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OPTIMAL DESIGN OF LAMINATED PLATE WITH OBSTACLE

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Summary. The aim of the present paper is to study problems of optimal design in mechanics, whose variational form is given by inequalities expressing the principle of virtual power in its inequality form. The elliptic, linear symmetric operators as well as convex sets of possible states depend on the control parameter. The existence theorem for the optimal control will be applied to the design problems for an elastic laminate plate where a variable thickness appears as a control variable.

Keywords: Optimal control, variational inequality, convex set, laminated plate, thickness-function, rigid obstacle

AMS classification: 49A29, 49A27, 49A34

INTRODUCTION

In this work we introduce an abstract framework for the theoretical study of the thickness optimization in the variational inequality context. As already the name implies it is devoted to the problem of finding the optimal thickness of a laminate plate. The omnipresence of plates and plate-like structures in modern technology is well known and needs no particular elaboration. Whether the concern is with aircraft and missile surface (skin) components, reinforced concrete floor slabs, glass-window panes, electric circuit boards, or certain layered geological formations, engineers and analysts are frequently called upon to predict deformations, stresses of elastic plates. We introduce an abstract framework for the theoretical study of the thickness optimization in the variational inequality context. Thus, we consider an optimal control problem in which the state variable of the system (which includes an elliptic, linear, symmetric operator, the coefficients of which are chosen as the design-control variables) is defined as the (unique) solution of a variational inequality. We give sufficient conditions for the existence of an optimal control. The most characteristic property of the variational inequalities is that their solution does not depend smoothly on the control, i.e. one cannot differentiate the solution of the state problem with respect to the control. This may lead to complications especially in the design of the solution procedures since the objective functional will not be smooth. The existence result proved in Section 1 can be applied to the optimal design of an elastic laminate plate with a rigid obstacle. The problem is formulated and solved in terms of the plate displacement components, an optimal design problem of a laminate is introduced and investigated. The role of a design variable is played by the thickness of the faces. The optimal control for a system, governed by an elliptic variational inequality, was proposed by J. L. Lions [13] and discussed in Mignot [15], V. Barbu [3], S. Shuzhong [24] and F. Mignot and Puel [16], J. P. Yvon [23].

1. On the convergence of sets

Let $V(\Omega)$ be as normed linear space. Following Mosco [17] we introduce a convergence of sequences of subsets of $V(\Omega)$.

Definition 1. A sequence $\{K_n(\Omega)_n\}$ of the spaces $V(\Omega)$ *converges* to a set $K(\Omega) \subset V(\Omega)$ if

$$\begin{cases} 1^\circ & K(\Omega) \text{ contains all weak limits of sequences } \{u_{n_k}\}_{n_k},\ u_{n_k} \in K_{n_k}(\Omega), \text{ where } \{K_{n_k}(\Omega)\} \text{ are arbitrary subsequences of } \{K_n(\Omega)\}; \\ 2^\circ & \text{every element } v \in K(\Omega) \text{ is the strong limit of some sequence } \{v_n\},\ v_n \in K_n(\Omega). \end{cases}$$

Notation. $K(\Omega) = \operatorname*{Lim}_{n\to\infty} K_n(\Omega)$.
Let $\mathscr{W}\colon V(\Omega) \to (-\infty, \infty]$ be a functional. The set

$$\operatorname{epi}\mathscr{W} := \{(v,\beta) \in V(\Omega) \times \mathbf{R}\colon \mathscr{W}(v) \leqslant \beta\}$$

is called the epigraf of $\mathscr{W}$, and the effective domain of $\mathscr{W}$ is a subset of $V(\Omega)$,

$$D\mathscr{W} \text{ (or } \operatorname{dom}\mathscr{W}) = \{v\colon \mathscr{W}(v) < +\infty, v \in V(\Omega)\}.$$

Moreover, the subdifferential $\partial\mathscr{W}$ is an operator from $V(\Omega)$ to $V^*(\Omega)$ given by $\partial\mathscr{W}(z) = \{z^* \in V^*(\Omega), \langle z^*, v - z\rangle_{V(\Omega)} \leqslant \mathscr{W}(v) - \mathscr{W}(z)$ for all $v \in V(\Omega)$, for $z \in V(\Omega)$ with $\mathscr{W}(z) < \infty$ and by $\partial\mathscr{W}(z) = \emptyset$ for $z \in V(\Omega)$ with $\mathscr{W}(z) = \infty\}$.

