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STRONG CONVERGENCE ESTIMATES FOR PSEUDOSPECTRAL METHODS

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Summary. Strong convergence estimates for pseudospectral methods applied to ordinary boundary value problems are derived. The results are also used for a convergence analysis of the Schwarz algorithm (a special domain decomposition technique). Different types of nodes (Chebyshev, Legendre nodes) are examined and compared.

Keywords: pseudospectral, collocation, Schwarz algorithm, strong convergence estimates

AMS classification: (MOS): 65N30, 65N35, 35J25; CR: 5.17

1. INTRODUCTION

We give strong convergence estimates (in $C[a,b]$) for pseudospectral (or collocation) methods applied to ordinary boundary value problems. The results are also used for a convergence analysis of the Schwarz algorithm which in complicated domains consists in resorting to a domain decomposition technique.

Our considerations follow the ideas of Vainikko [20], [21] and Witsch [24]. We go back to the investigation of the projection operator of the collocation method. If the space of the projection operators consists of global polynomials the projection operator coincides with the interpolation operator. For its norm in $C[a,b]$—known as the Lebesgue constant—many estimates dependent on the type of nodes can be found in literature (see Brutman [1], Ehlich and Zeller [19], Natanson [12], Powel [13], Rivlin [14]). This treatment directly allows a comparison of different types of nodes.

In the last few years spectral methods have become of great interest (see, e.g., Canuto et al. [2], [3], [4], [5]). In order to employ Fast Fourier Transforms (FFT's) they usually used the extrema of the Chebyshev polynomials as collocation points.

We show that in the strong estimates nearly a factor N (N = number of nodes) is gained by choosing the zeros of the Chebyshev polynomials.

Furthermore, we explain how FFT's can successfully be applied to these nodes. Hence we have found an attractive alterative to the common method.

The Schwarz algorithm [17] was already examined by Canuto et al. [6] for spectral (Legendre- and Chebyshev-) Galerkin methods. We give convergence estimates for pseudospectral methods with different types of nodes. The Schwarz alternating procedure is based on the decomposition of the domain into everlapping regions, coupled with an interactive solution procedure alternating over the subdomains. The purpose of this strategy is to retain the computational efficiency of spectral methods in each simple domain. Clearly, the Schwarz method is more relevant in the two-dimensional case. But the one-dimensional analysis yields a good prediction for the convergence behaviour in the case of two overlapping rectangles (see also [6]). Fast Fourier Transforms are available on each subdomain and the high accuracy of the method is retained. Recently the method has gained new popularity since it can easily be implemented in a parallel way (see Rodrigue et al. [15], [16]).

In Sect. 2 we give some general convergence estimates which depend on the norm of the projection operator and the approximation error. A further investigation of the projection operator is attained by an argument of compact perturbations. Using concrete estimates for the Lebesgue constants (Sect. 3) and the approximation error (Sect. 4) we derive concrete convergence results. The case of inhomogeneous boundary conditions is also treated. In Sect. 5 we adopt our analysis to the Schwarz algorithm. Finally, in Sect. 6 we present numerical results for an example with a smooth solution which show the high accuracy of spectral methods as compared to finite difference methods.

2. Pseudospectral method, convergence estimates

We consider ordinary boundary value problems, given as

$$(2.1)\qquad Lu = u^{(k)} + \sum_{j=0}^{k-1} a_j u^{(j)} = f \quad \text{on } (a,b),$$

$$B_i[u] = \sum_{j=0}^{k-1} \left(\alpha_{i,j} u^{(j)}(a) + \beta_{i,j} u^{(j)}(b)\right) = 0 \quad (i = 1,\ \dots,\ k),$$

where $U^{(j)}$ denotes the j-th derivative and a_j, $f \in C[a,b]$, $\alpha_{i,j}$, $\beta_{i,j} \in \mathbf{R}$. Let L be defined on

$$D = \left\{u \in C^k[a,b] \colon B_i[u] = 0 \quad (i = 1, \dots, k)\right\}.$$

In the following we assume that $L\colon D \to C[a,b]$ is non-singular. Let U_N denote an N-dimensional subspace of D and let $x_j \in (a,b)$ $(j = 1,\dots,N)$ be given nodes such that

$$(2.2) \qquad u_N \in U_N,\ Lu_N(x_j) = 0\ (j = 1,\dots,N) \implies u_n \equiv 0.$$

$u_N \in U_N$ is called the approximation of the pseudospectral (or collocation) method for problem (2.1) iff

$$(2.3) \qquad (Lu_N)(x_j) = f(x_j) \quad (j = 1,\dots,N).$$

For $V_N = LU_N$ the corresponding projection operator $P_N\colon C[a,b] \to V_N$ is for each $v \in C[a,b]$ defined by

$$(2.4) \qquad P_N v(x_j) = v(x_j) \quad (j = 1,\dots,N).$$

Now the approximation u_N can also be interpreted as the solution of $Lu_N = P_N f$.

