Jan Rataj; Ivan Saxl; Karol Pelikán

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CONVERGENCE OF RANDOMLY OSCILLATING POINT PATTERNS TO THE POISSON POINT PROCESS

JAN RATAJ, IVAN SAXL, Praha, KAROL PELIKÁN, Košice

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Summary. Oscillating point patterns are point processes derived from a locally finite set in a finite dimensional space by i.i.d. random oscillation of individual points. An upper and lower bound for the variation distance of the oscillating point pattern from the limit stationary Poisson process is established. As a consequence, the true order of the convergence rate in variation norm for the special case of isotropic Gaussian oscillations applied to the regular cubic net is found. To illustrate these theoretical results, simulated planar structures are compared with the Poisson point process by the quadrat count and distance methods.

Keywords: Poisson point process, asymptotically uniform distributions, weak convergence, variation distance, rate of convergence, Poisson hypothesis testing, distance method, quadrat count method

AMS classification: 60G55, 60D05

1. INTRODUCTION

In many natural situations, the aggregation proper to the Poisson point process is not permissible. Therefore, hard core point processes or, more generally, processes derived from the Poisson point process by dependent thinning (Stoyan et al. [11], Diggle [4]) are often used. Unfortunately, mathematical tractability of such processes is rather limited, which turns the attention to the extremal case of hard-core processes, namely to lattice point processes, or more generally, to deterministic point patterns. Their properties can be calculated relatively simply (cf. Persson [8], Holgate [5], Saxl and Rataj [9]), and the passage to less idealized but still tractable models can be realized by introducing random lattice faults (Brown and Holgate [2], Saxl and Rataj [9]) or by a superposition of a lattice and Poisson processes (Diggle [3]).

Whereas some faulted lattices retain their hard-core character even when substantially randomized (e.g. by very severe random thinning), the introduction of other faults can lead to the restoration of nearly complete or locally complete spatial randomness. An example of this case are patterns in which the points of the originally regular structure are allowed to oscillate independently randomly in space according to a prescribed probabilistic rule. The present paper is devoted to the estimation of the distance of such an oscillating point pattern from the Poisson point process.

If the probability distribution of the oscillation disperses in the limit over the whole space, then the oscillating point pattern converges weakly as well as in the variation norm to the stationary poisson point process. These facts are variations on more general results displayed in Kallenberg [6] and Matthes et al. [7]. To find upper bounds for the variation distance from the Poisson point process, Stein's method [10] has been successfully applied by Barbour [1]. In this paper, also a lower bound is found and the results are applied to determine the true order of the convergence rate in the special case of isotropic Gaussian oscillations of points of a regular cubic point set. The theoretical results are then illustrated in Section 5 by examining simulated planar point sets of this type.

2. Convergence to the Poisson point process

Let $\mathsf{E} = \mathbf{R}^d$ be the d-dimensional Euclidean vector space and $\mathscr{B}$, $\mathscr{B}_0$ the systems of all Borel and bounded Borel subsets of E, respectively. Let m be the d-dimensional Lebesgue measure on E, $m_B = m(B)^{-1}m\lfloor B$ the uniform probability distribution over $B \in \mathscr{B}$ with $0 < m(B) < \infty$ and δ_x the Dirac probability measure concentrated in x. Let $\|.\|$ denote the variation norm of signed measures and $\star$ the operation of convolution of measures.

Let $(\mathscr{M}, \mathfrak{M})$ be the measurable space of integer-valued Radon measures on E with the σ-algebra $\mathfrak{M}$ generated by the sets $\{\xi \in \mathscr{M} : \xi(B) = k\}$, $B \in \mathscr{B}_0$, $k = 0, 1, \ldots$. The support $S(\xi)$ of $\xi \in \mathscr{M}$ is a locally finite subset of E. A measure $\xi \in \mathscr{M}$ is called *simple* if $\xi(\{x\}) \leqslant 1$ for all $x \in \mathsf{E}$. The mapping $\xi \mapsto S(\xi)$ defines a unique correspondence between simple integer-valued Radon measures and locally finite sets in E. $\mathscr{M}$ can be endowed with the vague topology to become a Polish space, $\mathfrak{M}$ becoming the Borel σ-algebra [6].

Under a *point process* we shall understand a measurable mapping X from a probability space $(\Omega, \mathscr{A}, Pr)$ into $(\mathscr{M}, \mathfrak{M})$. The probability distribution of X will be denoted by $\mathscr{D}(X)$. The convergence in distribution of point processes should be understood as the vague (or, equivalently, weak) convergence of their probability distributions. For any Radon measure Λ on E, P_Λ will be the Poisson point process of the intensity measure Λ.

