Peter P. Klein

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INCLUDING EIGENVALUES OF THE PLANE ORR-SOMMERFELD PROBLEM

PETER P. KLEIN, Clausthal

Summary. In an earlier paper [5] a method for eigenvalue inclusion using a Gerschgorin type theory originating from Donnelly [2] was applied to the plane Orr-Sommerfeld problem in the case of a pure Poiseuille flow. In this paper the same method will be used to deal with the plane Orr-Sommerfeld problem where the basic flow is a linear combination of Poiseuille and Couette flow. Potter [6] has treated this case before with an approximative method.

Keywords: plane Orr-Sommerfeld problem, combination of Couette flow and Poiseuille flow, infinite dimensional matrix eigenvalue problem, inclusion of eigenvalues using a generalization of Gerschgorin's method

AMS classification: 65L15

1. THE ABSTRACT EIGENVALUE PROBLEM

A generalized linear eigenvalue problem of the form

$$(A_0 + \varrho K)u = \lambda Lu, \quad u \in D(A_0) \tag{1}$$

is considered, where ϱ is a given real number and A_0, K and L are linear differential operators, the domains of which are linear subspaces of a complex Hilbert space $(H, (\cdot,\cdot))$ with $D(A_0) \subset D(L)$, $D(A_0) \subset D(K)$, $\overline{D(A_0)} = H$. Let A_0 be a symmetric and positive definite operator. H_0 shall denote the completion of $D(A_0)$ with respect to the energy norm defined by the operator A_0. There exists a mapping $G\colon H \to H_0$ which is symmetric and positive and the inverse of which satisfies $A_0 \subset G^{-1}$. Applying G to eigenvalue problem (1) leads to

$$(I + \varrho GK)u = \lambda GLu, \quad u \in H_0. \tag{2}$$

Under the assumptions

(A1) $\lambda = 0$ is no eigenvalue of eigenvalue problem (2)

(A2) G, GK, GL are compact operators on H_0

eigenvalue problem (2) has a countably infinite number of eigenvalues with no accumulation point in the finite part of the complex plane.

In order to deal with eigenvalue problem (2) numerically it is transformed into an eigenvalue problem with infinite dimensional matrices. It is assumed that the eigenvalue problem

$$A_0 f = \lambda L_0 f, \quad f \in D(A_0) \tag{3}$$

with another symmetric and positive definite operator L_0 satisfying $D(A_0) \subset D(L_0)$ can be solved for eigenvalues λ_j, $j \in \mathbb{N}$ and eigenfunctions f_j, $j \in \mathbb{N}$ with $(L_0 f_j, f_k) = \delta_{jk}$, $j, k \in \mathbb{N}$. If the system of eigenfunctions of (3) is complete in H_0, eigenvalue problem (2) is equivalent to the following eigenvalue problem in Hilbert sequence space l_2

$$(I + \varrho B)v = \lambda C v, \quad v \in l_2 \tag{4}$$

with infinite matrices, I denoting the identity and

$$B = (b_{l,k}) = \left(\frac{(K f_k, f_l)}{\sqrt{\lambda_k}\sqrt{\lambda_l}} \right)_{l,k \in \mathbb{N}}, \qquad C = (c_{l,k}) = \left(\frac{(L f_k, f_l)}{\sqrt{\lambda_k}\sqrt{\lambda_l}} \right)_{l,k \in \mathbb{N}}.$$

The special case where $L = L_0$ leads to

$$C = \Lambda^{-1} = \operatorname{diag}\left(\frac{1}{\lambda_j} \right)_{j \in \mathbb{N}},$$

so that matrix eigenvalue problem (4) appears in the following form

$$(I + \varrho B)v = \lambda \Lambda^{-1} v, \quad v \in l_2 \tag{5}$$

with a diagonal matrix on the right hand side. In order to derive eigenvalue inclusions for (5) the following assumption is made

$$\sum_{j,k=1; j \neq k}^{\infty} |b_{jk}| < \infty, \quad \sup_{j \in \mathbb{N}} |b_{jj}| < \infty. \tag{6}$$

After a suitable transformation of a finite dimensional subsection of (5) employing the QZ-Algorithm, inclusion sets for the eigenvalues of this problem can be obtained using a generalization of the Gerschgorin theorem (cf. [2], [4]).