We remark that we consider more general differential equations than those treated by Canuto et al. [3], [4]. The above normalization (highest coefficient is equal to one) can easily be obtained by dividing through the highest coefficient, which is always supposed to be positive. Now the introduction of interpolation operators (as done in [3], [4]) for evaluating the derivatives of the coefficient functions is no longer needed.

We always suppose that u_N and P_N exist and are unique. This can often be shown by means of the perturbation results. We now introduce a nonnegative integrable function ω define in $[a,b]$ and satisfying

$$(2.5) \qquad \int_a^b \frac{\mathrm{d}x}{\omega(x)} < \infty.$$

Let $L^{2,\omega}(a,b)$ denote the space of all square integrable functions with respect to the weight function ω. Further let $C^s[a,b]$, $s \in \mathbf{R}$, $s \geqslant 0$ denote the space of functions with uniformly continuous derivatives up to order $[s]$, and the $[s]$-th derivative is required to be Hölder-continuous with exponent $s - [s]$, i.e.

$$\left[u^{([s])}\right]_{s-[s]} = \sup\left\{\frac{|u^{([s])}(x) - u^{([s])}(y)|}{\|x-y\|^{s-[s]}} : x, y \in (a,b),\ x \neq y\right\} < \infty.$$

The norm on $C^s[a,b]$, $s \notin \mathbf{N} \cup \{0\}$, $s \geqslant 0$ is given by

$$\|u\|_{C^s[a,b]} = \max\left\{\max\left\{|u^{(1)}(x)| : x \in [a,b];\ 1 = 0,\dots,[s]\right\},\ \left[u^{([s])}\right]_{s-[s]}\right\}.$$

The following error estimates describe the approximation error of f in V_N, i.e.

$$E_N(f, C[a,b]) = \inf\left\{\|f - f_N\|_{C[a,b]} \colon f_N \in V_N\right\}.$$

Theorem 1. *Let $u \in D$ denote the unique solution of (2.1). Condition (2.2) is fulfilled. Then u_N, P_N are uniquely determined and the following error estimates hold:*

$$\|Lu_N - f\|_{C[a,b]} \leqslant (1 + \|P_N\|_{C[a,b]\to C[a,b]}) E_N(f, C[a,b]),$$
$$\|Lu_N - f\|_{2,\omega_{(a,b)}} \leqslant \left(\left(\int_a^b \omega(x)\,\mathrm{d}x\right)^{1/2} + \|P_N\|_{C[a,b]\to L^{2,\omega}(a,b)}\right) E_N(f, C[a,b]).$$

The error $u - u_N$ is bounded by

$$\|u_N - u\|_{C^k[a,b]} \leqslant \gamma_0 \|Lu_N - f\|_{C[a,b]},$$
$$\|u_N - u\|_{C^{k-1}[a,b]} \leqslant \gamma_1 \|Lu_N - f\|_{L^{2,\omega}(a,b)}.$$

with positive constants γ_0, γ_1 independent of N.
In particular, we conclude that $Lu_N \to f$, $u_N \to u$ iff

$$\|P_N\|_{C[a,b]\to C[a,b]} E_N(f, C[a,b]) \to 0 \quad \text{or.}$$
$$\|P_N\|_{C[a,b]\to L^{2,\omega}(a,b)} E_N(f, C[a,b]) \to 0 \quad \text{for } N \to \infty.$$

Proof. From $Lu = f$ and $Lu_N = P_N f$ we deduce

$$L(u - u_N) = (I - P_N)f$$
$$= (I - P_N)(f - f_N) \quad \text{for } f_N \in V_N.$$

The estimates for the defect are now straightforward. The estimates for $u - u_N$ follow by means of the representation by Green's function (see Collatz [7] and Vainikko [20]). □

Results about compact perturbations (see Witsch [24], Theorem 2.5) allow further investigation of the projection operators P_N. For this reason we decompose the operator L into the form

$$L = \hat{L} + \tilde{L},$$

where $\hat{L} = u^{(k)}$ and $\tilde{L} = L - \hat{L}$.
We assume that L, $\hat{L} \colon D \to C[a,b]$ are invertible.

Using the Arzela-Ascoli theorem [11] we deduce that $\tilde{L}L^{-1}: C[a,b] \to C[a,b]$ and $\tilde{L}L^{-1}: L^{2,\omega}(a,b) \to C[a,b]$ (ω as in (2.5)) are compact. Let $\hat{P}_N$ denote the projection operator belonging to $\hat{L}$. In order to shorten the following explanations we introduce the abbreviation V for $V = C[a,b]$ or $V = L^{2,\omega}(a,b)$.

