mathrm{d}x$ exist for any $b' \in [a, b[$.

For determining convergence of the improper integral $\int_a^b f(x)\, g(x)\,\mathrm{d}x$ the two folowing theorems usually are applied.

[AB_2] Abel's test. If the integral $\int_a^b f(x)\,\mathrm{d}x$ converges and $g(x)$ is monotonic and bounded on $[a, b[$, then $\int_a^b f(x)\, g(x)\,\mathrm{d}x$ converges.

[DR_2] Dirichlet's test. If there is $\left| \int_a^{b'} f(x)\,\mathrm{d}x \right| \leqq M$ for some constant M and any $b' \in [a, b[$, $g(x)$ is monotonic on $[a, b[$ and $\lim_{x\to b^-} g(x) = 0$, then $\int_a^b f(x)\, g(x)\,\mathrm{d}x$ converges.

We wish to get two theorems on improper integrals which would correspond to the theorems [DBR_1] and [DD_1] about series.

2. The proofs of the theorems [AB_2] and [DR_2] are based on the Second Mean Value theorem for Riemann integrals. For our purpose we would need any mean value theorem in which no monotonic behaviour is supposed. Let us mention both classical theorems.

[L_1] First Mean Value theorem. Let proper Riemann integrals $\int_a^{b'} f(x)\,\mathrm{d}x$ and $\int_a^{b'} g(x)\,\mathrm{d}x$ exist. Let $g(x) \geqq 0$ and $m \leqq f(x) \leqq M$ on $[a, b']$, where m, M are real numbers. Then there exists a real number $s \in [m, M]$ such that $\int_a^{b'} f(x)\, g(x)\,\mathrm{d}x = s \int_a^{b'} g(x)\,\mathrm{d}x$.

[L_2] Second Mean Value theorem. Let proper Riemann integral $\int_a^{b'} f(x)\,\mathrm{d}x$ exist. Let $g(x)$ be monotonic on $[a, b']$. Then there exists $s \in [a, b']$ such that $\int_a^{b'} f(x)\, g(x)\, dx = g(a) \int_a^{s} f(x)\,\mathrm{d}x + g(b) \int_s^{b} f(x)\,\mathrm{d}x$.

The two theorems we are giving below also function as mean value theorems.

[L_3] Let $\int_a^{b'} f(x)\,\mathrm{d}x$ and $\int_a^{b'} g(x)\,\mathrm{d}x$ exist and $g(x) \geqq 0$ on $[a, b']$. Then there exists $s \in [a, b']$ such that $\int_a^{b'} f(x)\, g(x)\,\mathrm{d}x = \sup\limits_{x \in [a, b']} f(x) \int_a^{s} g(x)\,\mathrm{d}x + {}$ $+ \inf\limits_{x \in [a, b']} f(x) \int_s^{b'} g(x)\,\mathrm{d}x$.

Proof. Let us put $M = \sup\limits_{x \in [a, b']} f(x)$, $m = \inf\limits_{x \in [a, b']} f(x)$. For the continuous function $F(x) = \int_a^{x} (M - m)\, g(t)\,\mathrm{d}t - \int_a^{b'} f(t) - m)\, g(t)\,\mathrm{d}t$ there holds $F(a) \leqq 0$, $F(b') \geqq 0$. Hence there is a number $s \in [a, b']$ such that $F(s) = 0$.

[L_4] Let $f(x)$ and $g'(x) = \left(\dfrac{\mathrm{d}g(x)}{\mathrm{d}x}\right)$ be Riemann — integrable on $[a, b']$. Then (1) $\int_a^{b'} f(x)\, g(x)\,\mathrm{d}x = -\int_a^{b'} (g'(s) \int_a^{s} f(x)\,\mathrm{d}x)\,\mathrm{d}s + g(b') \int_a^{b'} f(x)\,\mathrm{d}x$.