Definition 2. A sequence $\{\mathscr{W}_n\}$ of functionals from $V(\Omega)$ into $(-\infty,\infty]$ *converges* to $\mathscr{W}\colon V(\Omega)\to(-\infty,\infty]$ in $V(\Omega)$, if $\operatorname{epi}\mathscr{W}=\operatorname*{Lim}_{n\to\infty}\operatorname{epi}\mathscr{W}_n$. We use the notation $\mathscr{W}=\operatorname*{Lim}_{n\to\infty}\mathscr{W}_n$.

Let us recall the following lemma of Mosco on the convergence of functionals in $V(\Omega)$.

$$\begin{cases} 1^\circ & \text{For every } v\in V(\Omega) \text{ there exists a sequence } \{v_n\}\subset V(\Omega) \text{ such that } \limsup_{n\to\infty}\mathscr{W}_n(v_n)\leqslant\mathscr{W}(\Omega).\\ 2^\circ & \text{For every subsequence } \{\mathscr{W}_{n_k}\} \text{ of } \{\mathscr{W}_n\} \text{ and every sequence } \{v_k\}\subset V(\Omega) \text{ weakly convergent to } v\in V(\Omega) \text{ the inequality } \mathscr{W}(v)\leqslant\liminf_{n\to\infty}\mathscr{W}_{n_k}(v_k) \text{ holds.}\end{cases}$$

Lemma 1. *Let $\mathscr{W}_n\colon V(\Omega)\to(-\infty,\infty]$, $n=1,2,\ldots$. Then $\mathscr{W}=\operatorname*{Lim}_{n\to\infty}\mathscr{W}_n$ and one of the following conditions holds:*

$$\begin{cases} 1^\circ & \textit{For every } v\in V(\Omega) \textit{ there exists a sequence } \{v_n\}\subset V(\Omega) \textit{ such that } \lim_{n\to\infty}v_n=v \textit{ (strongly) in } V(\Omega) \textit{ and } \limsup_{n\to\infty}\mathscr{W}_n(v_n)\leqslant\mathscr{W}(\Omega).\\ 2^\circ & \textit{For every subsequence } \{\mathscr{W}_{n_k}\} \textit{ of } \{\mathscr{W}_n\} \textit{ and every sequence } \{v_k\}\subset V(\Omega) \textit{ weakly convergent to } v\in V(\Omega) \textit{ the inequality } \mathscr{W}(v)\leqslant\liminf_{n\to\infty}\mathscr{W}_{n_k}(v_k) \textit{ holds.}\end{cases}$$

We shall denote by $\mathscr{E}_0(V(\Omega))$ the family of all lower semicontinuous convex functionals $\mathscr{W}\colon V(\Omega)\to(-\infty,\infty]$, not identically equal to $+\infty$. Moreover, $\mathscr{E}_0(V(\Omega))$ is a subset of the family $\mathscr{E}(V(\Omega))$ of all l.s.c. functionals, $\mathscr{W}\colon V(\Omega)\to[-\infty,+\infty]$.

Remark 1. Due to the previous lemma the condition $\mathscr{W}=\operatorname*{Lim}_{n\to\infty}\mathscr{W}_n$ implies that for every $v\in V(\Omega)$ there exists a sequence $\{v_n\}\subset V(\Omega)$ such that $\lim_{n\to\infty}v_n=v$ (strongly) in $V(\Omega)$ and $\lim_{n\to\infty}\mathscr{W}_n(v_n)=\mathscr{W}(v)$.

Let $U(\Omega)$ be a reflexive Banach space of controls with a norm $\|.\|_{U(\Omega)}$. Let $U_{\mathrm{ad}}(\Omega)\subset U(\Omega)$ be a set of admissible controls compact in $U(\Omega)$. Further, denote by $V(\Omega)$ a real Hilbert space with an inner product $(.,.)_{V(\Omega)}$ and a norm $\|.\|_{V(\Omega)}$, by $V^*(\Omega)$ its dual space with a norm $\|.\|_{V^*(\Omega)}$ and with the duality pairing $\langle.,.\rangle_{V(\Omega)}$.

Let constants ϱ_0, ϱ_1 $(0<\varrho_0<\varrho_1)$ be given. We denote by $E(\varrho_0,\varrho_1)$ the class of the linear, continuous and symmetric operators $\mathscr{A}\colon V(\Omega)\to V^*(\Omega)$ such that $\varrho_0\|v\|^2_{V(\Omega)}\leqslant\langle\mathscr{A}v,v\rangle_{V(\Omega)}\leqslant\varrho_1\|v\|^2_{V(\Omega)}$ for all $v\in V(\Omega)$.