Theorem 2. *Let L, $\hat{L}$: $D \to C[a,b]$ be invertible and let condition (2.2) be true for $\hat{L}$. Assume that $I - \hat{P}_N$ (weakly) converges on $\tilde{L}L^{-1}(V)$ to zero, i.e. for all $f \in \tilde{L}L^{-1}(V)$:*

$$\|(I - \hat{P}_N)f\|_V \to 0 \quad \text{for } N \to \infty.$$

Then for sufficiently large N the projections P_N are also uniquely determined and we get the estimate

$$\|P_N\|_{C[a,b]\to V} \leqslant c_N \|P_N\|_{C[a,b]\to V}$$

where $c_N \to 1$ for $N \to \infty$.

Proof. A proof of the above result in a more general situation is given in [24, Theorem 2.5]. From there it becomes clear that the projection P_N exists and is unique if

$$\hat{L} + \hat{P}_N \tilde{L} = (I - (I - \hat{P}_N)\tilde{L}L^{-1})L$$

is invertible, and it is then given by

$$P_N = L(\hat{L} + \hat{P}_N\tilde{L})^{-1}\hat{P}_N = \left(I - (I - \hat{P}_N)\tilde{L}L^{-1}\right)^{-1}\hat{P}_N.$$

In particular, the condition

$$\beta_N = \|(I - \hat{P}_N)\tilde{L}L^{-1}\|_{V\to V} < 1$$

is sufficient for this to hold, and we get the estimate

$$\|P_N\|_{C[a,b]\to V} \leqslant \frac{1}{1-\beta_N}\|\hat{P}_N\|_{C[a,b]\to V}.$$

By a result from functional analysis (see Gelfand [11, Th. 3 (1.IX)]) it follows that weakly convergent operators (on compact sets) are uniformly convergent, i.e., $\beta_N \to 0$ for $N \to \infty$. The constants c_N can now be defined as $c_N = \frac{1}{1-\beta_N} \to 1$ $(N \to \infty)$ and this concludes the proof. □

Remarks. (i) We get an estimate of the form

$$\|P_N\|_{C[a,b]\to V} \leqslant \frac{1}{1-q}\|\hat{P}_N\|_{C[a,b]\to V}$$

if we merely require that

$$\|(I-\hat{P}_N)\tilde{L}L^{-1}\|_{V\to V} \leqslant q < 1 \quad \text{for sufficiently large } N.$$

This means that the perturbation only has to be sufficiently small.

(ii) If in Theorem 2 we further require that

$$\|(I-\hat{P}_N)\tilde{L}L^{-1}\|_{V\to V} \cdot \|\hat{P}_N\|_{C[a,b]\to V} \to 0,$$

then it also follows that

$$\|P_N - \hat{P}_N\|_{C[a,b]\to V} \to 0 \quad \text{for } N \to \infty.$$

In order to show the weak convergence of $\hat{P}_N$ to I it is sufficient tc show that for all $f \in \tilde{L}L^{-1}(V)$

$$\|\hat{P}_N\|_{C[a,b]\to V}\hat{E}_N(f, C[a,b]) \to 0 \quad \text{for } N \to \infty, \tag{2.6}$$

where $\hat{E}_N(f, C[a,b]) = \inf\left\{\|f - f_N\|_{C[a,b]} : f_N \in \hat{L}U_N\right\}$. In what follow we derive concrete results in the case of

$$U_N = Q_N = \{p_N : p_N \text{ algebraic polynomial of degree } \leqslant N+k-1$$
$$\text{satisfying } B_i[p_N] = 0 \quad (i = 1, \ldots, k)\}.$$

If L is invertible we deduce that $P_N = \Pi_N$ where Π_N denotes the interpolation operator which maps into

$\mathbf{P}_{N-1} = \{p_N : p_N$ algebraic polynomial of degree $\leqslant N-1\}$. For some typical distributions of nodes we give the norms of Π_N in the next section.

3. Norms of interpolation operators

At the beginning we consider

$$\|\Pi_N\|_{C[a,b]\to L^{2,\omega}(a,b)}.$$

Let $\{\omega_1 : \omega_1$ polynomial of degree $1\}$ denote a system of polynomials which are orthogonal relative to the inner product $(\ ,\ )_\omega$ on $[a,b]$. It is known that the 1 zeros of ω_1 are simple and lie in (a,b) (see Szegö [18]). Using this notation we obtain

Lemma 1. *If the nodes x_i $(i = 1, \ldots, N)$ are the zeros of ω_N then*

$$\|\Pi_N\|_{C[a,b]\to L^{2,\omega}(a,b)} = \left(\int_a^b \omega(x)\,\mathrm{d}x\right)^{1/2}.$$

