Archivum Mathematicum

Anna Sekaninová; Milan Sekanina

Topologies compatible with ordering

Archivum Mathematicum, Vol. 2 (1966), No. 3, 113--126

Persistent URL: `http://dml.cz/dmlcz/104614`

TOPOLOGIES COMPATIBLE WITH ORDERING

Sekanina

A. AND M. SEKANINA (Brno)

Received March 31, 1066

I. INTRODUCTION

Many sets, occurring in the mathematical considerations, are simultaneously both ordered and topological spaces. Mostly investigated cases were cases when the topology was defined by means of the ordering. The general case of connection of an ordering with a topology occurs in studying topological lattices, ordered topological groups, semi-groups and similarly (see [14]). The general concept of compatibility of an ordering and a topology, the so called "Dedekind's compatibility", was delt in papers [4], [11]. In this paper, we are going to treat in details two kinds of compatibility of ordering and topology.

II. BASIC CONCEPTS

Under a *relation* on a set A we understand a subset of the cartesian product $A \times A$. Relations on A are ordered by means of set inclusion. When speaking of *ordering* of the set A we mean a reflexive, antisymmetrical and transitive relation on A. If in this relation every two elements from A are comparable, we say that the ordering is complete and A is called a *chain*; if every two different elements a and b are incomparable, i.e. $a \,||\, b$, A is called an *antichain*. In what follows, the terminology and notation are the same as in [10] as far as ordering of the set is concerned.

Under a *topological space* we shall understand a topological space (P, u) in the sense of Bourbaki. $C(u)$, $O(u)$, respectively, denotes the system of all closed, or open sets in (P, u). If (P, u) and (P, v) are two topological spaces with the same carrier we write $u \leqq v$ whenever $C(u) \supset C(v)$ and we say that $v(u)$ is a coarser (finer) topology than $u(v)$. By this relation the set $\mathscr{B}(P)$ of all Bourbaki's topologies on P is ordered. As for further concepts referring to topological spaces see [2].

III. COMPATIBILITY OF A TOPOLOGY WITH AN ORDERING

Definition 3.1. *Let A be an ordered set and u a topology on A. Say that u is compatible with the ordering, if u is a T_1-topology and if for every* *pair* $a, b \in A$, $a < b$, *there exist a neighbourhood* O_1 *of the point* a *and a neighbourhood* O_2 *of the point* b *so that*

$$x \in O_1 \Rightarrow x < b \text{ or } x \parallel b,$$
$$y \in O_2 \Rightarrow y > a \text{ or } y \parallel a \text{ hold.}$$

Definition 3.2. *Let* A *be an ordered set,* u *a topology on* A. *Say that* u *is strongly compatible with the ordering, if* u *is a* T_1*-topology and if for every pair* $a, b \in A$, $a < b$ *there exist a neighbourhood* O_1 *of a point the and a neighbourhood* O_2 *of the point* b *such that*

$$x \in O_1, y \in O_2 \Rightarrow x < y \text{ or } x \parallel y.$$

Theorem 3.3. *Let* A *be an ordered set,* u *a topology on* A. *If* u *is strongly compatible with ordering then* u *is compatible with the ordering, too.*

Proof is evident.

Theorem 3.4. *Let* A *be an ordered set and* u, v, $v \leqq u$ *two topologies on* A. *Let* u *be compatible (strongly compatible) with the ordering. Then* v *is compatible (strongly compatible) with the ordering.*

Proof is evident.

Theorem 3.5. *Let* π_1, π_2 *be two orderings on* A. *Let* $\pi_1 \subset \pi_2$. *Let* u *be compatible with* π_2. *Then* u *is compatible with* π_1, *too.*

Proof is evident.

Theorem 3.6. *Let* A *be an ordered set,* $B \subset A$. *Let* u *be a topology on* A *compatible (strongly compatible) with the ordering. Let* u/B *be the topology induced by means of* u *on* B. *Then* u/B *is a topology on* B *compatible (strongly compatible) with the ordering.*

Proof is evident.

Definition 3.7. *We say that a subset* B *in an ordered set* A *is densely imbedded, if for every* $x \in B$, $y \in A - B$, $x < y$ *or* $y < x$ *there exists* $b \in B$ *such that* $x < b < y$ *or* $y < b < x$.

Theorem 3.8. *Let* B *be densely imbedded in a ordered set* A. *Let* u *be a topology on* B *compatible (strongly compatible) with the ordering. Then, there exists a topology* v *on* A *compatible (strongly compatible) with the ordering, for which* $v/B = u$.

Proof. Let us define a topology u_1 on A—B as the discrete topology. Let $(A, v) = (B, u) + (A—B, u_1)$ be the sum of two topological spaces. Then v is evidently compatible (strongly compatible) with the ordering and $u = v/B$.

IV. SPECIAL TYPES OF TOPOLOGIES ON AN ORDERED SET

Let A be a given ordered set. Let us set $[x) = \{y \mid y \in A,\ y \geqq x\}$, $(x] = \{y \mid y \in A,\ y \leqq x\}$ for $x \in A$. For $x, y \in A$, $x \leqq y$, $[x, y] =$

$=\{z \mid x \leqq z \leqq y\}$, $N(a, b) = \{x \mid$ it does not hold $x \leqq a, x \leqq b$ or $x \geqq a$, $x \geqq b\}$. A set $C \subset A$ is called convex if

$$x, y \in C,\ x \leqq y \Rightarrow [x, y] \subset C.$$

Let B be an up-directed ordered set, (i.e. for $x, y \in B$ z exists such that $x \leqq z$, $y \leqq z$), $\{x_\beta\}_{\beta \in B}$ a net in A. Say that a net $\{x_\beta\}_{\beta \in B}$ converges to x if there exists a set M of majorants for $\{x_\beta\}_{\beta \in B}$ and a set N of minorants for $\{x_\beta\}_{\beta \in B}$ with the property $\inf M = x = \sup N$ (see [3], O_2 — convergence in [13]). We write $\lim x_\beta = x$. A minorant or a majorant of the net $\{x_\beta\}_{\beta \in B}$ is an element y such that $y \leqq x_\beta$ or $x_\beta \leqq y$, respectively, for all indices $\beta \geqq \beta_1$, for a suitable β_1.