2. Application to the plane Orr-Sommerfeld problem

The plane Orr-Sommerfeld problem reads (cf. [1], [3]):

$$(7)\quad (-D^2+a^2)^2y+\mathrm{i}aR[U(-D^2+a^2)+U'']y=\lambda(-D^2+a^2)y \quad \text{in} \quad -1\leqslant x\leqslant 1$$

with boundary conditions $y(\pm1)=Dy(\pm1)=0$ and the denotations $D=\mathrm{d}/\mathrm{d}x$, $\mathrm{i}=\sqrt{-1}$, a real valued function $U\in C^2([-1,1])$, called basic flow, and positive real numbers a, R (a wave number, R Reynolds number). If λ denotes an eigenvalue of (7), the sign of $\mathrm{Re}\,\lambda$ depending on the parameters a and R shall be considered. A pair of parameters (a,R) shall belong to a domain of instability of the Orr-Sommerfeld problem, if there exists an eigenvalue λ of (7) with $\mathrm{Re}\,\lambda<0$. On the other hand a pair of parameters (a,R) shall belong to a domain of stability of the Orr-Sommerfeld problem, if all eigenvalues of (7) satisfy $\mathrm{Re}\,\lambda\geqslant 0$. Using the denotations:

$$A_0f=(-D^2+a^2)^2f,\quad D(A_0)=\{f\in C^4([-1,1])\mid f(\pm1)=Df(\pm1)=0\}$$

$$\varrho=R,\quad Kf=\mathrm{i}a[U(-D^2+a^2)+U'']f,\quad D(K)=C^2([-1,1])$$

$$Lf=L_0f=(-D^2+a^2)f,\quad D(L_0)=\{f\in C^2([-1,1])\mid f(\pm1)=0\}$$

the Orr-Sommerfeld problem (7) fits into the framework given above (see [5]). In the special case $U\equiv0$ the Orr-Sommerfeld problem (7) turns into an eigenvalue problem with constant coefficients

$$(8)\qquad (-D^2+a^2)^2y=\lambda(-D^2+a^2)y \quad \text{in} \quad -1\leqslant x\leqslant 1,$$

the eigenvalues and eigenfunctions of which can be stated explicitly after the solution of transcendental equations. The eigenfunctions of this problem being properly normalized are used to calculate the matrix elements of matrix B in eigenvalue problem (5).

In case the basic flow U is a linear combination of plane Poiseuille flow U_p and plane Couette flow U_c

$$U(x)=c_1U_p+c_2U_c,\quad U_p(x)=1-x^2,\quad U_c(x)=x;\quad c_1,c_2\in\mathbb{R}$$

M.C.Potter stated in [6] that the eigenvalues of the Orr-Sommerfeld problem are not depending on the sign of the Couette parameter c_2. This conclusion, which M. C. Potter drew from his numerical calculations, may be deduced theoretically. As the basic flow U is a linear combination of U_p and U_c, likewise the operator K may be represented as a linear combination of operators K_p and K_c

$$Kf = c_1 K_p f + c_2 K_c f \quad \text{with} \quad \begin{cases} K_p f = \mathrm{i}a[U_p(-D^2 + a^2) + U_p''] f \\ K_c f = \mathrm{i}a[U_c(-D^2 + a^2) + U_c''] f \end{cases}$$

for $f \in D(K)$. Let y be an eigenfunction corresponding to the eigenvalue λ of the Orr-Sommerfeld problem with basic flow $U = c_1 U_p + c_2 U_c$ and let $y = y_e + y_o$, where y_e denotes the even and y_o the odd part of y.