Proof. Using the interpolation formula of Lagrange we get

$$\Pi_N u = \sum_{j=1}^{N} u(x_j)\ell_j^{(N)} \quad \text{for } u \in C[a,b],$$

where $\ell_j^{(N)}$ denotes the j-th Lagrange factor, given by

$$\ell_j^{(N)}(x) = \frac{\omega_N(x)}{\omega_N(x_j)(x - x_j)}.$$

As shown in the Lemma of Grunwald and Turan [12, §2] the polynomials $\ell_i^{(N)}$ and $\ell_j^{(N)}$ are orthogonal with respect to $(\ ,\ )_\omega$. Hence we get

$$\begin{aligned}
\|\Pi_N u\|_{L^{2,\omega}}^2 &= \sum_{j=1}^{N} |u(x_j)|^2 \int_a^b \omega(x)\big(\ell_j^{(N)}(x)\big)^2 \mathrm{d}x \\
&\leqslant \left(\sum_{j=1}^{N} \int_a^b \omega(x)\big(\ell_j^{(N)}(x)\big)^2 \mathrm{d}x\right)\|u\|_{C[a,b]}^2 \\
&= \left[\int_a^b \left(\sum_{j=1}^{N} \ell_i^{(N)}(x)\right)\left(\sum_{j=1}^{N} \ell_j^{(N)}(x)\right)\omega(x)\,\mathrm{d}x\right]\|u\|_{C[a,b]}^2 \\
&= \left(\int_a^b \omega(x)\,\mathrm{d}x\right)\|u\|_{C[a,b]}^2
\end{aligned}$$

We have used the orthogonality of $\ell_i^{(N)}$ and $\sum_{j=1}^{N} \ell_j^{(N)} \equiv 1$.

For $u \equiv 1$ equality is attained and this concludes the proof. □

For example, in the case $a = -1$, $b = 1$, $\omega(x) = (1 - x^2)^{-1/2}$ with the Chebyshev nodes

$$x_j = \cos\frac{(2j-1)\pi}{2N}, \quad (j = 1, \ldots, N), \tag{3.1}$$

we get

$$\|\Pi_N\|_{C[-1,1]\to C[-1,1]} = \sqrt{\pi}.$$

We now consider norms of the type

$$\|\Pi_N\|_{C[a,b]\to C[a,b]}.$$

Here we fix $a = -1$, $b = 1$ and briefly write λ_N instead of $\|\Pi_N\|_{C[-1,1]\to C[-1,1]}$. In the literature the constants λ_N are often called the Lebesgue constants. First, Natanson [12] gave quite rough estimates for λ_N which have been improved by Brutman [1], Ehlich and Zeller [10], Powel [13] and Rivlin [14]. It is known that the constants λ_N grow logarithmically; Brutman [1, ineq. (41)] present a quite sharp lower bound,

$$\lambda_N > \frac{2}{\pi} \ln N + 0.5212.$$

We now want to give upper bounds for λ_N for different types of nodes. For this purpose we introduce the extreme of the Chebyshev polynomials (without ± 1)

$$x_j = \cos\left(\frac{j\pi}{N+1}\right) \quad (j = 1, \ldots, N). \tag{3.2}$$

In connection with spectral methods [2, 3, 4, 5, 6, 25, 26] this type of grid is recommended with $N+1$ equal to a power of 2. Then fast cosine transforms based on real FFT's are available and can be efficiently employed for solving the linear spectral systems. Above all for system arising from elliptic equations this aspect is of great interest (see Zang et al. [25], [26]).
Upper bounds for different nodes are

- zeros of ω_N (see [12, Chap. III, §2, Th. 1]):
  $\lambda_N \leqslant K \cdot N$, where $K = \left(\int_{-1}^{1} \omega(x)\,\mathrm{d}x/(2\hat{\omega})\right)^{1/2}$, $\omega(x) \geqslant \hat{\omega} > 0$ on $[-1,1]$.
- Legendre nodes (see [12]):
  $\lambda_N \leqslant C\sqrt{N}$, $C > 0$ independent of N.
- Chebyshev nodes (3.1) (see [14]):
  $\lambda_N \leqslant \frac{2}{\pi} \ln N + 1$.
- Chebyshev nodes (3.2) (see [1, eq. (47)]:
  $\lambda_N = N$.

For the Chebyshev nodes (3.2) where the endpoints ± 1 are added a logarithmic estimate also exists. But for collocation on $(-1,1)$ this bound is not relevant. Furthermore, for equidistant collocation points the Lebesgue constants increase exponentially fast.

In particular, the results show that for estimates in $C[-1,1]$, the Chebyshev nodes (3.1) yield a higher accuracy than the nodes (3.2). Obviously, the nodes (3.1) also admit a fast computation of truncated Chebyshev series using FFT's. This can be achieved by means of a fast cosine and sine transform. Hence the computational effort is twice as high as for the nodes (3.2) but still increases logarithmically. Because of the higher accuracy the nodes (3.1) yield an attractive alternative to the nodes (3.2). For completeness we give in Appendix a stable version of the fast cosine transform (see also Temperton [19] for the fast sine transform).