A subset $C \subset A$ is called an ideal (see [6], p. 227) (a dual ideal) in A if it holds

$F \subset C$, $F \neq 0$, (0 denotes the empty set), F finite $\Rightarrow (F^*)^+ \subset C$ ($(F^+)^* \subset C$), when

$$F^* = \{z \mid z \geqq x \quad \text{for all} \quad x \in F\}, \qquad F^+ = \{z \mid z \leqq x \quad \text{for all} \quad x \in F\}.$$

An ideal C (a dual ideal) is called totally irreducible if it is not an intersection of ideals (dual ideals) different from C.

A set $C \subset A$ is called finite separable if there exist $x_1, \ldots, x_n \in C$ such that

$$C \subset (x_1] \cup \ldots \cup (x_n] \cup [x_1) \cup \ldots \cup [x_n).$$

Definition 4.1. *We call a topology on A which has as subbasis of closed sets intervals* $[x)$, $(x]$ *for* $x \in A$, *an interval topology. We shall denote this topology by* ι *or* ι_A (see [5]).

Note. The interval topology is evidently always a T_1-topology.

Theorem 4.2. (see [7]). *The interval topology on A is Hausdorff exactly when the set* $N(a, b)$ *for* $a, b \in A$, $a \neq b$ *is finitely separable.*

Note. The condition for a topology to be Hausdorff can be easily transcribed into the notation of closed sets in the following way: A topology on A is Hausdorff exactly when for every two points $b, c \in A$, $b \neq c$ there exist closed sets B and C, $b \in B$, $b \notin C$, $c \in C$, $c \notin B$ such that $B \cup C = A$.

Theorem 4.3. *If the interval topology* ι *on A is Hausdorff then it is strongly compatible with the ordering.*

Proof. Let an ordered set A with a property that $N(a, b)$ is finitely separable for all $a \neq b$ be given.

We are interested in pairs of comparable elements, thus let $a < b$, and we seek a neighbourhood O_1 of the point a and a neighbourhood O_2 of the point b so that for every $x \in O_1$, $y \in O_2$ there were $x < y$ or $x \parallel y$.

The set $N(a, b)$ is finitely separable, so there exists a finite set of elements $c_1, \ldots, c_n$ from $N(a, b)$ such that every element from $N(a, b)$

belongs to any of the intervals $[c_1), (c_1], \ldots, [c_n), (c_n]$. Let us construct closed sets $X = \cup (c_i] \cup (a]$, $Y = \cup [c_i) \cup [b)$, for which $a \in X$, $a \notin Y$, $b \in Y$, $b \notin X$, $X \cup Y = A$ hold. Put $O_1 = A—Y$, $O_2 = A—X$. Then $a \in O_1$, $b \in O_2$, $O_1 \cap O_2 = 0$. Suppose that $x \in O_1$, $y \in O_2$ exist such that $x > y$; $x \in O_1 \Rightarrow x \notin O_2 \Rightarrow x \in X \Rightarrow c_i$ exists so that $x \in (c_i]$ or $x \in (a] \Rightarrow c_i \geqq x$ or $a \geqq x \Rightarrow c_i > y$ or $a > y \Rightarrow y \in (c_i]$ or $y \in (a] \Rightarrow y \in X \Rightarrow y \notin A—X = O_2$ which is a contradiction.

Theorem 4.4. *The interval topology is compatible with the ordering.*

Proof. For every pair of points $a < b$ we look for a neighbourhood O_1 of the point a and for a neighbourhood O_2 of the point b so that for all $x \in O_1$ and all $y \in O_2$ there hold $x < b$ or $x \| b$ and $y > a$ or $y \| a$. Let us put $O_2 = A—(a]$, $O_1 = A—[b)$. O_1, O_2 are open sets and $a \in O_1$, $b \in O_2$.

If there existed $x \in O_1$, $x > b$ then $x \in [b)$. But $x \in O_1 = A—[b)$. Analogously for O_2.

Definition 4.5. *A topology on A in which a set is closed exactly when it contains with every convergent net simultaneously its limit is called the convergent topology* (see [3], [10], this topology is sometimes called order-topology). *Let us denote it by $\varkappa$, more precisely by $\varkappa_A$.*

Note 4.6. The convergent topology is always a T_1 — topology.

Theorem 4.7. *The convergent topology is compatible with the ordering.*

Proof. It holds $\varkappa \leqq \iota$ (see e.g. [3], [5]). By Theorem 3.4 and Theorem 4.4 $\varkappa$ is compatible with the ordering.

Before showing that there exists an ordered set A with Haudorff convergent topology which is not strongly compatible with the ordering, we shall introduce some auxiliary statements.

Lemma 4.8. Let $\{x_\beta\}_{\beta \in B} \to x$. Then, for the set M of all minorants and the set N of all majorants $\bigvee M = x = \bigwedge N$ hold (see [3]).

Lemma 4.9. Let A be an ordered set. Then for $x_1 \leqq \ldots \leqq x_n \leqq \ldots$ $(x_1 \geqq \ldots \geqq x_n \geqq \ldots)$

$\lim x_n = x \equiv x = \bigvee x_n$ $(x = \bigwedge x_n)$ holds.

Proof. Let $\lim x_n = x$ hold, for $x_1 \leqq \ldots \leqq x_n \leqq \ldots$. x_n is a minorant for $\{x_n\}_{n=1}^{\infty}$; consequently if M denotes the set of all minorants it holds $\bigvee M = x \geqq x_n$ by lemma 4.8. On the contrary $m \geqq x_n$, for all $n \Rightarrow m$ is a majorant, thus $m \geqq x$. Hence $\bigvee x_n = x$.

Let $\bigvee x_n = x$. Then x_n's form a certain set of minorants, $\{x\}$ is a one element set of majorants. Thus, $\lim x_n = x$. In the similar way for $x_1 \geqq x_2 \geqq \ldots$.