Proof. Let us compute the derivation at an arbitrarily fixed $x \in \,]a, b'[$ of the function $F(x) = \int_a^{x} (g'(s) \int_a^{s} f(t)\,\mathrm{d}t + f(s)\, g(s))\,\mathrm{d}s - g(x) \int_a^{x} f(s)\,\mathrm{d}s$ according to the definition of derivation. We have $F(x + h) - F(x) =$ $= \int_x^{x+h} (g'(s) \int_a^{s} f(t)\,\mathrm{d}t + f(s)\, g(s))\,\mathrm{d}s - (g(x + h) - g(x)) \int_a^{x+h} f(s)\,\mathrm{d}s -$

$$-g(x)\int_x^{x+h} f(s)\,ds = \int_x^{x+h} (g'(s)\int_a^s f(t)\,dt + f(s)g(s))\,ds - \int_x^{x+h} g'(s)\,ds \int_a^{x+h} f(s)\,ds -$$

$$-g(x)\int_x^{x+h} f(s)\,ds = \int_x^{x+h} [g'(s)\int_{x+h}^s f(t)\,dt + f(s)\,(g(s) - g(x))]\,ds = \int_x^{x+h} (g'(s)$$

$$\int_{x+h}^s f(t)\,dt + f(s)\int_x^s g'(t)\,dt)\,ds.$$

We can suppose that $|f(s)| \leqq M$ and $|g'(s)| \leqq M$ on $[a, b']$. Because s lies between x and $x + h$ we obtain $|F(x + h) - F(x)| \leqq |h|(M|h|M + M|h|M) = 2M^2h^2$. Hence $F'(x) = 0$. Thus $F(x)$ is constant on $]a, b'[$.

From the continuity of $F(x)$ on $[a, b']$ there follows that for $x \in [a, b']$ there is $F(x) = F(a) = 0$. Putting $x = b'$ we obtain the formula of the theorem.

Theorem $[L_4]$ presents integration by parts, see [3], and implies the formula of $[L_2]$ provided moreover $g(x)$ is monotonic, see [2]. E.g; if $g(x)$ is nondecreasing, it is $g'(x) \geqq 0$ and $[L_1]$ can be applied to the integral $\int_a^{b'} g'(s)\int_a^s f(x)\,dx ds$. We get $\int_a^{s_0} f(x)\,dx \int_a^{b'} g'(s)\,ds = (g(b') -$ $- g(a))\int_a^{s_0} f(x)\,dx$ where s_0 is a convenient number in $[a, b']$. Hence the formula of $[L_2]$ follows.

We do not give here any application of $[L_3]$, whereas $[L_4]$ will be applied to obtain the following theorems $[DBR_2]$ and $[DD_2]$.

3. Let $f(x)$ and $g(x)$ be functions defined in an interval $[a, b[$ on the set of real numbers. Let the proper Riemann integrals $\int_a^{b'} f(x)\,dx$ and $\int_a^{b'} g'(x)\,dx$ exist for any $b' \in [a, b[$, where $g'(x) = \dfrac{dg(x)}{dx}$.

$[DBR_2]$ **If the integrals $\int_a^b f(x)\,dx$ and $\int_a^b |g'(x)|\,dx$ converge then the integral $\int_a^b f(x)\,g(x)\,dx$ converges.**

Proof. For arbitrary $b' \in [a, b[$ there holds formula (1). For $b' \to b^-$ the integral $\int_a^{b'} (g'(s)\int_a^s f(x)\,dx)\,ds$ converges absolutely, because the function $\int_a^s f(x)\,dx$ of the variable s is bounded and the integral $\int_a^{b'} g'(s)\,ds$ converges absolutely. As to the term $g(b')\int_a^{b'} f(x)\,dx$, there exists $\lim\limits_{b'\to b^-}$

$g(b') = g(a) + \int_a^b g'(s)\,ds$ and hence a finite limit of $g(b')\int_a^{b'} f(x)\,dx$ exists.