$$[A_0 + \varrho(c_1 K_p + c_2 K_c)](y_e + y_o) = \lambda L(y_e + y_o)$$

If the symmetry properties of the operators A_0, L, K_p and K_c are taken into account, A_0, L, K_p are preserving symmetry, i.e., mapping even functions into even functions and odd functions into odd functions, K_c is changing symmetry, i.e., mapping even functions into odd functions and vice versa and if furthermore even and odd parts in the above equation are separated the following two equations result:

$$A_0 y_e + \varrho(c_1 K_p y_e + c_2 K_c y_o) = \lambda L y_e$$

$$A_0 y_o + \varrho(c_1 K_p y_o + c_2 K_c y_e) = \lambda L y_o .$$

These two equations are equivalent to

$$A_0 y_e + \varrho(c_1 K_p y_e - c_2 K_c(-y_o)) = \lambda L y_e$$

$$A_0(-y_o) + \varrho(c_1 K_p(-y_o) - c_2 K_c y_e) = \lambda L(-y_o).$$

The addition of both of these equations leads to

$$[A_0 + \varrho(c_1 K_p - c_2 K_c)](y_e - y_o) = \lambda L(y_e - y_o),$$

which means that $\tilde{y} = y_e - y_o$ is an eigenfunction corresponding to the eigenvalue λ of the Orr-Sommerfeld problem with basic flow $\tilde{U} = c_1 U_p - c_2 U_c$.

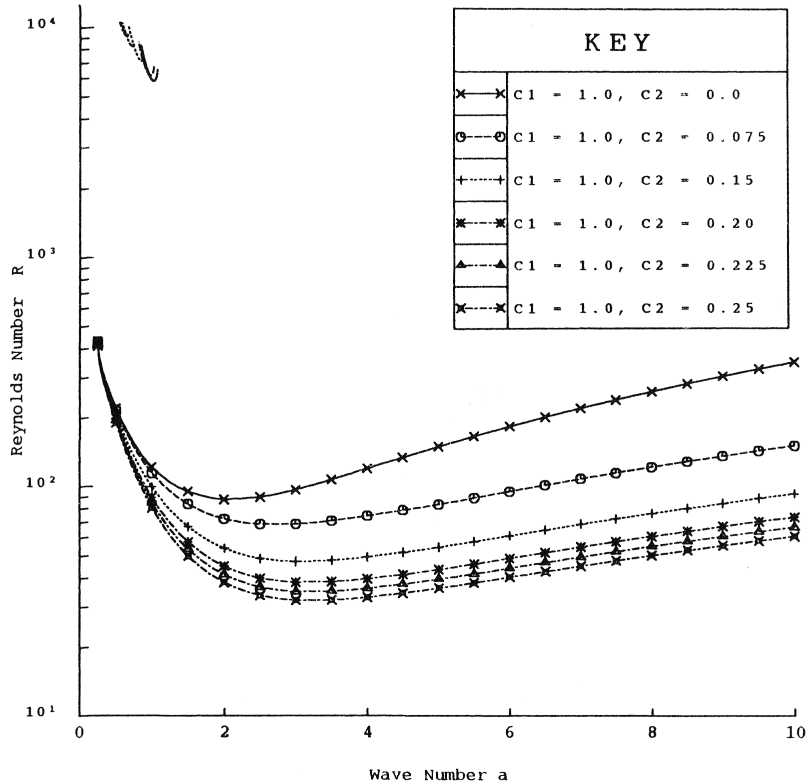

Fig. 1. Stability/instability domains for Poiseuille-Couette flow

3. Numerical results

The numerical results shown in Figures 1 and 2 were obtained using finite dimensional subproblems of (5) of order $N = 160$. Employing the sufficient condition for stability stated in [5] the curves in the lower part of Figure 1 were obtained. The region below each curve is belonging to a domain of stability for the parameters of Poiseuille flow and Couette flow indicated.

The Poiseuille parameter c_1 is kept to be 1.0 and the Couette parameter c_2 ranges from 0.0 to 0.25. Increasing the Couette parameter c_2 the curves obtained are moving downward, indicating that the domain of stability guaranteed by the sufficient condition used is becoming smaller.

Figure 2 is showing a magnification of the upper part of Figure 1 with a linear scale on the y-axis. For the same combinations of Poiseuille and Couette parameters as before boundary curves to domains of instability were determined, using the criterion already employed in [5]: A pair (a, R) belongs to a domain of instability, if for (a, R)

the eigenvalue inclusion of eigenvalue problem (5) yields a Gerschgorin disk being isolated from the other inclusion sets of eigenvalues and lying completely in the left half plane ($\mathrm{Re}(z) < 0$), so that the included eigenvalue has a negative real part. This works well as long as the radius of the enclosing disk is neglegibly small, of magnitude 10^{-4} or 10^{-5}. Pursuing the right branch of each boundary curve further the radius of the enclosing disk is increasing however. In order to fulfill the above criterion still one would have to go deeper into the region of instability, meaning that the curve obtained would begin to turn backward slightly.