Lemma 4.10. A net $\{x_\beta\}_{\beta \in B}$, for which there exists β_1 such that for β, $\beta_1 \leqq \beta$ is $x_\beta = a$ for a suitable a, converges to a.

Definition 4.11. *A net, described in lemma 4.10, is called an almost stationary net.*

Lemma 4.12. Let $\{x_\beta\}_{\beta\in B}$ be a net converging to x which is not almost stationary and let its elements form a chain $y_1 < y_2 < \ldots < y_n < \ldots$ Then $x = \bigvee y_n$. Similarly for a decreasing chain.

Proof. For every n β_n exists such that for $\beta \geqq \beta_n$ $x_\beta \notin \{y_1, \ldots, y_n\}$ (it follows from the theorem on convergence of a subnet). Thus, y_n for all n are minorants of $\{x_\beta\}_{\beta\in B}$. Hence $y_n \leqq x$.

Let $z \geqq y_n$ for all n, so z is a majorant to $\{x_\beta\}_{\beta\in B}$. Then $z \geqq x$, because x is an infimum of the set of all majorants. For that reason $x = \sup y_n$.

Lemma 4.13. Let A be a set where no two incomparable elements possess the upper und the lower bound at the same time. Then every convergent net $\{x_\beta\}_{\beta\in B}$ in A has the property that for a certain β_1, $\{x_\beta\}_{\beta>\beta_1}$ is a chain.

Proof. Let $x = \lim \{x_\beta\}_{\beta\in B}$. Elements a and b exist such that starting from a certain index β_1 it is $a \leqq x_\beta \leqq b$, $\beta > \beta_1$. Thus, for $\beta, \gamma > \beta_1$ a, b are common upper and lower bounds consequently x_β and x_γ are comparable.

Theorem 4.14. *Let A be a set where every chain is finite or of a type ω or ω^* and no two incomparable elements from A possess simultaneously the upper and the lower bounds. Then, a set $M \subset A$ is in $\varkappa_A$ closed exactly when for every chain $x_1 < x_2 < \ldots < x_n < \ldots$, $x_n \in M$ there is $\bigvee x_n \in M$ and every chain $x_1 > x_2 > \ldots > x_n > \ldots$, $x_n \in M$ is $\bigwedge x_n \in M$ (in the case when $\bigvee x_n$ or $\bigwedge x_n$, respectively, exists).*

Proof. Let M be a closed set. Then the statement on chains holds by Lemma 4.9.

Let M contain with every convergent, increasing or decreasing chain of a type ω or ω^* its limit. Let $\{x_\beta\}_{\beta\in B}$ be a convergent net in M with a limit a and $x_\beta \in M$. According to the preceding Lemma β_1 exists such that $\{x_\beta\}_{\beta\geqq\beta_1}$ is a chain. In consequence of Lemma 4.10 we can suppose that the net $\{x_\beta\}_{\beta\in B}$ is not almost stationary so that the set formed by the elements x_β with $\beta \geqq \beta_1$ is a chain of the type ω or ω^*. Furthermore by Lemma 4.12. Thereby the proof of Theorem 4.14 is finished.

Let F_i be a chain of the type ω, $i \in N$, where N is the set of all positive integers, $F_i \cap F_j = 0$ for $i, j \in N$, $i \neq j$. Let G_j be a chain of the type ω^*, $j \in R$, where R is the set of all real numbers, $G_j \cap G_k = 0$, for $j, k \in R$, $j \neq k$. Let $\bigcup_{i\in N} F_i \cap \bigcup_{i\in R} G_i = 0$.

Furthermore, let a, b be two different elements not belonging to $\bigcup_i F_i \cup \bigcup_i G_j$. F_i be $\{x_{i,1} < \ldots < x_{i,k} < \ldots\}$ and $G_j = \{y_{j,1} > \ldots y_{j,k} > > \ldots\}$. Put $H = [\underset{i\in N}{S} F_i \oplus \{a\}] + [\{b\} \oplus \underset{j\in R}{S} G_j]$, where S, $+$ denote a cardinal sum, $\oplus$ an ordinal sum.

The ordering on H will be completed in such a way:

First we put $a \geqq b$. Let $A = \prod_{i\in I} N_i$, where $N_i = N$, card $I = \aleph_0$.

Card $A = 2^{\aleph_0}$. Let f be a one-to-one mapping of A on the set R. Let $\alpha = (\ldots, a_i, \ldots) \in A$. Then, let us put $x_{i,k} < y_{f(\alpha),j}$ for $k \leqq a_i$ and $j \leqq i$. The set H, the ordering of which is completed in the above described way, will be denoted by H^*.

Theorem 4.15. *The convergent topology on H^* is Hausdorff and fails to be strongly compatible with the ordering.*

Proof. First we are going to show that H^* fulfills the conditions of Theorem 4.14. An arbitrary chain in H^* is either finite or of the type ω or ω^*. If c and d are incomparable elements in H^* then one of these possibilities occurs under a suitable notation.

1. $c \in F_i$, $d \in F_i'$, $i \neq i'$.
2. $c \in G_j$, $d \in G_j'$, $j \neq j'$.
3. $c \in F_i$, $d \in G_j$.
4. $c = a$, $d \in G_j$.
5. $d = b$, $c \in F_i$.

Ad 1. Elements c and d have no common lower bound.

Ad 2. Elements c and d have not an upper bound in common.

Ad 3. Let us admit that there exist elements e, g such that $e \leqq c, d$ and $c, d \leqq g$. Then $e \in F_i$. If it is $c = x_{i,s}$, then $e = x_{i,k}$ for $k \leqq s$. Similarly $g \in G_j$, if $d = y_{j,l}$ then $g = y_{j,m}$ for $m \leqq l$.

According to the definition of ordering in H^* there holds furthermore: if $f(\alpha) = j$, then $a_i \geqq s$, because $g \geqq c$. As $e \leqq d$ it is $l \leqq i$. Thus $x_{i,s} \leqq y_{j,l}$. Hence $c \leqq d$ which is a contradiction.

Ad 4. and 5. a is a maximal and b a minimal element from H^*.