Given any measure $\xi \in \mathscr{M}$ and any probability distribution μ on E we define the point process

$$T_\mu(\xi) = \int \delta_{x+Y_x}\xi(\mathrm{d}x), \tag{1}$$

where $\{Y_x : x \in \mathsf{E}\}$ is a family of i.i.d. random vectors with distribution μ. Note that the support of $T_\mu(\xi)$ is obtained from $S(\xi)$ by shifting independently randomly each point according to the distribution μ. The aim of this paper is to investigate the quality of approximation of $T_\mu(\xi)$ by a Poisson process.

Following Matthes et al. [7] (Chap. 5.2) we shall say that a family $\{\mu_r : r > 0\}$ of probability distributions on E is *asymptotically uniform* (abbr. a.u.) if

$$\lim_{r\to\infty} \|\mu_r - \mu_r \star \delta_a\| = 0 \quad \text{for any } a \in \mathsf{E}.$$

The intuitive meaning of asymptotical uniformity is that of the mass being dispersed over the whole space as $r \to \infty$. As a consequence of the asymptotical uniformity we get (see Matthes et al. [7], Sect. 5.2.15)

$$\lim_{r\to\infty} \sup_{x\in\mathsf{E}} \mu_r(B - x) = 0 \quad \text{for any } B \in \mathscr{B}_0. \tag{2}$$

Example 1. Let $\{C_r : r > 0\}$ be a family of bounded Borel sets such that

$$\lim_{r\to\infty} m_{C_r}(C_r - a) = 1 \quad \text{for any } a \in \mathsf{E}. \tag{3}$$

Then the family $\{m_{C_r}\}$ is a.u. (see Matthes et al [7], Sect. 5.2.1). Remark that condition (3) is fulfilled e.g. for any family of convex bodies with inradii growing to infinity.

Example 2. The family $\{\gamma_r : r > 0\}$ of d-dimensional centred normal distributions with variances $r^2 I$ is a.u. (see Matthes et al. [7], Sect. 5.2.13).

Let $\{\mu_r : r > 0\}$ be an a.u. family of probability distributions on E, $\{Y_x^r : x \in \mathsf{E}\}$ a family of i.i.d. random vectors with distribution μ_r for each $r > 0$, and let $\xi \in \mathscr{M}$ be simple. Then the family of point processes

$$\{\beta_{r,a} = \delta_{a+Y_a^r} : r > 0,\ a \in S(\xi)\}$$

forms a null-array in the sense of Kallenberg [6], Chap. 6, since $\beta_{r,a}$ are independent for fixed r and

$$\lim_{r\to\infty} \sup_{a\in S(\xi)} Pr\{\beta_{r,a}(B) > 0\} = 0$$

by (2) (more exactly, Kallenberg's null array will be obtained when choosing any sequence $r_n \to \infty$ and realizing that the set $S(\xi)$ is countable). Corollary 7.5 in Kallenberg [6] states that the family $T_{\mu_r}(\xi)$ converges in distribution to $P_{\lambda m}$ as $r \to \infty$ for a given $\lambda > 0$ iff

$$\lim_{r\to\infty} \sum_{a\in S(x)} Pr\{\beta_{r,a}(B) > 0\} = \lambda m(B), \quad B \in \mathscr{B}_0,$$

which is equivalent to

$$\lim_{r\to\infty} \int \mu_r(B-a)\xi(\mathrm{d}x) = \lambda m(B), \quad B \in \mathscr{B}_0. \tag{4}$$

A natural condition on ξ ensuring (4) for any a.u. family $\{\mu_r\}$ can be found. We shall call a measure $\xi \in \mathscr{M}$ *homogeneous* if there is $\lambda \geqslant 0$ such that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{E}} \left|(\omega_d r^d)^{-1}\xi(\mathrm{B}(x,r)) - \lambda\right| = 0,$$

where $\mathrm{B}(x,r)$ is the closed d-ball of radius r centred at x. The number λ will be called the *sample intensity* of ξ. The following result is a consequence of Matthes et al. [7], Proposition 6.5.9.

Lemma 1. *For any $\xi \in \mathscr{M}$ and $\lambda > 0$, ξ is homogeneous with sample intensity λ if and only if (4) holds for any a.u. family $\{\mu_r\}$ of probability distributions on $\mathbb{E}$.*

As an immediate consequence we obtain

Theorem A. *Let $\xi \in \mathscr{M}$ be simple and homogeneous with sample intensity $\lambda \in (0,\infty)$ and let $\{\mu_r : r > 0\}$ be an a.u. family of probability distributions on $\mathbb{E}$. Then the family $T_{\mu_r}(\xi)$ converges in distribution to $P_{\lambda m}$ as $r \to \infty$.*

A substantially stronger result replacing the weak convergence of distributions by the convergence in variation norm of restrictions to a bounded subset can be obtained as a special case of Matthes et al. [7], Theorem 7.4.1.

Theorem B. *Under the assumptions of Theorem A, the relation*

$$\lim_{r\to\infty} \left\|\mathscr{D}(T_{\mu_r}(\xi)\lfloor B) - \mathscr{D}(P_{\lambda m}\lfloor B)\right\| = 0$$

holds for any $B \in \mathscr{B}_0$.