We introduce the systems $\{\mathscr{K}(e,\Omega)\}$, $\{A(e)\}$ of convex closed subsets $\mathscr{K}(e,\Omega)\subset V(\Omega)$ and linear bounded operators $A(e)\in L(V(\Omega),V^*(\Omega))$, $e\in U_{\mathrm{ad}}$, satisfying the following assumptions:

$$
(\mathrm{H1}) \quad \begin{cases} 1^\circ & \bigcap_{e\in U_{\mathrm{ad}}(\Omega)} \mathscr{K}(e,\Omega) \neq \emptyset; \\ 2^\circ & e_n \to e_0 \text{ (strongly) in } U(\Omega) \Rightarrow \mathscr{K}(e_0,\Omega) = \operatorname*{Lim}_{n\to\infty} \mathscr{K}(e_n,\Omega); \\ 3^\circ & \|A(e)\|_{L(V(\Omega),V^*(\Omega))} \leqslant M \text{ for all } e \in U_{\mathrm{ad}}(\Omega); \\ 4^\circ & \langle A(e)v,v\rangle_{V(\Omega)} \geqslant \alpha\|v\|^2_{V(\Omega)},\ \alpha > 0, \text{ for all } e \in U_{\mathrm{ad}}(\Omega) \text{ and} \\ & v \in V(\Omega) \text{ (a real number } \alpha \text{ not depending on } e \text{ and } v,\ A(e) \text{ is} \\ & \text{said to be uniformly coercive with respect to } U(\Omega)); \\ 5^\circ & e_n \to e_0 \text{ (strongly) in } U(\Omega) \Rightarrow A(e_n) \to A(e_0) \\ & \text{in } L(V(\Omega),V^*(\Omega)),\ e_n \in U_{\mathrm{ad}}(\Omega). \end{cases}
$$

Thus, by virtue of ((H1), 3°, 4°), $A(e_n)$, $n = 1, 2, \ldots$ and $A(e_0)$ are elements of the class $E(\alpha, M)$ for each sequence $\{e_n\}_n$, where $e_n \to e_0$ (strongly) in $U(\Omega)$.

Moreover, we suppose:

$$
(\mathrm{E1}) \quad \begin{cases} 1^\circ & \text{There is a system of functionals } \{\Phi(e_n,.)\}_n \text{ on } V(\Omega) \text{ with val-} \\ & \text{ues in } (-\infty,\infty] \text{ (not identically equal to } +\infty) \text{ semicontinuous} \\ & \text{and convex on } V(\Omega),\ \{v \in V(\Omega)\colon \Phi(e_n,v) < \infty\} \subset \mathscr{K}(e_n,\Omega), \\ & \Phi(e,.) = \operatorname*{Lim}_{n\to\infty} \Phi(e_n,.) \text{ as } e_n \to e \text{ (strongly) in } U(\Omega). \\ 2^\circ & \{L(e_n)\}_n \text{ is a sequence in } V^*(\Omega) \text{ such thast } L(e_n) \to L(e) \\ & \text{(strongly) in } V^*(\Omega) \text{ as } e_n \to e \text{ (strongly) in } U(\Omega). \end{cases}
$$

Further we assume that for each sequence $\{e_n\}$, $e_n \to e$ (strongly) in $U(\Omega)$ there is a bounded sequence $\{a_n\}_n$ with $a_n \in \mathscr{K}(e_n,\Omega)$ and $\Phi(e_n, a_n) < \infty$ for all n, $e_n \in U_{\mathrm{ad}}(\Omega)$ such that

$$
\limsup_{n\to\infty} \Phi(e_n, a_n) < \infty. \tag{1.1}
$$

There exist two possible constants c_1, c_2 such that for each sequence $\{e_n\}$, $e_n \to e$ (strongly) in $U(\Omega)$,

$$
\Phi(e_n, v_n) \geqslant -c_1\|v_n\|_{V(\Omega)} - c_2 \quad \text{for } n = 1, 2, \ldots \quad \text{(see [18])}. \tag{1.2}
$$

Then, since $A(e_n) \in E(\alpha, M)$ for any sequence of pairs $\{[e_n, v_n]\}_n$, $e_n \in U_{\mathrm{ad}}(\Omega)$, $n = 1, 2, \ldots$ with $\|v_n\|_{V(\Omega)} \to \infty$ and $e_n \to e$ (strongly) in $U(\Omega)$ we have

$$
\frac{[\langle A(e_n)v_n, v_n - a_n\rangle_{V(\Omega)} + \Phi(e_n, v_n)]}{\|v_n\|_{V(\Omega)}} \to \infty. \tag{1.3}
$$