By the statement of Theorem 4.14 it is necessary to deal with convergent chains of types ω or ω^*. The only such convergent chains are subchains of the F_i (converging to a) and subchains of the G_j (converging to b). All points except for a and b are, thus, isolated in $\varkappa_{H^*}$. $\cup G_j \cup \{b\}$, $\cup F_i \cup \{a\}$ are in $\varkappa_{H^*}$ closed sets, fulfilling for points a and b the properties mentioned in the remark following Theorem 4.2. Consequently $\varkappa_{H^*}$ is Hausdorff. We shall show that it is not strongly compatible with the ordering. Let O be a neighbourhood of the point a. Then there exists $\alpha = (\ldots, a_i, \ldots) \in A$ such that $x_{i,k}$ for $k \geqq a_i$ is an element of O. Let O_1 be some neighbourhood of the point b. Then there exists j_1 such that for $j > j_1$, $y_{f(\alpha),j} \in O_1$. Then $x_{j,a_j} < y_{f(\alpha),j}$. At the same time $a > b$. Consequently $\varkappa_{H^*}$ is not strongly compatible.

Definition 4.16. *Let A be an ordered set. The topology in which totally irreducible ideals and totally irreducible dual ideals form a subbasis of open sets is called the ideal topology* ([6] p. 232).

Lemma 4.17. Let $A_1 \subset A_2 \subset \ldots \subset A_i \subset \ldots$ be a transfinite sequence of ideals or of dual ideals, respectively. Then $\cup A_i$ is an ideal or an dual ideal, respectively.

Proof is evident from the definition of ideal.

Lemma 4.18. Every ideal (dual ideal, respectively) is an intersection of totally irreducible ideals (dual ideals).

Proof. Let A_1 be an ideal in B. Let $b \notin A_1$. By Lemma 4.17 and Zorn lemma there exists a maximal ideal $A_2 \supset A_1$ not containing b. Admit that this ideal is not totally irreducible. Then $A_2 = \cap A_n$, A_n are ideals different from A_2. At least one of them, let us denote it by A, does not contain b. This is a contradiction with the maximality of A_2.

Corollary of Lemma 4.18. Ideal topology of A is a T_1-topology.

Proof. Let $x \in A$, $y \neq x$. At least one of the sets $[y)$ and $(y]$ does not contain x. Let it be $[y)$. $[y)$ is a dual ideal. By 4.18 an totally irreducible dual ideal I exists for which $I \supset [y)$, x non $\in I$. I is an open set in the ideal topology of A. Similarly for $(y]$. Hence $A - \{x\}$ is an open set in the ideal topology, so the ideal topology is a T_1-topology.

Lemma 4.19. Every ideal (dual ideal) is convex.

Proof. Let A_1 be an ideal, $x < y < z$, $x, z \in A_1$. Then $(\{z\}^*)^+ = (z]$ thus, $y \in A_1$. Similarly for a dual ideal.

Theorem 4.20. *Let A be an ordered set, u its ideal topology. Then u is compatible with the ordering.*

Proof. Let $a < b$. $(a]$ is an ideal in A. As $b \notin (a]$ there exists by Lemma 4.18, a totally irreducible ideal A_1 which contains $(a]$ and fails to contain b. As A_1 is convex, it does not contain $z > b$. Thus A_1 is a neighbourhood of a with the demanded properties. Similarly for b.

Problem 4.21. Let the ideal topology be Hausdorff. Is it strongly compatible?

Naito in [8] has defined P-ideal topology (CP-ideal, MP-ideal topology) on a lattice. From Lemma 2 and 3 on page 242 in [8] there follows

Theorem 4.22. *Let the P-ideal (CP-, MP-ideal) topology on a lattice A be a $T_1(T_2)$-topology. Then, this topology is compatible (strongly compatible) with the ordering.*

Definition 4.23. *Let A be an up-and-down directed set. Let the sets of a form $[a, b]$ form a subbasis of the closed sets. Then, let us denote by ν_A the topology defined in such a way and let us call it the B-interval topology* (see related concepts in [1]).

Theorem 4.24. *Let A be an up-and-down directed set. Then ν_A is compatible with the ordering exactly when A has the smallest and the greatest elements.*

Proof. Let ν_A be compatible with the ordering. For card $A = 1$ Theorem is evident. Let card $A \geqq 2$. Let $a > b$ (such two elements exist, the set A is directed). Let $[c_1, d_1], \ldots, [c_n, d_n]$ be such intervals that, when putting $A_1 = [c_1, d_1] \cup \ldots \cup [c_n, d_n]$, $O = A - A_1$ is an open set for which $b \in O$ and $x \in O \Rightarrow x < a$ or $x \parallel a$. Consequently $y \geqq a \Rightarrow y \in A_1$. Thus, especially if $c \geqq d_1, \ldots, d_n$ (c exists, A is directed) it is $c \geqq y$ for $y \geqq a$ i.e. c is a maximal element over a, i.e. a maximal element of the set A. But the directed set has at most one maximal element. In a similar way we can prove the existence of the smallest element.

When A has the greatest and smallest element then $\nu_A = \iota_A$.

Consequently by Theorem 4.4 B-interval topology is compatible with the ordering exactly when it coincides with the interval topology.

Definition 4.25. *Let S be a lattice and u a T_1-topology on it. Let there exist to every neighbourhood O of a point $a \vee b$ ($a \wedge b$) a neighbourhood O_1 of the point a and a neighbourhood O_2 of the point b such that for arbitrary $x \in O_1$, $y \in O_2$ $x \vee y \in O$ ($x \wedge y \in O$). Then S together with the topology u is called a topological lattice.*

Theorem 4.26. *Let (S, u) be a topological lattice, u a T_1-topology. Then u is compatible with ordering.*

Proof. Let $a < b$. Let O_1 and O_2 be neighbourhoods of the point a or b respectively, not containing b or a, respectively. Then there exist neighbourhoods O_3 and O_4 of the point a or b, respectively, not containing b or a, respectively, such that

$$x \in O_3,\ y \in O_4 \Rightarrow x \wedge y \in O_1$$
$$x \in O_3,\ y \in O_4 \Rightarrow x \vee y \in O_2.$$

Let us admit that there exists $z \in O_3$ such that $z > b$. Then $z \wedge b = b \in O_1$ which is a contradiction. Similarly for O_4.