Remark. If X is a point process, the set $T_\mu(X)$ can be also defined by (1), assuming $\{Y_x\}$ to be independent of X. Theorem B remains valid for $T_{\mu_r}(X)$ if X is stationary and ergodic with intensity λ.

3. Rate of convergence

For estimating the rate of convergence of $T\mu_r(\xi)$ in the variation norm, Stein's method developed originally for the rate of convergence of sums of random variables to the normal distribution [10] can be used. Let $\{\mu_r : r > 0\}$ and ξ be as in Theorem A, and for given $r > 0$ and $B \in \mathscr{B}_0$ set

$$\sigma_r(B) = \sup_{a \in S(\xi)} \mu_r(B - a),$$

$$\Lambda_r(B) = (\mu_r \star \xi)(B) = \int \mu_r(B - a)\xi(\mathrm{d}a),$$

$$\eta_r(B) = \Lambda_r(B) - \lambda m(B)$$

and

$$W_r(B) = \int \left(\mu_r(B - a)\right)^2 \xi(\mathrm{d}a).$$

By (2) and Lemma 1 we know that both $\sigma_r(B)$ and $\eta_r(B)$ tend to zero with r tending to infinity, and the obvious inequality

$$W_r(B) \leqslant \Lambda_r(B)\sigma_r(B) \tag{5}$$

implies also $W_r(B) \to 0$, $r \to \infty$. Note that Λ_r is a Radon measure on E.

Barbour [1] has applied Stein's method to obtain estimates of the rate of convergence to the multivariate Poisson distribution. From his result (Theorem 1) it follows that

$$\left\| \mathscr{D}(T_{\mu_r}(\xi)\lfloor B) - \mathscr{D}(P_{\Lambda_r}\lfloor B) \right\| \leqslant W_r(B). \tag{6}$$

To obtain an upper bound for the variation distance expressed in Theorem B, it remains to estimate the distance between two Poisson processes. Matthes et al. [7], Proposition 1.6.26, have proved that

$$\left\| \mathscr{D}(P_\Lambda) - \mathscr{D}(P_{\Lambda'}) \right\| \leqslant 2\|\Lambda - \Lambda'\|$$

for any finite Borel measures Λ, Λ' on $\mathbf{E}$. Hence we get

$$\left\|\mathscr{D}(P_{\Lambda_r}\lfloor B) - \mathscr{D}(P_{\lambda m}\lfloor B)\right\| \leqslant 4 \sup_{C\subseteq B} |\eta_r(C)|. \tag{7}$$

By a minor modification of the proof of Proposition 6.5.9 in Matthes et al. [7], it can be shown that the right hand side of (7) tends to zero. Thus (6) and (7) give a reasonable upper bound for the variation distance between $T\mu_r(\xi)$ and $P_{\lambda m}$ restricted to B.

To obtain a lower bound for the variation distance, consider the "void probabilities"

$$H_r(B) = Pr\{T_{\mu_r}(\xi)(B) = 0\}$$

in order to compare them with the Poisson void probabilities $\exp\left(-\lambda m(B)\right)$. If $\left\{Y_a^r : a \in S(\xi)\right\}$ is a family of i.i.d. random vectors with distribution μ_r, we have by (1)

$$\begin{aligned} H_r(B) &= Pr\left\{a + Y_a^r \notin B \text{ for all } a \in S(\xi)\right\} \\ &= Pr\Big(\bigcap_{a\in S(\xi)} \{Y_a^r \notin B - a\}\Big) \\ &= \prod_{a\in S(\xi)} \left(1 - \mu_r(B - a)\right) \end{aligned}$$

and, using the Taylor expansion of the logarithm, we get

$$\begin{aligned} \log H_r(B) &= \int \log\left(1 - \mu_r(B - a)\right)\xi(\mathrm{d}a) \\ &= -\Lambda_r(B) - \frac{1}{2}W_r(B) + o\big(W_r(B)\big) \end{aligned}$$

and, consequently,

$$\begin{aligned} &\left|H_r(B) - \exp\left(-\lambda m(B)\right)\right| = \\ &\qquad = \exp\left(-\lambda m(B)\right)\left|\log H_r(B) + \lambda m(B)\right|\left(1 + o(1)\right) \\ &\qquad = \exp\left(-\lambda m(B)\right)\left|\Lambda_r(B) + \tfrac{1}{2}W_r(B) + o\big(W_r(B)\big) - \lambda m(B)\right|\left(1 + o(1)\right) \\ &\qquad = \exp\left(-\lambda m(B)\right)\left\{\left|\eta_r(B) + \tfrac{1}{2}W_r(B)\right| + o\big(|\eta_r(B)| + W_r(B)\big)\right\}, \quad r \to \infty. \end{aligned}$$

The upper and lower bounds for the variation distance

$$\Delta_r(B) = \left\|\mathscr{D}(T_{\mu_r}(\xi)\lfloor B) - \mathscr{D}(P_{\lambda m}\lfloor B)\right\|$$

are summarized in the following theorem.