4. Approximation error, convergence results

We consider the approximation property of $\mathbf{P}_N$ for a given function $f \in C[a,b]$, i.e. $\tilde{E}_N(f, C[a,b]) = \inf\{\|f - p_N\|_{C[a,b]} : p_N \in \mathbf{P}_N\}$. For $f \in C^s[a,b]$, $s \in \mathbf{N}$ Jackson's theorems can be applied. A generalization for $s \in \mathbf{R}$, $s \geqslant 0$ was given by Witsch [23, Lemma 3.4]:

Lemma 2. *Let $s \in \mathbf{R}$, $s \geqslant 0$ be a given constant. Then there exists a positive constant $K = K(s)$ independent of N such that for $f \in C^s[a,b]$*

$$\tilde{E}_N(f, C[a,b]) \leqslant K\|f\|_{C^s[a,b]} N^{-s}.$$

If $s \geqslant 1$ then there exists a polynomial $p_N \in \mathbf{P}_N$ with

$$\|f - p_N\|_{C[a,b]} \leqslant K\|f\|_{C^s[a,b]} N^{-s}.$$

If $s = 0$ then $\tilde{E}_N(C[a,b]) \to 0$ for $N \to \infty$.

For the proof Witsch introduces an approximation operator which is constructed according to an idea of De Vore [9]. By using the smoothness assumptions on the coefficient functions a_j similar approximation estimates are also available for $E_N(f, C[a,b])$. The result for $s = 0$ is due to the theorem of Weierstrass.

Summarizing the above results, we derive for $U_N = Q_N$ and different types of nodes the following convergence estimates:

Theorem 3. *Let L, $\hat{L}$: $D \to C[a,b]$ be invertible and let $u \in D$ be the unique solution of (2.1). Then for sufficiently large N, the pseudospectral approximation*

$u_N \in Q_N$ is uniquely determined, and the following error estimates hold:

– *zeros of othogonal polynomials, Chebyshev nodes* (3.2)
if "$a_{k-1} = 0$" or "a_{k-1} sufficiently small"
and $a_j \in C^{1+\varepsilon}[a,b]$, $\varepsilon > 0$
$\|u_N - u\|_{C^k[a,b]} \leqslant K_1 N E_N(f, C[a,b])$;
– *Legendre nodes if $a_j \in C^{1/2+\varepsilon}[a,b]$, $\varepsilon > 0$*
$\|u_N - u\|_{C^k[a,b]} \leqslant K_2 \sqrt{N} E_N(f, C[a,b])$;
– *Chebyshev nodes* (3.1) *if $a_j \in C^{\varepsilon}[a,b]$, $\varepsilon > 0$*
$\|u_N - u\|_{C^k[a,b]} \leqslant K_3 \ln(N) E_N(f, C[a,b])$.

Without any further assumption on $a_j \in C[a,b]$ we get

$$\|Lu_N - f\|_{L^{2,\omega}(a,b)} \leqslant K_4 E_N(f, C[a,b])$$

and

$$\|Lu_N - f\|_{C^{k-1}[a,b]} \leqslant K_5 E_N(f, C[a,b]).$$

$K_1, \ldots, K_5$ denote positive constants independent of N.

Proof. The smoothness assumptions on a_j are an immediate consequence of (2.6) and Theorem 2. For estimates in $L^{2,\omega}(a,b)$ we do not need similar conditions since the projection operators are now uniformly bounded. □

We consider problems with inhomogeneous boundary conditions, given as

$$Lu = f \quad \text{on } (a,b) \tag{4.1}$$
$$B_i[u] = r_i \quad (i = 1, \ldots, k)$$

where L, B_i, f are defined as in (2.1) and $r_i \in \mathbf{R}$ are constants. We reduce the investigation of (4.1) to a problem with homogeneous boundary conditions. Let $u^1 \in C^k[a,b]$ be given, satisfying

$$B_i[u^1] = r_i \quad (i = 1, \ldots, k). \tag{4.2}$$

It is obvious that (4.2) can be fulfilled, e.g., by a polynomial of degree $\leqslant k-1$. Using $v = u^1 - u$, problem (4.1) is equivalent to

$$Lv = f - Lu^1 \quad \text{on } (a,b), \tag{4.3}$$
$$B_i[v] = 0 \quad (i = 1, \ldots, k)$$

If v_N denotes the pseudospectral approximation of (4.3) then $u_N = u^1 + v_N$ is the pseudospectral approximation of (4.1) and we get the estimate

$$\begin{aligned}\gamma_0^{-1}\|u_N - u\|_{C^k[a,b]} &\leqslant \|Lu_N - f\|_{C[a,b]} \\ &\leqslant \left(1 + \|P_N\|_{C[a,b]\to C[a,b]}\right) E_N\left(f - Lu^1, C[a,b]\right).\end{aligned}$$

Different from the estimate in Theorem 1, the approximation error is now taken for $f - Lu^1$ instead of f. If u^1 is a polynomial then $f - Lu^1$ has the same smoothness properties as f and the order of convergence is the same as for problem (2.1).