Theorem 4.27. *If the topology of a topological lattice is Hausdorff, then it is strongly compatible with the ordering.*

Proof. Let a lattice S be a topological lattice (S, u) with a topology u which is Hausdorff. Let us choose a, $b \in S$, $a < b$. Then there exist a neighbourhood O_1 of the point a and a neighbourhood O_2 of the point b, $O_1 \cap O_2 = 0$.

It holds $a \vee b = b$. Furthermore there exist a neighbourhood O_3 of the point a and a neighbourhood O_4 of the point b so that for all $x \in O_3$, $y \in O_4$ $x \vee y \in O_2$ holds. Now, let us denote $O' = O_1 \cap O_3$, $O'' = O_2 \cap O_4$. For $x \in O'$, $y \in O''$ it holds $x \vee y \in O_2$, $O' \cap O'' = 0$. If $x \in O'$, $y \in O''$, $x > y$ existed, then $x \vee y = x \in O_2$, so $O_2 \cap O' \neq 0$ but $O' \subset O_1$ and $O_2 \cap O_1 = 0$ which is a contradiction. Thus no such x, y exist and consequently the topology u is strongly compatible with the ordering.

V. EXTREMAL PROPERTIES OF THE INTERVAL TOPOLOGY

Let A be an ordered set, $\mathscr{S}(A)$ be the system of all topologies on A which are compatible with the ordering.

Let $C = \{c_1, \ldots, c_m\}$, $D = \{d_1, \ldots, d_n\}$, $C \subset A$, $D \subset A$, $C \cup D \neq 0$.

Put $P(C, D) = P(c_1, \ldots, c_m; d_1, \ldots, d_n) = (A - [c_1)) \cap \ldots \cap (A - [c_m)) \cap (A - (d_1]) \cap \ldots \cap (A - (d_n])$.

Let $b \in A$. Let us call an ordered pair of subsets C and D ($C \neq 0$, or $D \neq 0$) for which $P(C, D) \cap [b) \neq 0$ holds up-admissible (more concisely an admissible pair) with respect to b. Let $\mathscr{D}(b)$ be the system of all such admissible pairs.

Let $a \parallel b$. Let us call $0 \neq \mathscr{D}_1 \subset 2^{\mathscr{D}(b)}$ a system up-admissible with regard to $\langle a, b\rangle$ (more concisely an admissible system) when it holds:

(1) $S \in \mathscr{D}_1 \Rightarrow S \neq 0$.

(2) $S_1, \ldots, S_m \in \mathscr{D}_1 \Rightarrow$ there exists $S \in \mathscr{D}_1$ such that $\langle C_1, D_1\rangle \in S_1, \ldots, \langle C_m, D_m\rangle \in S_m$ exist for which $\langle C_1 \cup \ldots \cup C_m, D_1 \cup \ldots \cup D_m\rangle \in S$.

(3) a) $x > a \Rightarrow$ there exists $S \in \mathscr{D}_1$ such that $\langle C, D\rangle \in S \Rightarrow P(C, D) \subset A - [x)$.

b) $x < a \Rightarrow$ there exists $S \in \mathscr{D}_1$ such that $\langle C, D\rangle \in S \Rightarrow P(C, D) \subset A - (x]$.

(4) $x \neq a \Rightarrow$ there exists $S \in \mathscr{D}_1$ such that for every $\langle C, D\rangle \in S$ $x \notin P(C, D)$ holds.

In a similar way we define down-admissible pairs and a down-admissible system. In what follows we are going to deal with properties of the up-admissible system. By means of dualization we get corresponding statements for the down-admissible system. Let $\mathscr{D}_1$ be a system up-admissible with regard to $\langle a, b\rangle$.

Let us put

$O(\mathscr{D}_1) = \{X \mid X \in O(\iota_A),\ a \notin X\} \cup \{X \mid X = O \cup O_1,\ O \in O(\iota_A),\ a \in O$ and $O_1 = \bigcup_{\langle C,D\rangle \in S} P(C, D)$ for certain $S \in \mathscr{D}_1\}$.

Let $O(\mathscr{D}_1)$ be a subbase of open sets of a topology $u(\mathscr{D}_1)$. Then it holds

a) $u(\mathscr{D}_1) > \iota_A$.

Proof. $u(\mathscr{D}_1) \geqq \iota_A$ follows from the fact that O_1 is an open set in ι_A. $u(\mathscr{D}) \neq \iota_A$ follows from the fact that $[b)$ is not closed set in $u(\mathscr{D}_1)$. A—$[b)$ is not a neighbourhood of a point a in $u(\mathscr{D}_1)$, because for $O_1', \ldots, O_1^{(k)} \in O(\mathscr{D}_1)$ constructed by means of $S_1, \ldots, S_k \in \mathscr{D}_1$, it is by (2) possible to find S and $\langle C_1, D_1\rangle \in S_1 \ldots \langle C_k, D_k\rangle \in S_k$ so that $\langle C_1 \cup \ldots \cup C_k, D_1 \cup \ldots \cup D_k\rangle \in S$ and $P(C_1 \cup \ldots \cup C_k; D_1 \cup \ldots \cup D_k) \cap [b) \neq 0$, hence $O_1' \cap \ldots \cap O_1^{(k)} \cap [b) \neq 0$.

b) $u(\mathscr{D}_1)$ is a T_1-topology.

Proof. Let $y \neq x$.

b_1) Let $y \neq a$. Then there exists a set not containing either a or x and containing y in $O(\iota_A)$. This set belongs to $O(u(\mathscr{D}_1))$, as well.

b_2) Let $y = a$. Then by (4) there exists $S \in \mathscr{D}_1$ such that for $\langle C, D\rangle \in S$ it is $x \notin P(C, D)$, thus, for $O_1 = \bigcup_{\langle C,D\rangle \in S} P(C, D)$ it is x non $\in O_1$.

Furthermore, $O \in O(\iota_A)$ exists for which $a \in O$, $x \notin O$. Consequently $a \in O \cup O_1$, $x \notin O \cup O_1$.

c) $u(\mathscr{D}) \in \mathscr{S}(A)$.