Theorem 1. *Let $\{\mu_r : r > 0\}$ and ξ be as in Theorems A, B. Then for any $B \in \mathscr{B}$*

(i) $\Delta_r(B) \leqslant W_r(B) + 4 \sup_{C \subseteq B} |\eta_r(C)|$,

(ii) $\Delta_r(B) \geqslant \exp\left(-\lambda m(B)\right)\left\{|\eta_r(B) + \frac{1}{2}W_r(B)| + o\left(|\eta_r(B)| + W_r(B)\right)\right\}$, $r \to \infty$.

To conclude this section, we will show that Theorem 1 gives the true order of the rate of convergence for the special case of Gaussian distributions γ_r from Example 2.

Theorem 2. *Let $\xi = \xi_0$ be the simple integer-valued measure supported by $\mathbf{Z}^d$ and let $\mu_r = \gamma_r$ be the centred normal distribution of variance $r^2 I$, $r > 0$. Then for any $B \in \mathscr{B}_0$ with $m(B) > 0$ there exist constants $c_1(B)$, $c_2(B) > 0$ such that*

$$c_1(B) \leqslant \liminf_{r \to \infty} r^d \Delta_r(B) \leqslant \limsup_{r \to \infty} r^d \Delta_r(B) \leqslant c_2(B).$$

Theorem 2 can be proved using the estimates summarized in the following lemma. Let ω_d be the volume of the unit d-ball in $\mathbf{E}$ and $b = \frac{1}{2}\operatorname{diam}(B)$.

Lemma 2. *Under the assumptions of Theorem 2*

(i) $\limsup_{r \to \infty} r^d \sigma_r(B) \leqslant \omega_d (2\pi)^{-d/2} b^d$,

(ii) $\limsup_{r \to \infty} r^d W_r(B) \leqslant \omega_d (2\pi)^{-d/2} m(B) b^d$,

(iii) $\liminf_{r \to \infty} r^d W_r(B) \geqslant \omega_d (2\pi)^{-d} e^{-1} m(B)^2$,

(iv) $\sup_{C \subseteq B} |\eta_r(C)| = O\left(\exp(-2\pi^2 r^2)\right)$, $r \to \infty$.

Proof. (i) Since $\varphi_r(x) = (2\pi)^{-d/2} r^{-d} \exp\left(-\frac{1}{2}|x|^2\right)$ is the density of γ_r w.r.t. m, we have

$$\begin{aligned}\sigma_r(B) \leqslant \gamma_r\left(\mathrm{B}(o, b)\right) &= \int_{\mathrm{B}(o,b)} \varphi_r(x) m(\mathrm{d}x) \\ &= d\omega_d (2\pi)^{-d/2} r^{-d} \int_0^b \varrho^{d-1} \exp\left(-\frac{\varrho^2}{2r^2}\right) \mathrm{d}\varrho \\ &= d\omega_d (2\pi)^{-d/2} r^{-d} \left(\frac{b^d}{d} + o(1)\right),\end{aligned}$$

which implies (i). Further, (ii) immediately follows by (i), using (5) and Lemma 1.

(iii). Without loss of generality it can be assumed that $B \cap [0,1]^d \neq \emptyset$ (the set B can be shifted by any $k \in \mathbf{Z}^d$ without changing the value of $W_r(B)$). Then we have for any $x \in \mathbf{E}$ with $|x| \leqslant r$

$$\begin{aligned}\gamma_r(B - x) &\geqslant m(B) \inf\left\{\varphi_r(x) : |x| \leqslant r + 2b + \sqrt{d}\right\} \\ &= (2\pi)^{-d/2} r^{-d} \exp\left(-\frac{(r + 2b + \sqrt{d})^2}{2r^2}\right) m(B).\end{aligned}$$

On the other hand, one can easily check that there are at least $m(\mathrm{B}(o, r-\sqrt{d}))=\omega_d(r-\sqrt{d})^d$ points $k \in \mathbb{Z}^d$ with $|k| \leqslant r$. Putting these two facts together, we obtain

$$\begin{aligned}
W_r(B) &\geqslant \sum_{|k|\leqslant r} \left(\gamma_r(B-k)\right)^2 \\
&\geqslant \omega_d(r-\sqrt{d})^d(2\pi)^{-d}r^{-2d}\exp\Big(-\frac{(r+2b+\sqrt{d})^2}{r^2}\Big)m(B)^2 \\
&= \omega_d(2\pi)^{-d}r^{-d}\Big(1-\frac{\sqrt{d}}{r}\Big)^d \exp\Big(-\big(1+r^{-1}(2b+\sqrt{d})\big)^2\Big)m(B)^2 \\
&= \omega_d(2\pi)^{-d}r^{-d}e^{-1}m(B)^2+o(r^{-d}), \quad r\to\infty,
\end{aligned}$$

which proves (iii).