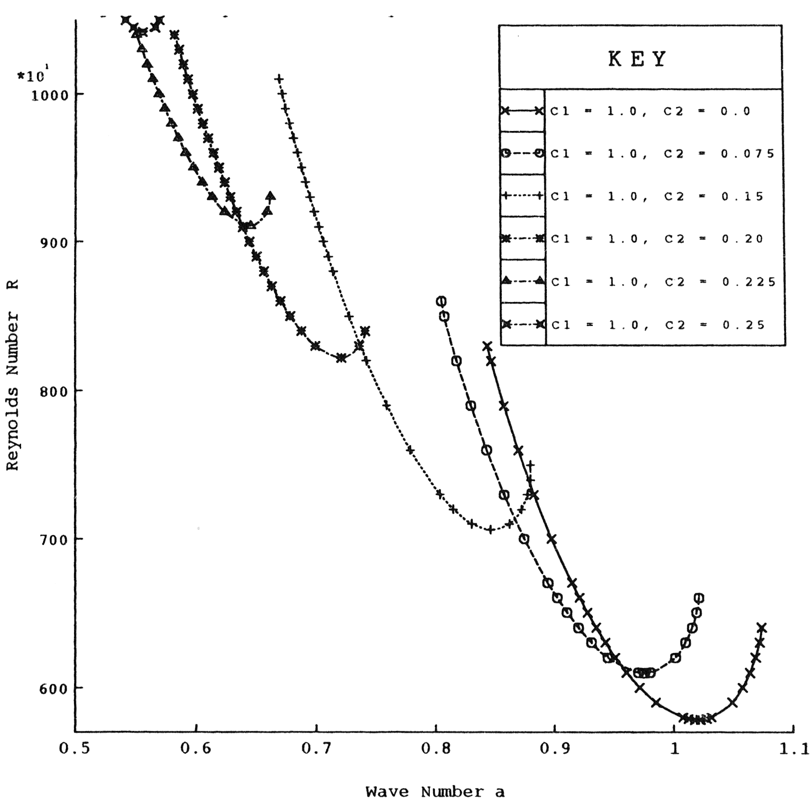

Fig. 2. Boundary to instability domain for Poiseuille-Couette flow

The following table shows the approximate minima of the different boundary curves in Figure 2 together with the corresponding combination of Poiseuille and

Couette parameters.

c_1	c_2	$a_{\min}$	$R_{\min}$
1.0	0.0	1.0198	5784
1.0	0.075	0.97573	6097
1.0	0.15	0.8474	7058
1.0	0.20	0.72229	8220
1.0	0.225	0.64618	9107
1.0	0.25	0.555925	10418

Table 1: Approximate minima of boundary curves in Figure 2 and corresponding Couette parameters c_2

Looking at Table 1 the wave number $a_{\min}$ decreases monotonously as the Couette parameter c_2 increases and the Reynolds number $R_{\min}$ increases monotonously as the Couette parameter c_2 increases. In order to compare the results in Table 1 here with those in Table 4 of [7], one has to observe, that the basic flow there is defined as

$$U(x) = \frac{3}{2}(1 - x^2) + u_w x = \frac{3}{2}\left[(1 - x^2) + \frac{2}{3}u_w x\right].$$

Thus Couette parameter c_2 here should correspond to $\tilde{u}_w = \frac{2}{3}u_w$ there, $a_{\min}$ here should correspond to α_{cr} there and $R_{\min}$ here should correspond to $\tilde{R}_{cr} = \frac{3}{2}R_{cr}$ there. In the case of pure Poiseuille flow ($c_2 = u_w = 0$) both results are in agreement, but in Table 4 of [7] R_{cr} does not increase monotonously when u_w does.

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Author's address: *Peter P. Klein*, Rechenzentrum der TU Clausthal, Erzstraße 51, D-3392 Clausthal-Zellerfeld, Federal Republic of Germany.