[DD$_2$] Let $|\int_a^{b'} f(x)\,dx| \leqq M$ for every $b' \in [a, b[$. Let the integral $\int_a^b |g'(x)|\,dx$ converges and $\lim_{x\to b^-} g(x) = 0$. Then the integral $\int_a^b f(x)\,g(x)\,dx$ converges.

Proof. For all $b' \in [a, b[$ there holds formula (1). For $b' \to b^-$ the integral $\int_a^{b'} (g'(s)\int_a^s f(x)\,dx)\,ds$ converges for the same reasons as in [DBR$_2$] and $g(b')\int_a^{b'} f(x)\,dx$ converges to 0.

4. For $i = 1$ or $i = 2$ there are certain relations among the theorems $[AB_i]$, $[DR_i]$, $(DBR_i]$, $[DD_i]$, see [4].

$[DG]$ For $i = 1,2$ there holds the implication-diagram

$$\begin{array}{ccc} [DD_i] & \Rightarrow & [DBR_i] \\ \Downarrow & & \Downarrow \\ [DR_i] & \Rightarrow & [AB_i] \end{array}$$

Proof. a) $[DR_1] \Rightarrow [AB_1]$. Under the suppositions of $[AB_1]$ there exists a finite limit $\lim_{n\to\infty} b_n = b$. Let us write $a_n b_n = a_n b + a_n(b_n - b)$. The series $\Sigma\, a_n(b_n - b)$ converges according to $[DR_1]$, because $|\sum_{n=0}^{N} a_n| \leqq$ $\leqq M$ and $(b_n - b)$ converges to 0.

b) $[DR_2] \Rightarrow [AB_2]$. Under the assumptions of $[AB_2]$ there exists a finite limit $\lim_{x\to b^-} g(x) = s$. Let us write $f(x)\,g(x) = sf(x) + f(x)\,(g(x) - s)$. The integral $\int_a^b f(x)\,(g(x) - s)\,dx$ converges according to $[DR_2]$, because $|\int_a^{b'} f(x) dx| \leqq M$ and $g(x) - s$ converges monotonically to 0 if $x \to b^-$.

c) $[DBR_1] \Rightarrow [AB_1]$. Under the assumptions of $[AB_1]$ the series $\Sigma(b_n - b_{n+1})$ converges absolutely.

d) $[DBR_2] \Rightarrow [AB_2]$. Under the assumtions of $[AB_2]$ a finite limit $\lim_{x\to b^-} g(x) = s$ exists. Let us put $z = 1$ or $z = -1$ according to $g(x)$

being nondecreasing or nonincreasing. Then $\int_a^{b'} |g'(x)|\,dx = z\int_a^{b'} g'(x)\,dx =$ $= z(g(b') - g(a))$. Hence the integral $\int_a^b |g'(x)|\,dx$ converges.

e) $[DD_1] \Rightarrow [DR_1]$. Under the assumptions of $[DR_1]$ the convergence of the series $\Sigma\, |b_n - b_{n+1}|$ follows.

f) $[DD_2] \Rightarrow [DR_2]$. Under the assumptions of $[DR_2]$ the integral $\int_a^b |g'(x)|\,dx$ converges.

g) $[DD_1] \Rightarrow [DBR_1]$. Under the assumptions of $[DBR_1]$ from the convergence of the series $\Sigma(b_n - b_{n+1})$ the existence of a finite limit $\lim\limits_{n\to\infty} b_n =$ $= b$ follows. Let us put $a_n b_n = a_n b + a_n(b_n - b)$. The series $\Sigma\, a_n(b_n - b)$ converges according to $[DD_1]$ because $|\sum\limits_{\nu=0}^{n} a_\nu| \leqq M$, and $\Sigma\, |(b_n - b) -$ $- (b_{n+1} - b)| = \Sigma\, |b_n - b_{n+1}|$ converges. The sequence $(b_n - b)$ converges to 0.