(iv). For any $C \subseteq B$ we have (all summations will be taken over $\mathbb{Z}^d$, χ_C is the characteristic function of the set C)

$$\begin{aligned}
\Lambda_r(C) &= \sum_k \gamma_r(C-k) = \sum_k \int \varphi_r(x+k)\chi_C(x)\,\mathrm{d}x \\
&= \sum_k\sum_l \int_{[0,1]^d} \varphi_r(y+k+l)\chi_C(y+l)\,\mathrm{d}y \\
&= \int_{[0,1]^d} \Big(\sum_n \varphi_r(y+n)\Big)\Big(\sum_l \chi_C(y+l)\Big)\,\mathrm{d}y.
\end{aligned}$$

Let $F\varphi_r$, $F\chi_C$ be the Fourier transforms of φ_r, χ_C, respectively. Then $F\varphi_r(-2\pi k)$, $F\chi_C(-2\pi k)$ are the k-th Fourier coefficients of the 1-periodic functions $\sum_n \varphi_r(y+n)$, $\sum_l \chi_C(y+l)$, respectively ($k \in \mathbb{Z}^d$), and the Parseval identity yields

$$\Lambda_r(C) = \sum_k F\varphi_r(2\pi k)F\chi_C(2\pi k).$$

Using the fact that $F\chi_C(o)=m(C)$ and $F\varphi_r(y)=\exp\left(-\frac{1}{2}r^2|y|^2\right)$ we have

$$\eta_r(C) = \sum_{k\neq o} F\chi_C(2\pi k)\exp(-2\pi^2r^2|k|^2).$$

The relation

$$\exp(-2\pi^2r^2|k|^2) = \exp(-2\pi^2r^2)\exp\left(-2\pi^2r^2(|k|^2-1)\right)$$

implies that

$$\exp(-2\pi^2r^2|k|^2) \leqslant \exp(-2\pi^2r^2)\exp(-|k|^2+1)$$

if $2\pi^2 r^2 \geqslant 1$. Thus we have for great r

$$|\eta_r(C)| \leqslant m(C)\exp(-2\pi^2 r^2)\sum_{k\neq o}\exp(-|k|^2+1)$$

and, since $m(C) \leqslant m(B)$ and the last sum is convergent, (iv) follows and the proof of Lemma 2 is complete. □

4. Examples

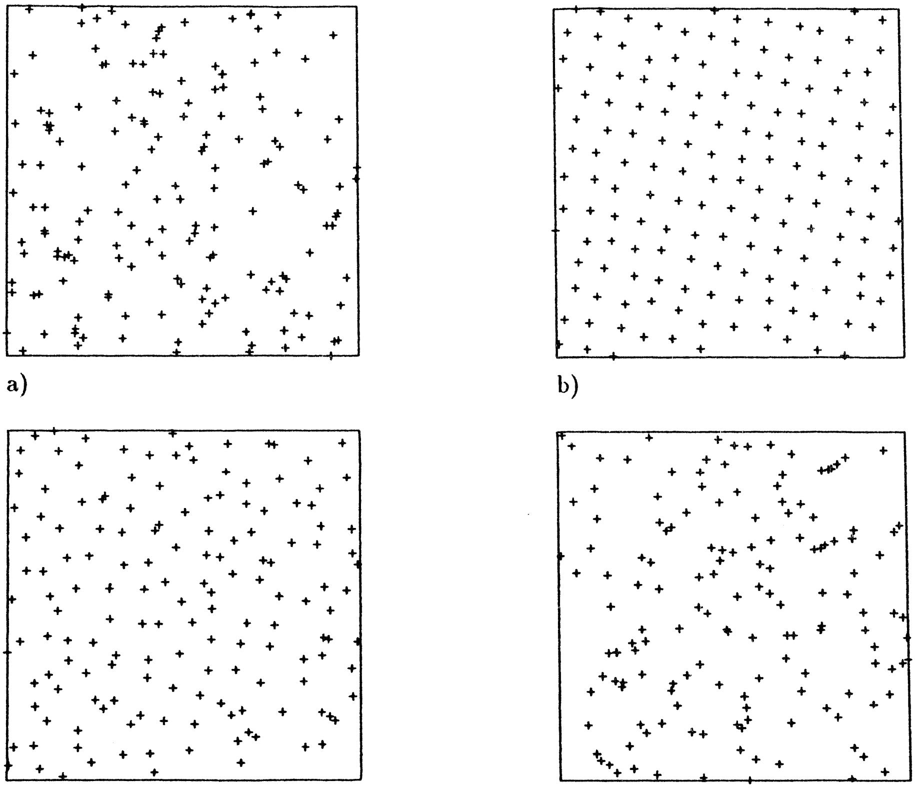
The aim of this section is to give some illustrations to the theoretical results presented in the previous sections. Consider the planar case ($d = 2$) of the point process

$$O_r = T_{\gamma_r}(\xi_0)$$

from Theorem 2 (i.e. ξ_0 is the simple integer-valued measure supported by the regular planar integer lattice $\mathbb{Z}^2$ and γ_r is the centered planar normal distribution with variance $r^2 I$). Theorem 2 states that the variation distance between the distributions of O_r and $P \equiv P_{1.m}$ is of order r^{-2} as $r \to \infty$. In order to demonstrate and more closely investigate this convergence, several simulated realizations of O_r will be examined by the techniques of quadrat count and distance methods [11], [4].