5. The Schwarz algorithm

We consider the Schwarz algorithm for the pseudospectral approximation of the following simple problem (see Canuto et al. [6]):

$$\begin{aligned} Lu = -u'' = f \quad \text{on } \Omega = (-1,1), \\ u(-1) = u(1) = 0. \end{aligned} \tag{5.1}$$

In order to explain the algorithm we decompose Ω into two overlapping intervals, given as

$$\Omega_1 = (-1, \beta), \quad \Omega_2 = (\alpha, 1) \quad \text{for } -1 < \alpha < \beta < 1.$$

We introduce the spaces

$$U^{(i)} = \left\{u \in C^2(\bar{\Omega}_i) \colon u(-1) = 0 \ (i = 1) \text{ or } u(1) = 0 \ (i = 2)\right\}, \quad i = 1, 2.$$

Let $x_j^{(1)}$ and $x_j^{(2)}$ ($j = 1, \ldots, N$) denote the collocation nodes in Ω_1 and Ω_2, respectively.

Furthermore, we introduce the following subspaces of $U^{(i)}$:

$$U_N^{(i)} = U^{(i)} \cap \mathbf{P}_{N+1}, \quad i = 1, 2.$$

Now we are able to show how the discrete pseudospectral Schwarz algorithm can be applied to problem (5.1). Given an arbitrary initial function $u_N^1 \in U_N^{(2)}$ we construct sequences $u_N^{2n+1} \in U_N^{(2)}$ and $u_N^{2n} \in U_N^{(1)}$ as follows:

$$\begin{aligned} Lu_N^{2n}(x_j^{(1)}) = f(x_j^{(1)}) \quad (j = 1, \ldots, N), \\ u_N^{2n}(-1) = 0, \quad u_N^{2n}(\beta) = u_N^{2n-1}(\beta), \end{aligned} \tag{5.2}$$

and

$$(5.3) \qquad \begin{aligned} &Lu_N^{2n+1}(x_j^{(2)}) = f(x_j^{(2)}) \quad (j = 1, \ldots, N), \\ &u_N^{2n+1}(1) = 0, \quad u_N^{2n+1}(\alpha) = u_N^{2n}(\alpha). \end{aligned}$$

It is quite easy to prove that the discrete Schwarz algorithm yields convergent sequences (for $n \to \infty$).

Theorem 4. *Let $N \geqslant 2$ and let $u_N^{2n} \in U_N^{(1)}$, $u_N^{2n+1} \in U_N^{(2)}$ be defined as in (5.2), (5.3). Then there exist polynomials $u_N^{(i)} \in U_N^{(i)}$, $i = 1, 2$ satisfying $u_N^{(1)}(\beta) = u_N^{(2)}(\beta)$, $u_N^{(1)}(\alpha) = u_N^{(2)}(\alpha)$ such that*

$$\|u_N^{2n} - u_N^{(1)}\|_{C^2(\bar{\Omega}_1)} + \|u_N^{2n+1} - u_N^{(2)}\|_{C^2(\bar{\Omega}_2)} \leqslant Ck^n.$$

where C is a positive constant and $k = \frac{1+\alpha}{1-\alpha}\frac{1-\beta}{1+\beta} < 1$.

Proof. For the proof we introduce polynomials

$$w_N^{2n} = u_N^{2n+2} - u_N^{2n}, \quad w_N^{2n+1} = u_N^{2n+3} - u_N^{2n+1}.$$

Since $Lw_N^{2n} \equiv 0$, $w_N^{2n}(-1) = 0$ and $Lw_N^{2n+1} \equiv 0$, $w_N^{2n+1}(1) = 0$ we get

$$\begin{aligned} w_N^{2n}(x) &= (1+\beta)^{-1} w_N^{2n}(\beta)(1+x) \text{ and} \\ w_N^{2n+1}(x) &= (1-\alpha)^{-1} w_N^{2n+1}(\alpha)(1-x). \end{aligned}$$

Further we have

$$\begin{aligned} |w_N^{2n}(\beta)| &= |w_N^{2n-1}(\beta)| = \frac{1-\beta}{1-\alpha}|w_N^{2n-1}(\alpha)| \text{ and} \\ |w_N^{2n+1}(\alpha)| = |w_N^{2n}(\alpha)| &= \frac{1+\alpha}{1+\beta}|w_N^{2n}(\beta)| = \frac{1+\alpha}{1-\alpha}\frac{1-\beta}{1+\beta}|w_N^{2n-1}(\alpha)|. \end{aligned}$$