h) $[DD_2] \Rightarrow [DBR_2]$. Under the assumptions of $[DBR_2]$ the existence of a finite limit $\lim\limits_{b'\to b^-} g(b') = s$ follows from the convergence of the integral $\int_a^b g'(x)dx$. Let us put $f(x)\,g(x) = sf(x) + f(x)\,(g(x) - s)$. The integral $\int_a^b f(x)\,(g(x) - s)\,dx$ converges according to $[DD_2]$ because $|\int_a^{b'} f(x)\,dx| \leqq M$, the integral $\int_a^b |(g(x) - s)'|\,dx = \int_a^b |g'(x)|\,dx$ converges and $\lim\limits_{x\to b^-} (g(x) - s) = 0$

5. For the uniform convergence of the series $\Sigma\, a_n(y)\, b_n(y)$, $y \in Y$, where Y is an arbitrary set of real numbers, the tests analogous to $[AB_1]$, $[DR_1]$, $[DBR_1]$, $[DD_1]$ are used. For uniform convergence of improper Riemann integrals $\int_a^b f(x, y)\, g(x, y)\,dx$, $y \in Y$ Abel's and Dirichlet's tests $[AB_4]$ and $[DR_4]$ are used.

Suppose that the functions $f(x, y)$ and $g(x, y)$ are defined for $x \in [a, b[$ and $y \in Y$. Let the proper Riemann integrals $\int_a^{b'} f(x, y)\,dx$ and $\int_a^{b'} \frac{\partial g(x, y)}{\partial x}\,dx$ exist for any $b' \in [a, b[$ and $y \in Y$.

$[AB_4]$ Abel's test. If the integral $\int_a^b f(x, y)\,dx$ converges uniformly for $y \in Y$, $|g(x, y)| \leqq M$ for $x \in [a, b[$ and $y \in Y$, and for any $y \in Y$ the function $g(x, y)$ of the variable x is monotonic in $[a, b[$, then $\int_a^b f(x, y)\, g(x, y)\,dx$ converges uniformly for $y \in Y$.

$[DR_4]$ Dirichlet's test. If $\left|\int_a^{b'} f(x, y)\,dx\right| \leqq M$ for any $b' \in [a, b[$ and $y \in Y$, $\lim\limits_{x \to b^-} g(x, y) = 0$ uniformly on Y and $g(x, y)$ as a function of x is monotonic in $[a, b[$ for any $y \in Y$, then $\int_a^b f(x, y)\, g(x, y)\,dx$ converges uniformly on Y.

Below we give two tests for uniform convergence of improper Riemann integrals analogous to $[DBR_2]$ and $[DD_2]$.

$[L_5]$ Let $f(x)$ and $g'(x)$ be integrable in $[a, b']$ for any $b' \in [a, b[$. Let the integral $\int_a^b f(x)\,dx$ converge. Then for every $b', b'' \in [a, b[$ there holds the formula

$$(2)\ \int_{b'}^{b''} f(x)\, g(x)\,dx = \int_{b'}^{b''} \Big(g'(s) \int_s^b f(t)\,dt\Big)\,ds - g(b'') \int_{b''}^{b} f(t)\,dt + g(b') \int_{b'}^{b} f(t)\,dt.$$

Proof. Applying formula (1) for the intervals $[a, b']$ and $[a, b'']$ and subtracting we get a formula containing $\int_a^s f(t)\,dt$, $\int_a^{b''} f(t)\,dt$ and $\int_a^{b'} f(t)\,dt$. For $s' = s, b'', b'$ let us put $\int_a^{s'} f(t)\,dt = \int_a^{b''} f(t)\,dt - \int_{s'}^{b'} f(t)\,dt$. Then we obtain (2).