Simulation. The simulation of O_r for $r = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 6, 7$ within the chosen square window B were carried out in two steps: first the points of the regular unit square lattice O_0 lying in a region B' much greater than B (we have chosen $m(B) = 5^4 = 0.16\, m(B')$) were shifted by independently sampled values of γ_r, then an isotropic random direction (with respect to the lattice direction) of the square edge was selected and the position of the window B in the central part of B' was chosen. This procedure ensured that the mean number of points falling into B was 625 independently of the size of r and no edge effect corrections were necessary when using the distance method. For comparison, also a binomial point process of N points (with N being the Poisson random variable with the mean $m(B)$) was generated to simulate P in B. Four chosen sections of B (of 156 points in the mean) are shown in Fig. 1.

Testing. For the quadrat count technique, the sampling window was divided into $q = 2^i$ translation equivalent rectangles Q_j, $j = 1, \ldots, q$ (dividing a square window, we obtain rectangles with the edge length ratio 2 for j odd and squares for j even), and the numbers n_j of points falling into Q_j were determined. Under the Poisson hypothesis, the number n_j is Poisson of mean $m(B)/q$ and the counts in disjoint rectangles are independent [11]. The following statistics have been used in order to



a) b) c) d)

Fig. 1. Section of the simulated realizations of the point pattern: a) P, b) $O_{1/8}$, d) $O_{3/8}$, d) $O_{3/2}$.

determine the minimum number $r_0(i)$ such that the Poisson hypothesis cannot be rejected for O_r with $r \geqslant r_0(i)$ at the 5% confidence level:

a) The standard χ^2-*goodness-of-fit* test comparing the distribution of $n_j(r)$ with the Poisson distribution of mean $m(B)/q$ (note that the size of the basic rectangle roughly equals that of the lattice cell of O_0 for $i = 9$). The test statistic $\chi^2(r)$ for $i = 8$ (seven size classes) is plotted in Fig. 2 together with the corresponding critical value $\chi^2_{6;0.05}$. The stationary Poisson process hypothesis is not rejected at the level $\alpha = 0.05$ for $r \geqslant r_0(8) = 1.25$ and the observed value of $\chi^2(r)$ lies even below the critical value of $\chi^2_{6;0.5}$ for $r \geqslant 3$. The test was repeated also for $i = 7$ and 9 and the values $r_0(7) = 2.5$ and $r_0(9) = 1$ were obtained.

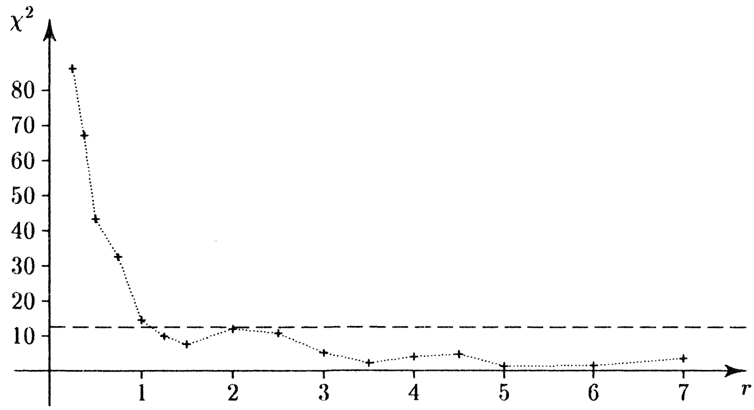


Fig. 2. The χ_6^2 statistics of the goodness-of-fit test for the distribution of the quadrat counts $n_j(r)$ in $2^8 = 256$ translation equivalent squares for a simulation of O_r and the critical value of $\chi^2_{6;0.05}$ (dashed line).

b) The *index-of-dispersion* test is based on the statistic

$$\mathrm{ID}_q = \frac{1}{\bar{n}} \sum_j (n_j - \bar{n})^2,$$

where $\bar{n}$ is the mean number of points per rectangle. Under the Poisson hypothesis, ID_q follows the χ^2-distribution of $q-1$ degrees of freedom. The two-sided test has been carried out for the divisions of the window B corresponding to $i = 8$, 9, 10 by examining the condition

$$\chi^2_{q-1;0.975} \leqslant \mathrm{ID}_q \leqslant \chi^2_{q-1;0.025}.$$

ID_q has never exceeded the second level but has not attained the first at $r < r_0(q)$—a demonstration of the residual regularity in the pattern. The values $r_0(10) = 1$, $r_0(9) = 2.5$ and $r_0(8) = 3$ show, similarly as in the preceding test, that the coarser is the examined division the more sensitive to the residual regularity is the test and, consequently, the higher must be r_0. The values corresponding to the division $i = 8$ are shown in Fig. 3.