Hence $|w_N^{2n+1}(\alpha)| \leqslant k|w_N^{2n-1}(\alpha)|$, $k < 1$ and $\|w_N^{2n+1}\|_{C(\bar{\Omega}_2)} = |w_N^{2n+1}(\alpha)| \to 0$ ($n \to \infty$). Since $\|\frac{d}{dx}w_N^{2n+1}\|_{C(\bar{\Omega}_2)} \leqslant (1-\alpha)^{-1}\|w_N^{2n+1}\|_{C(\bar{\Omega}_2)}$ and $\frac{d^2}{dx^2}w_N^{2n+1} \equiv 0$ it easily follows that u_N^{2n+1} forms a Cauchy sequence in $C^2(\bar{\Omega}_2)$. Therefore there exists a unique polynomial $u_N^{(2)} \in U_N^{(2)}$ such that $u_N^{2n+1} \to u_N^{(2)}$ in $C^2(\bar{\Omega}_2)$. Using this result we conclude that

$$\|u_N^{2n+1} - u_N^{(2)}\|_{C^2(\bar{\Omega}_2)} \leqslant C_0 \sum_{m \geqslant n} \|w_N^{2n+1}\|_{C(\bar{\Omega}_2)} \leqslant C_1 k^n$$

with positive constants C_0, C_1. A similar argument holds for $u_N^{2n} \in U_N^{(1)}$ and the theorem is proved. □

Now it remains to show that the discrete approximations $u_N^{(i)} \in U_N^{(i)}$ converge to u in $C^2(\bar{\Omega}_i)$ for $i = 1, 2$. Then we obtain

Theorem 5. *Let u be the solution of (5.1) for $f \in C^s(\bar{\Omega})$, $s \geqslant 0$. Then for the Schwarz sequence (u^{2n}, u^{2n+1}) as in (5.2), (5.3) the following estimate holds:*

$$\begin{aligned}&\|u - u_N^{2n}\|_{C^2(\bar{\Omega}_1)} + \|u - u_N^{2n+1}\|_{C^2(\bar{\Omega}_2)}\\ &\leqslant c_1 \pi_N N^{-s} \|f\|_{C^s(\bar{\Omega})} + c_2 k^n,\end{aligned}$$

where c_1, c_2 are positive constants and $k = \frac{1+\alpha}{1-\alpha}\frac{1-\beta}{1+\beta} < 1$. π_N denotes the maximum of $\|\Pi_N^{(i)}\|_{C(\bar{\Omega}_i)\to C(\bar{\Omega}_i)}$, $i = 1, 2$. Here $\Pi_N^{(i)}$ is the interpolation operator on Ω_i relative to $x_j^{(i)}$ $(j = 1, \ldots, N)$. If the nodes $x_j^{(i)}$ are those of Theorem 3 transformed to Ω_i we get the following asymptotic behaviour of Π_N:

- *zeros of orthogonal polynomials:* $\pi_N = 0(N)$
- *Chebyshev nodes (3.2):* $\pi_N = 0(N)$
- *Legendre nodes:* $\pi_N = 0(\sqrt{N})$
- *Chebyshev nodes (3.1):* $\pi_N = 0(\ln N)$

P r o o f. The approximations $u_N^{(1)}$ and $u_N^{(2)}$ are given as

$$\begin{aligned}u_N^{(1)}(x) &= (1+\beta)^{-1} u_N^{(2)}(\beta)(1+x) + \tilde{u}_N^{(1)}(x),\\ u_N^{(2)}(x) &= (1-\alpha)^{-1} u_N^{(1)}(\alpha)(1-x) + \tilde{u}_N^{(2)}(x),\end{aligned}$$

where $u_N^{(1)}$, $u_N^{(2)} \in \mathbf{P}_{N+1}$ satisfy

$$\begin{aligned}L\tilde{u}_N^{(1)}(x_j^{(1)}) &= f(x_j^{(1)}) \ (j = 1, \ldots, N), \quad \tilde{u}_N^{(1)}(-1) = \tilde{u}_N^{(1)}(\beta) = 0,\\ L\tilde{u}_N^{(2)}(x_j^{(2)}) &= f(x_j^{(2)}) \ (j = 1, \ldots, N), \quad \tilde{u}_N^{(2)}(1) = \tilde{u}_N^{(2)}(\alpha) = 0.\end{aligned}$$

Because of the identity

$$\begin{aligned}u(x) - u_N^{(2)}(x) &= \big(u(x) - (1-\alpha)^{-1}u(\alpha)(1-x)\big) - \tilde{u}_N^{(2)}(x)\\ &\quad + (1-\alpha)^{-1}\big(u(\alpha) - u_N^{(1)}(\alpha)\big)(1-x)\end{aligned} \tag{5.4}$$

we derive using Theorem 3 and Lemma 2

$$|u(\beta) - u_N^{(2)}(\beta)| \leqslant C_1 \pi_N N^{-s} \|f\|_{C^s(\bar{\Omega})} + \frac{1-\beta}{1-\alpha}|u(\alpha) - u_N^{(1)}(\alpha)|.$$

By the same argument we get

$$|u(\alpha) - u_N^{(1)}(\alpha)| \leqslant C_2 \pi_N N^{-s} \|f\|_{C^s(\bar\Omega)} + \frac{1+\alpha}{1+\beta} |u(\beta) - u_N^{(2)}(\beta)|.$$

Since $k < 1$ we obtain

$$|u(\alpha) - u_N^{(1)}(\alpha)| \leqslant C_3 \pi_N N^{-s} \|f\|_{C^s(\bar\Omega)}.$$