$[L_6]$ Assume that the integral $\int_a^b f(x, y)\,dx$ converges for any $y \in Y$, that for any $b'' > b'$, $b' \to b^-$ the integral $\displaystyle\int_{b'}^{b''} \left(\frac{\partial g(s, y)}{\partial s} \int_s^b f(t, y)\,dt\right) ds$ converges for $y \in Y$ uniformly to 0, and for $b' \to b^-$ the function $g(b', y)$ $\int_{b'}^{b} f(t, y)\,dt$ converges for $y \in Y$ uniformly to 0. Then the integral $\int_a^b f(x, y)\, g(x, y)\,dx$ converges uniformly on Y.

Proof. We are to show that for $b'' > b'$, $b' \to b^-$ the integral

$\int_{b'}^{b''} f(x, y)\, g(x, y)\, dx$ converges uniformly to 0 on Y. For any $y \in Y$ according to (2) we have $\int_{b'}^{b''} f(x, y)\, g(x, y)\, dx = \int_{b'}^{b''} \left(\frac{\partial g(s, y)}{\partial s} \int_s^b f(t, y)\, dt\right) ds -$ $- g(b'', y) \int_{b''}^{b} f(t, y)\, dt + g(b', y) \int_{b'}^{b} f(t, y)\, dt$. All terms on the right converge uniformly to 0 on Y whenever $b'' > b'$, $b' \to b^-$.

$[DBR_4]$. Let the integrals $\int_a^b f(x, y)\, dx$ and $\int_a^b \left| \frac{\partial g(x, y)}{\partial x} \right| dx$ converge uniformly on Y. If $| g(x, y) | \leqq M$ for $x \in [a, b[$ and $y \in Y$, then $\int_a^b f(x, y)\, g(x, y)\, dx$ converges uniformly on Y.

Proof. The integral $\int_{b'}^{b''} \left(\frac{\partial g(s, y)}{\partial s} \int_s^b f(t, y)\, dt\right) ds$ converges uniformly to 0 on Y whenever $b'' > b'$, $b' \to b^-$, because for $s \to b^-$ the integral $\int_s^b f(t, y)\, dt$ converges uniformly to 0 and thus there exists a constant K such that $| \int_s^b f(t, y)\, dt | \leqq K$ for $x \in [a, b[$ and $y \in Y$. Hence

$$\left| \int_{b'}^{b''} \left(\frac{\partial g(s, y)}{\partial s} \int_s^b f(t, y)\, dt\right) ds \right| \leqq K \int_{b'}^{b''} \left| \frac{\partial g(s, y)}{\partial s} \right| ds$$

and the right-hand side converges uniformly to 0. The affirmation follows from $[L_6]$.

$[DD_4]$ Let $| \int_a^{b'} f(x, y)\, dx | \leqq M$ for $b' \in [a, b[$, $y \in Y$. Let the integral $\int_a^b \left| \frac{\partial g(x, y)}{\partial x} \right| dx$ converge uniformly on Y and let be $\lim_{x \to b^-} g(x, y) = 0$ uniformly on Y. Then integral $\int_a^b f(x, y)\, g(x, y)\, dx$ converges uniformly on Y.

Proof. For any $y \in Y$ and any $b' \in [a, b[$ we can apply $[L_4]$ and write according to (1)

$$\int_a^{b'} f(x, y,) g(x, y) \, \mathrm{d}x = -\int_a^{b'} \frac{\partial g(s, y)}{\partial s} \int_a^{s} f(t, y) \, \mathrm{d}t) \, \mathrm{d}s + g(b', y) \int_a^{b'} f(x, y) \, \mathrm{d}x.$$

The first term on the right converges uniformly because $\left| \int_a^s f(t, y) \, \mathrm{d}t \right| \leqq$ $\leqq M$ and $\int_a^b \left| \frac{\partial g(s, y)}{\partial s} \right| \mathrm{d}s$ converges uniformly. The second term converges uniformly to 0.

As to the implication-diagram for the uniform convergence of integrals we can prove only the following propositions. Analogous results may be obtained for series.

[P_1] If $\left| \int_a^b f(x, y), \mathrm{d}x \right| \leqq K$ for $y \in Y$, then [DD_4] $\Rightarrow$ [DBR_4].