c) In contrast to the previous two tests, which are insensitive to the position of rectangles, the *Greig-Smith* test examines the degree to which the independence of n_j in neighbouring rectangles is fulfilled. It is based on the statistic

$$CS_q = \frac{1}{\bar{n}} \Big(\sum_j n_j^2 - \frac{1}{2} \sum_{[j,k]} (n_j + n_k)^2 \Big),$$

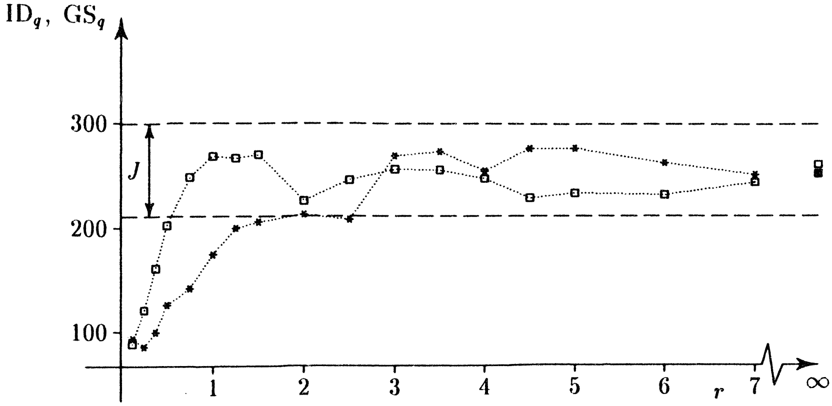


Fig. 3. The statistics $ID_{255}(*)$ and $GS_{256}(\square)$ evaluated for O_r and the double-sided confidence interval $J = [\chi^2_{255;0.975}, \chi^2_{255,0.025}]$ for the both statistics at the 5%-level the slight difference between the distributions χ^2_{255} and χ^2_{256} is omitted here). The mean values of $ID_{255}(\blacksquare)$ and $GS_{256}(\square)$ obtained from five simulations of $\mathcal{P}$ are plotted for the comparison.

where the second summation is taken over pairs of neighbouring rectangles merging into one rectangle in the subdivision of the window into $\frac{1}{2}q$ regions. Under the Poisson hypothesis, GS_q follows the χ^2-distribution with $\frac{1}{2}q$ degrees of freedom (note that if we calculate GS_q for a sequence of subdivisions q, $\frac{1}{2}q$, $\frac{1}{4}q$ etc., we obtain the quantities I_1, I_2, I_3, ... in the notation used in Stoyan et al. [11], Chap. 2.7). The testing has been carried out for $i = 10$ and 9 (Fig. 3) and the Poisson hypothesis cannot be rejected at the 5% level for $r \geqslant r_0 = 1$. As we are comparing the neighbouring areas of the size of one half of the lattice cell at $i = 10$ and of the size of the cell at $i = 9$, this result simply says that when r attains the magnitude of the lattice parameter the Poissonian positional independence of points is established at the lattice cell size scale.

For the application of the distance method, uniform random points were generated in the window B and the corresponding spherical contact distances σ_r (distances to the nearest points of the pattern) were calculated for the above given values of r. The maximum value of σ_0 is 0.71 in the unit square lattice and $\sigma < 3$ with probability greater that $1 - 10^{-12}$ in P; consequently, the protective frame of width 5 secures that practically all spherical contact distances are observable and no edge corrections need be considered. The qualitative behaviour of the distribution functions $F_r(b) = 1 - H_r\big(B(o, b)\big)$ of σ_r (cf. Sect. 3) may be seen from Fig. 4. The main differences between spherical contact distribution functions in O_r and P are the excess of medium paths and the absence of long paths in O_r.

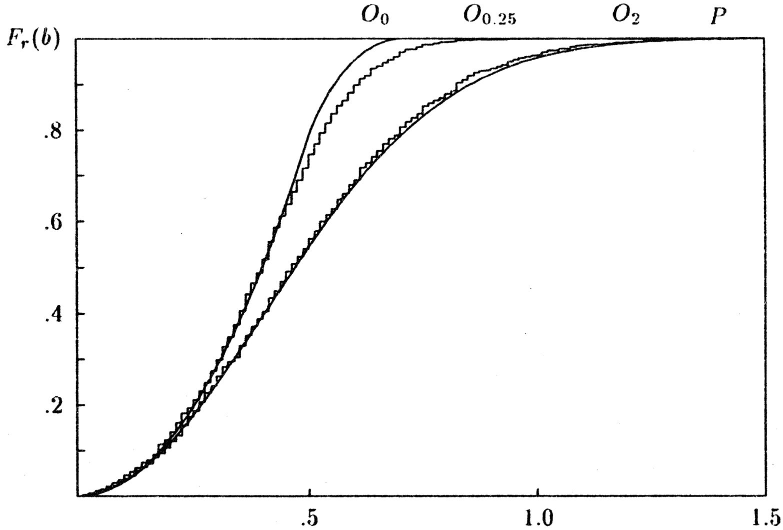


Fig. 4. The theoretical spherical contact distribution functions $F(b)$ for O_0 and P and the estimated $F_r(b)$ for $r = 0.25$, 2.