Proof. Let x, $y \in A$, $x > y$. For $x \neq a \neq y$ the existence of the neighbourhoods with the demanded properties is evident.

Let $y = a$. Then evidently there exists a demanded neighbourhood of the point x. The demanded neighbourhood of the point a can be constructed in the following way. According to (3) a) $S \in \mathscr{D}_1$ exists such that $O_1 = \bigcup_{\langle C,D\rangle \in S} P(C, D) \subset A - [x)$. Let $O \in O(\iota_A)$, $O \subset A - [x)$, $a \in O$ (O exists because $\iota_A \in \mathscr{S}(A)$). Then $O \cup O_1 \subset A - [x)$.

For $x = a$ we can proceed similarly.

Thus following Theorem holds

Theorem 5.1. *Let A be an ordered set and let for suitable points $a \parallel b$ there exist an up or down admissible system $\mathscr{D}_1$ to $\langle a, b\rangle$. Then $u(\mathscr{D}_1) \in \mathscr{S}(A)$ and $u(\mathscr{D}_1) > \iota_A$.*

There holds in a certain sense the converse of the introduced Theorem.

Theorem 5.2. *Let A be an ordered set and $u > \iota_A$, $u \in \mathscr{S}(A)$. Then there exists a pair of points a, b $a \parallel b$ such that there exists a system $\mathscr{D}_1$ up or down admissible to this pair $\langle a, b\rangle$.*

Proof. As the sets of the type $(x]$, $[x)$ for $x \in A$ form a subbasis of closed sets in ι_A, $b \in A$ exists such that $[b)$ or $(b]$ does not belong to $C(u)$. Let the first case occur, in the second one proceeds in a dual way. Thus $a \in A$ exists such that $A - [b)$ fails to be a neighbourhood of a in u i.e. any open set for the topology u containing a has with $[b)$ the non-empty intersection. We shall show that $a \parallel b$. If $a < b$, then the fact that every neighbourhood of a in u contains an element from $[b)$, would be a contradiction with $u \in \mathscr{S}(A)$.

$O(u) \subset O(\iota_A)$. Let $O \neq A$, $O \in O(u)$. Then it is $O = \bigcup_i \bigcap_{k=1}^{n_i} O_{i,k}$, where $O_{i,k} = A - (c_{i,k}]$ or $A - [c_{i,k})$ for suitable $c_{i,k}$. Naturally, the system of elements $c_{i,k}$ is not, in general, uniquely defined in this way. Let $a \in O$. Then at least one i_0 exists such that $\bigcap_{k=1}^{n_{i_0}} O_{i_0,k} \cap [b) \neq 0$. Let $O_{i_0,k} = A - [c_k)$ for $c_1, \ldots, c_m$, $O_{i_0,k} = A - (d_k]$ for $d_{m+1}, \ldots, d_{n_{i_0}}$. Put $C = \{c_1, \ldots, c_m\}$, $D = \{d_{m+1}, \ldots, d_{n_{i_0}}\}$. $\langle C, D\rangle$ is up-admissile pair to b. It is $\bigcap_{k=1}^{n_{i_0}} O_{i_0,k} = P(C, D)$. Let us denote by $S(O)$ the system of all ordered pairs $\langle C, D\rangle$ which can be constructed in the mentioned way. Put $\mathscr{D}_1 = \{S(O) \mid O \in O(u), a \in O, O \neq A\}$. Evidently $\mathscr{D}_1 \neq 0$. We shall show that $\mathscr{D}_1$ is up-admissible system to $\langle a, b\rangle$.

Ad (1) $S(O) \in \mathscr{D}_1$ is evidently a non-empty set.

Ad (2) Let $S_1, \ldots, S_m \in \mathscr{D}_1$. Let $S_1 = S(O_1), \ldots, S_m = S(O_m)$.
Let $O_1 \cap \ldots \cap O_m = O$. $O \in O(u)$, $a \in O$, $O \neq A$, thus $S(O) \in \mathscr{D}_1$.
Put $S = S(O)$. As $O = \bigcup_{i'} \bigcap_{k=1}^{n'_{i'}} O_{i',k} \cap \ldots \cap \bigcup_{i^{(m)}} \bigcap_{k=1}^{n^{(m)}_{i^{(m)}}} O_{i^{(m)},k}$ there exist indices $i'_0 \ldots, i_0^{(m)}$ such that

$$\bigcap_{k=1}^{n'_{i'_0}} O_{i'_0,k} \cap \ldots \cap \bigcap_{k=1}^{n^{(m)}_{i_0^{(m)}}} O_{i_0^{(m)},k} \cap [b) \neq 0,$$

i.e. $\langle C_1, D_1 \rangle \in S_1, \ldots, \langle C_m, D_m \rangle \in S_m$ exist such that $\langle C_1 \cup \ldots \cup C_m, D_1 \cup \ldots \cup D_m \rangle \in S$,

ad (3) a) Let $x > a$. Then there exists $O \in O(u)$ such that $a \in O$, $O \subset A$—$[x)$. Put $S = S(O)$. As $P(C, D) \subset O$ for $\langle C, D \rangle \in S$, it is $P(C, D) \subset A$—$[x)$ for $\langle C, D \rangle \in S$.

(3) b) can be proved dually.

Ad (4). Let $x \neq a$. As u is a T_1-topology, there exists $O \in O(u)$ such that $a \in O$, $x \notin O$. Consequently for $S = S(O)$ it is $\langle C, D \rangle \in S \Rightarrow \Rightarrow P(C, D) \subset O \Rightarrow x$ non $\in P(C, D)$. Thereby the proof of Theorem is accomplished.

Example 5.3. Let $A = B \oplus C$, where B is a two element antichain $\{a, b\}$, C a chain of the type $\omega^* c_1 > c_2 > \ldots$.

Let $\mathscr{A} = \{\langle D, \{b\} \rangle \mid D \subset C$ is a finite set$\}$. Let $\mathscr{D}_1$ be the system of all one point subsets from $\mathscr{A}$. Then $\mathscr{D}_1$ is up-admissible with respect to $\langle a, b \rangle$. Thus ι_A fails to be a maximal element in $\mathscr{S}(A)$.