Proof. Under the suppositions of [DBR_4] there exists a uniform limit $\lim\limits_{x \to b^-} g(x, y) = s(y)$ and there is $| s(y) | \leqq M$. Let us write $\int_a^b f(x, y)$ $g(x, y) \, \mathrm{d}x = s(y) \int_a^b f(x, y) \, \mathrm{d}x + \int_a^b f(x, y) \, (g(x, y) - s(y)) \, \mathrm{d}x$. The first term on the right converges uniformly because $s(y)$ is bounded. The second term converges uniformly according to [DD_4] because $\left| \int_a^{b'} f(x, y) \, \mathrm{d}x \right| \leqq K'$ for $y \in Y$ and every b' near b, the integral $\int_a^b \left| \frac{\partial}{\partial x} (g(x, y) - s(y)) \right| \mathrm{d}x$ converges uniformly and there is uniformly on Y $\lim\limits_{x \to b^-} (g(x, y) - s(y)) = 0$.

[P_2] It holds [DD_4] $\Rightarrow$ [DR_4].

Proof. Under the assumptions of [DR_0] let us put $z(y) = 1$ or $z(y) =$ $= -1$ for any $y \in Y$ according to $g(x, y)$ being a nondecreasing or nonincreasing function of x. Then the integral $\int_a^b \left| \frac{\partial g(x, y)}{\partial x} \right| \mathrm{d}x$ converges uniformly on Y, because $\int_a^{b'} \left| \frac{\partial g(x, y)}{\partial x} \right| \mathrm{d}x = z(y) \int_a^{b'} \frac{\partial g(x, y)}{\partial x} \, \mathrm{d}x =$ $= z(y) \, [g(b', y) - g(a, y)]$ which converges uniformly to the limit $-z(y) \, g(a, y)$. Hence the assumptions of [DD_4] are fulfilled.

$[P_3]$ Let Y be a bounded and closed set of real numbers. Let $g(x, y)$ be continuous on Y for any $x \in [a, b[$ and $s(y) = \lim_{x \to b^-} g(x, y)$ be continuous on Y. Then $[DBR_4] \Rightarrow [AB_4]$.

Proof. Under the assumptions of $[AB_4]$ there exists a finite limit $\lim_{x \to b^-} g(x, y) = s(y)$ for any $y \in Y$ because $g(x, y)$ is monotonic in x for any $y \in Y$ and $| g(x, y) | \leqq M$. Let $z(y)$ be defined like in $[P_2]$. The integral $\int_a^{b'} \left| \frac{\partial g(x, y)}{\partial x} \right| dx = z(y)\, (g(b', y) - g(a, y)]$ converges uniformly if and only if $g(x, y) \to s(y)$ does so. This follows from DINI'S theorem.

$[P_4]$ Let $\left| \int_a^b f(x, y)\, dx \right| \leqq K$ for $y \in Y$ where Y is a bounded and closed set of real numbers. Let $g(x, y)$ be continuous on Y for any $x \in [a, b[$ and $s(y) = \lim_{x \to b^-} g(x, y)$ be also continuous on Y. Then $[DR_4] \Rightarrow$ $\Rightarrow [AB_4]$.

Proof. Under the suppositions of $[AB_4]$ there exists a finite limit $\lim_{x \to b^-} g(x, y) = s(y)$ for any $y \in Y$ and $| s(y) | \leqq M$. Let us write $\int_a^b f(x, y)$ $g(x, y)\, dx) = s(y) \int_a^b f(x, y)\, dx + \int_a^b f(x, y)\, (g(x, y) - s(y))\, dx$. The first term on the right converges uniformly. So does the second term according to $[DR_4]$ because for b' near b and $y \in Y$ there is $\left| \int_a^{b'} f(x, y)\, dx \right| \leqq K'$ and $\lim_{x \to b^-} [g(x, y) - s(y)] = 0$ on Y uniformly by Dini's theorem.

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