The results of the χ^2-*goodness-of-fit* test comparing the distribution of the spherical contact distances in O_r and P are shown in Fig. 5 for numbers of classes $k = 10$, 14 and 17; again the 5%-level was chosen. In order to keep the theoretical frequencies at a reasonably high level (~ 5), the corresponding numbers of examined paths were 100, 500 and 10 000 for $k = 10$, 14 and 17, respectively. To suppress the effect of chosen realization in more detailed measurements, the number n of paths per point was not allowed to surpass one. The value of $n = 1$ is recommended by Diggle [4] as a minimum for an estimation of $F(b)$, so that the sample for $k = 10$ is heavily undersized; nevertheless the result obtained is reasonable. Consequently, one realization was examined at $k = 10$, 14 and twenty realizations were evaluated at $k = 17$. It may be seen from Fig. 5 that the Poisson hypothesis was not rejected at 5%-level only for $r \geqslant 4$ at $k = 17$, whereas the values of $r \geqslant 1.5$ and $r \geqslant 0.75$ were not rejected at $k = 14$ and 10, respectively.

Summarizing these results, we conclude that in realizations of the oscillating point patterns of the kind considered (normally distributed oscillations with zero mean value in a unit square lattice) the regularity of the node pattern is considerably destroyed at the area scale of the order of the variance r^2. As a consequence of overlapping of these areas at $r > 1$, the differences between the Poisson point process and oscillating patterns with standard deviation exceeding 4 are very small and highly

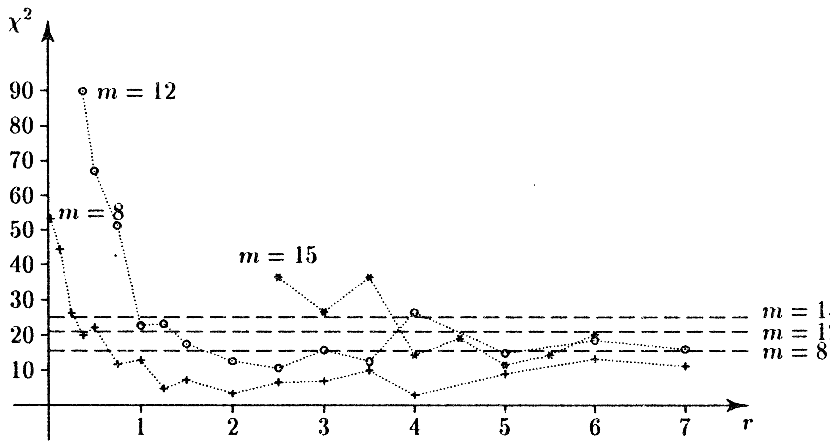


Fig. 5. The χ^2_m statistics of the goodness-of-fit test for the distribution of spherical contact distances in O_r and the critical values $\chi^2_{m;0.05}$ at the 5 %-level. The degree of freedom $m = 8$ corresponds to the sample of 100 measured distances, $m = 12$ to 500 distances, $m = 15$ to 10 000 distances (measured in 20 independent realizations of the point set).

sensitive and detailed statistical approaches requiring sizeable sections of several realizations would be needed to reveal the residual regularity in such patterns.

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Souhrn

KONVERGENCE NÁHODNĚ OSCILUJÍCÍCH BODOVÝCH SYSTÉMŮ K POISSONOVU BODOVÉMU PROCESU

JAN RATAJ, IVAN SAXL, KAROL PELIKÁN

Oscilující bodové systémy jsou bodové procesy odvozené z lokálně konečné množiny v prostoru konečné dimenze nezávislými stejně rozloženými kmity jednotlivých bodů. V práci je odvozena horní mez variační vzdálenosti oscilujícího bodového systému od limitního případu Poissonova bodového procesu. Odtud je dále určena skutečná rychost konvergence ve variační normě pro speciální případ Gaussovských kmitů pravidelné kubické bodové mřížky. Pro ilustraci teoretických výsledků jsou porovnány simulované oscilující bodové systémy v rovině s Poissonovým bodovým procesem.

Authors' addresses: *Jan Rataj*, Mathematical Institute, Charles University, Sokolovská 83, 186 00 Praha 8, ČR; *Ivan Saxl*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, ČR; *Karol Pelikán*, Institute of Materials Research, Slovak Academy of Sciences, Watsonova 47, 043 53 Košice, SR.