Theorem 5.4. *Let A be an ordered set. Then ι_A is the greatest element in $\mathscr{S}(A)$ exactly when there exist to every two points a, b $a \parallel b$ two groups of elements $a_1, \ldots, a_n > a$ and $a'_1, a'_2, \ldots, a'_m < a$ such that $[b)$ — $-\bigcup_{i=1}^{n} [a_i)$ and $(b]$ — $\bigcup_{i=1}^{m} (a'_i]$ are finite sets.*

Proof. α) Let $u \in \mathscr{S}(A)$, $u \nleq \iota_A$. A point $b \in A$ exists such that $[b)$, or $(b]$ is not in $C(u)$. Let us consider the first case, the second one is dual to it. Thus A—$[b) \notin O(u)$. So $a \in A$—$[b)$ exists such that A—$[b)$ fails to be a neighbourhood of a in u, i.e. every neighbourhood of the point a in u has a non-empty intersection with $[b)$. As $u \in \mathscr{S}(A)$, it is $a \parallel b$. Let us admit that there exist points $a_1, \ldots, a_n > a$ such that $[b)$ — $-\bigcup_i [a_i) = N$ is a finite set.

Let O_i be an open set in u containing a and contained in A—$[a_i)$. Such a set exists because $u \in \mathscr{S}(A)$. $O = \bigcap_i O_i \subset \bigcap_i (A$—$[a_i)) = A - \bigcup_i [a_i)$ is an open set in u containing a. $[b) \cap O \subset [b) \cap (A - -\bigcup_i [a_i)) = [b) - \bigcup_i [a_i) \cap [b) = N$. N is a finite set, consequently

$[b) \cap O$ is finite, too, and therefore closed. $O' = O - [b) \cap O \in O(u)$ and $a \in O'$. At the same time $O' \cap [b) = 0$ which is a contradiction to the statement that every neighbourhood of a has a non-empty intersection with $[b)$. Consequently N is not finite.

β) Let $a, b \in A$, $a \parallel b$. Let for every group of elements $a_1, \ldots, a_n > a$ be the set $N(a_1, \ldots, a_n) = [b) - \bigcup_i [a_i)$ infinite. Evidently it holds $N(a_1, \ldots, a_n) \cap N(c_1, \ldots, c_k) = N(a_1, \ldots, a_n, c_1, \ldots, c_k)$. Construct on $[b)$ a free filter S such that it contains all $N(a_1, \ldots, a_n)$. Such a filter really exists because $([b) - \{x\}) \cap N(a_1, \ldots, a_n)$ is an infinite set for $x \in [b)$ and every $N(a_1, \ldots, a_n)$. Put $T^* = \{X \mid X \in O(\iota_A),\ a \notin X\} \cup \cup \{X \mid X = Y_1 \cup Y_2,\ Y_1 \in O(\iota_A),\ a \in Y_1,\ Y_2 \in S\}$. T^* be a subbasis of the system of the open sets of a topology u.

1. u is a T_1-topology.

Let $a \neq x \neq y$. $O \in O(\iota_A)$ exists for which $a \notin O, y \notin O$, $x \in O$. At the same time $O \in T^*$.

Let $y \neq x = a$. $O \in O(\iota_A)$, $a \in O$, $y \notin O$ and $Y_2 \in S$ exists such that $y \notin Y_2$, because S is a free filter. Then $O \cup Y_2 \in O(u)$ and y non $\in O \cup Y_2$, $a \in O \cup Y_2$.

2. $u \in \mathscr{S}(A)$.

For $x \neq a \neq y$, $x < y$ the existence of required neighbourhoods is evident from the existence of analogous neighbourhoods in ι_A.

Let $x > a$. In $O(\iota_A)$ X exists such that $x \in X$ and $y \in X \Rightarrow y > a$ or $y \parallel a$. There exists $Y_1 \in O(\iota_A)$ for which $a \in Y_1$ and $y \in Y_1 \Rightarrow y < x$ or $y \parallel x$. In S there is contained as element the set $N(x) = [b) - [x)$ for which $y \in N(x) \Rightarrow y < x$ or $y \parallel x$ holds. Then $Y_1 \cup N(x)$ is the sought neighbourhood of a in u.

Let $a > x$. There exists $Y_1 \in O(\iota_A)$, $a \in Y_1 \subset A - (x]$. Let $Y_2 \in S$. If $y \in Y_2$ existed for which $y < x$ then $a > y$ but $y \in [b)$, consequently $a > b$, which is a contradiction to $a \parallel b$. Then $Y_1 \cup Y_2$ is the sought neighbourhood of a in u.

There exists $O \in O(\iota_A)$, $x \in O$, $O \subset A - [a)$. Then $O \in O(u)$, and it is the sought neighbourhood of x.

3. $u \not\geq \iota_A$.

We shall show that $[b)$ non $\in C(u)$. Let $O \in O(u)$, $a \in O$, $O \neq A$. Then $O = \bigcup_i \bigcap_{k=1}^{n_i} X_{i,k}$, where $X_{i,k} \in T^*$. Thus i_0 exists such that $a \in \bigcap_{k=1}^{n_{i_0}} X_{i_0,k}$, consequently $a \in X_{i_0,k}$ for all k, i.e. $X_{i_0,k} = Y_1^k \cup Y_2^k$, $\bigcap_{k=1}^{n_{i_0}} X_{i_0,k} = \bigcap_{k=1}^{n_{i_0}} [Y_1^k \cup Y_2^k] = [Y_1^1 \cup Y_2^1] \cap \ldots \cap [Y_1^{n_{i_0}} \cup Y_2^{n_{i_0}}] = [Y_1^1 \cap Y_1^2 \cap \ldots \cap Y_1^{n_{i_0}}] \cup \ldots \cup [Y_2^1 \cap Y_2^2 \ldots \cap Y_2^{n_{i_0}}]$. Simultaneously $Y_2^1 \cap \ldots \cap$

$\cap Y_2^{n_{i_0}} \in S$. Therefore $[\bigcap_{k=1}^{n_{i_0}} X_{i_0,k}] \cap [b) \supset Y_2^1 \cap \ldots \cap Y_2^{n_{i_0}} \neq 0$. Then $O \cap [b) \neq 0$, too. For that reason $A - [b)$ fails to be a neighbourhood of a in u, i.e. $[b) \notin C(u)$. Consequently ι_A is not the greatest element in $\mathcal{S}(A)$.