Inserting this result into equation (5.4) we get the estimate

$$\|u - u_N^{(2)}\|_{C^2(\bar\Omega_2)} \leqslant C \pi_N N^{-s} \|f\|_{c^s(\bar\Omega)}.$$

A similar estimate holds for $\|u - u_N^{(1)}\|_{C^2(\bar\Omega_1)}$. Using the result of Theorem 4 and the triangle inequality we conclude the proof. □

6. Numerical example

Here we consider the boundary value problem

$$\begin{aligned} lu = u'' - e^x u' = f \quad &\text{in } (-1,1), \\ u(-1) = u(1) = 0. & \end{aligned}$$

where the exact solution is given by $u(x) = \sin(\pi x)$ and $f = Lu$. We compare our pseudospectral method with the second and the fourth order finite difference (FD) methods. The pseudospectral approximation u_N is determined by using the Chebyshev nodes (3.2). For the FD discretization we use equidistant nodes $x_i = -1 + ih$, $h = \frac{2}{N+1}$, $i = 1, \ldots, N$. For the second order FD method we employ the following stencils:

(FD2) $$u' \simeq \frac{1}{2h}[-1\,0\,1]u, \quad u'' \simeq \frac{1}{h^2}[1\ -2\,1]u.$$

The FD2 approximation is denoted by u_h^2.
For the fourth order FD method (FD4) we employ the above stencils at the points next to the boundary while at the other inner points we use

(FD4) $$u' \simeq \frac{1}{12h}[1\ -8\,0\,8\ -1]u, \quad u'' \simeq \frac{1}{12h^2}[-1\,16\ -30\,16\ -1]u.$$

The FD4 approximation is written as u_h^4.
For measuring the error we further introduce the discrete L^2-norm given by

$$\|z\|_2 = \frac{1}{\sqrt{N}} \sqrt{\sum_{i=1}^{N} z^2(x_i)}.$$

Now we define the following discretization errors:

$$E_2 = \|u - u_h^2\|_2, \quad E_4 = \|u - u_h^4\|_2, \quad E_{sp} = \|u - u_N\|_2.$$

The numerical results for E_2, E_4, E_{sp} are presented in Table I. They show the second and fourth order accuracy of the methods FD2 and FD4. For the pseudospectral method we observe a spectral accuracy where the error decay is exponentially fast. The results substantiate the usefulness of spectral methods.

$N+1$	E_2	E_4	E_{sp}
8	$1.29 \cdot 10^{-1}$	$3.51 \cdot 10^{-2}$	$4.17 \cdot 10^{-4}$
16	$3.12 \cdot 10^{-2}$	$4.52 \cdot 10^{-3}$	$7.53 \cdot 10^{-12}$

Table I. Errors E_2, E_4 and E_{sp}.

APPENDIX

We want to evaluate

$$\text{(A.1)} \qquad y_j = \sum_{n=0}^{N} a_n \cos \frac{nj\pi}{N} \quad (j = 1, \ldots, N),$$

$$\text{(A.2)} \qquad z_j = \sum_{n=0}^{N} b_n \sin \frac{nj\pi}{N} \quad (j = 1, \ldots, N).$$

by means of real FFT's.

Cooley et al. [8] proposed an algorithm for the fast sine and cosine transform. For the fast sine transform it was already observed by Temperton [19] that this version is not very stable against round-off errors. The reason is that factors $(1/\sin\frac{j\pi}{N})$ $(j = 1, \ldots, N-1)$ appear which strongly propagate the errors for j near 1 and $N-1$. This is avoided in the "inverse" form which is given here for the cosine transform (A.1):

1. Calculation of b_n:

$$b_0 = a_0 + a_N,$$
$$b_n = \frac{1}{2}(a_n + a_{N-n}) - \sin\frac{n\pi}{N}(a_n - a_{N-n}), \quad n = 1, \ldots, N-1,$$

2. Real Fast Fourier Transform for the evaluation of

$$x_j^R = \sum_{n=0}^{N-1} b_n \cos\frac{2nj\pi}{N} \quad \text{for } j = 1, \ldots, \tfrac{1}{2}N,$$
$$x_j^I = \sum_{n=0}^{N-1} b_n \sin\frac{2nj\pi}{N} \quad \text{for } j = 1, \ldots, \tfrac{1}{2}N - 1.$$

Set

$$x_0^I = x_{N/2}^I = 0.$$

3. Calculation of y_j $(j = 1, \ldots, N)$:

$$y_{2j} = x_j^R \quad \text{for } j = 1, \ldots, \tfrac{1}{2}N,$$
$$y_{2j+1} = y_{2j-1} + x_j^I \quad \text{for } j = 1, \ldots, \tfrac{1}{2}N - 1$$

with $y_1 = \sum_{n=0}^{N} a_n \cos\frac{n\pi}{N}$ calculated, e.g., by Clenshaw recursion [22, p. 106].

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