Similarly for $a_1', \ldots, a_m' < a$.

Consequence 5.5. (see [12], p. 44). If A is a chain then ι_A is the greatest topology in $\mathcal{S}(A)$.

Proof. $a, b \in A$, $a \parallel b$ do not exist.

Corollary 5.6. ι_S for a lattice S is the greatest compatible topology exactly when for $a \parallel b$ $Y = [b) - [a \vee b)$, $Z = (b] - (a \wedge b]$ are finite sets.

Proof. Let ι_S be the greatest element in $\mathcal{S}(S)$, $a \parallel b$. Then for a certain finite group of elements $a_1, \ldots, a_n > a$ ($<a$ respectively) it is $[b) - \cup [a_i)$, ($(b] - \cup (a_i]$, respectively) finite. Let us consider the first case. The second case is dual. Let $x \in \cup [a_i) \cap [b)$. Then $x \geqq a \vee b$, i.e. $x \in [a \vee b) \cap [b)$. Hence $\cup [a_i) \cap [b) \subset [a \vee b) \cap [b)$, i.e. $[b) - \cup [a_i) \supset$ $\supset [b) - [a \vee b) = Y$. Thus Y is finite.

If Y is finite, the condition of Theorem 3.4 is fulfilled for $a_1 = a \vee b$. Similarly for Z.

Problem 5.7. Let A be an ordered set. When does the greatest element in $\mathcal{S}(A)$ exist?

For illustration of this problem let us introduce two examples.

Example 5.8. Let $A = B \oplus C$ as in example 5.3. Let u be a topology on A, defined in the following way: every point c_i is isolated and $\{(c_i] - \{b\}\}$ and $\{(c_i] - \{a\}\}$, respectively, are bases of the system of all neighbourhoods of the point a or b, respectively. According to 3.8 and 5.5, if v is the greatest topology in $\mathcal{S}(A)$, then v induces the interval topology on C, i.e. in this case the discrete topology. Hence it follows easily that u is the greatest element in $\mathcal{S}(A)$.

Example 5.9. Let Z denote an Euclidian plane, provided with a Cartesian system of coordinates, ordered in the following way

$$\langle x, y\rangle \leqq \langle u, v\rangle \Rightarrow x = u \quad \text{and} \quad y \leqq v.$$

Let $A \in Z$, p_1, p_2 be distinct closed rays with the end-point A, p_1 parallel (even as the sense is concerned) with the negative ray in the axis x, p_2 directs to the second quadrant, or it lies there. Let us denote the interior of the angle $\sphericalangle p_1Ap_2$ both with the points of the ray p_2 by $\Delta(A, p_1, p_2)$. Let all, in such a way gained sets form a subbasis of open sets for the topology u_l. $u_l \in \mathcal{S}(Z)$. Similarly (p_1 parallel with the positive half axis x and p_2 directed to the first quadrant) a topology u_p is defined, $u_p \in \mathcal{S}(Z)$, too. Let $O \in O(u_l) \cap O(u_p)$. Then there exists a half-plane π directed to the first and the second quadrants such that O contains a dense set in π formed by straight lines parallel to axis x. Consequently the topology $u_l \vee u_p$ (supremum in $\mathscr{B}(Z)$ — see e.g. [9]) does not lie in $\mathscr{S}(Z)$. So $\mathscr{S}(Z)$ has not the greatest element.

Let us finally show that the statement of Theorem 3.8 does not hold for general subsets.

Example 5.10. Let $A = (-\infty, 0] \cup [1, \infty)$, $B = (-\infty, 0] \cup (1, \infty)$, A, B ordered as subsets of real numbers. Every neighbourhood in ι_B O_1 of the point 0 in B intersects $(1, \infty)$. Thus, for u^* on A for which $u^* / B = u$, $u^* \notin \mathscr{S}(A)$ holds because a neighbourhood O in u^* with the required properties does not exist for the couple 0 and 1.

REFERENCES

[1] Birkhoff G., *A new interval topology for dually directed sets.* Univ. Nac. Tucuman. Rev. Ser. A 14 (1962), 325—331.

[2] Bourbaki N., *Topologie générale, Paris, deuxième édition.*

[3] Flachsmayer J., *Einige topologische Fragen in der Theorie der Booleschen Algebren,* Archiv der Mathematik, XVI (1965), 25—33.

[4] Floyd E. E., *Boolean algebras with pathological order topology.* Pac. Journal, 5 (1955), 687—689.

[5] Frink O., *Topology in lattices,* Transactions of A.M.S., 51 (1942), 569—582.

[6] Frink O., *Ideals in partially ordered sets,* Am. Math. Monthly 61 (1954), 223 to 234.

[7] Kolibiar M., *Bemerkungen über Intervalltopologie in halbgeordneten Mengen,* Gen. Topology and its Relations to Modern Analysis and Algebra, Prague, 1962, 252—253.

[8] Naito T., *Lattices with P-ideal topologies,* Tôhoku Math. Journal, 12 (1960), 235—261.

[9] Sekanina M., *Sistěmy topologij na dannom množestvě,* Czech. Math. Journal, 15 (1965), 9—29.

[10] Szász G., *Einführung in die Verbandstheorie,* Leipzig, 1962.

[11] Wolk E. S., *Order-compatible topologies on a partially ordered set,* Proc. of Am. Math. Soc., 9 (1958), 524—529.

[12] Eilenberg S., *Ordered topological spaces,* Am. J. of Math., LXIII (1941), 39—46.

[13] Rennie B. C., *Lattices,* Proc. of the London Math. Soc., 52 (1951), 386—400.

[14] Nachhin L., *Topology and order, Nostrand,* Toronto, 1965.