Moreover, for each n

$$\frac{[\langle A(e_*)v, v - a_n\rangle_{V(\Omega)} + \Phi(e_n, v)]}{\|v\|_{V(\Omega)}} \to \infty. \tag{1.4}$$

as $\|v\|_{V(\Omega)} \to \infty$, $v \in \mathscr{K}(e_n, \Omega)$ where $e_* \in U_{\mathrm{ad}}(\Omega)$ is arbitrary but fixed in $U_{\mathrm{ad}}(\Omega)$, $e_n \in U_{\mathrm{ad}}(\Omega)$, $n = 1, 2, \dots$ and $A(e_*) \in E(\alpha, M)$.

Remark 2. By virtue of ((H1), 3°, 4°) and (1.1) we can write

$$[\langle A(e_n)v_n, v_n - a_n\rangle_{V(\Omega)} + \Phi(e_n, v_n)] \geqslant \alpha\|v_n - a_n\|^2_{V(\Omega)} - c_3\|v_n - a_n\|_{V(\Omega)} - c_4$$

where a_n is bounded in $\mathscr{K}(e_n, \Omega)$ $(n = 1, 2, \dots)$ and when $\|v_n\|_{V(\Omega)} \to \infty$ then also $\|v_n - a_n\|_{V(\Omega)} \to \infty$. In a similar way (for each n) we obtain relation (1.4).

Let $B \in L(U(\Omega), V^*(\Omega))$, $f \in V^*(\Omega)$. It is well known [3] that for every $e \in U_{\mathrm{ad}}(\Omega)$ there exists a unique solution

$$\begin{cases} u(e) \in \mathscr{K}(e, \Omega) \\ \langle A(e)u(e), v - u(e)\rangle_{V(\Omega)} + \Phi(e, v) - \Phi(e, u(e)) \geqslant \langle L(e), v - u(e)\rangle_{V(\Omega)} \end{cases} \tag{1.5}$$

for all $v \in \mathscr{K}(e, \Omega)$, where $L(e) = f + Be$.

Further, consider a functional $\mathscr{L}\colon U(\Omega) \times V(\Omega) \to \mathbf{R}$ for which the following condition holds:

$$\begin{cases} e_n \to e \text{ (strongly) in } U(\Omega),\ v_n \rightharpoonup v \text{ in } V(\Omega) \text{ (weakly)} \Rightarrow \\ \Rightarrow \mathscr{L}(e, v) \leqslant \lim_{n\to\infty} \inf \mathscr{L}(e_n, v_n). \end{cases} \tag{E2}$$

We shall formulate the optimal control in the following way:

Problem $(\mathscr{P}_0)$. Find a control $e_0 \in U_{\mathrm{ad}}(\Omega)$ such that

$$\begin{aligned} &\langle A(e_0)u(e_0), v - u(e_0)\rangle_{V(\Omega)} + \Phi(e_0, v) - \Phi(e_0, u(e_0)) \\ &\qquad \geqslant \langle L(e_0), v - u(e_0)\rangle_{V(\Omega)} \quad \text{for all } v \in \mathscr{K}(e_0, \Omega), \end{aligned} \tag{1.6}$$

$$\mathscr{L}(e_0, u(e_0)) = \min_{e \in U_{\mathrm{ad}}(\Omega)} \mathscr{L}(e, u(e)). \tag{1.7}$$

Theorem 1. *Let the assumptions* (H0), (E1), (E2), (1.1), (1.2), (1.3) *be satisfied. Then there exists at least one solution e_0 of the optimal control problem $(\mathscr{P}_0)$.*

Proof. As the solution $u(e)$ of the variational inequality (1.5) is uniquely determined for every $e \in U_{\mathrm{ad}}(\Omega)$, we can introduce the functional $J(e)$ as

(1.8) $$J(e) = \mathscr{L}(e, u(e)) \quad e \in U_{\mathrm{ad}}(\Omega).$$

Due to the compactness of $U_{\mathrm{ad}}(\Omega)$ in $U(\Omega)$, there exists a sequence $\{e_n\} \subset U_{\mathrm{ad}}(\Omega)$ such that

(1.9) $$\lim_{n\to\infty} J(e_n) = \inf_{e\in U_{\mathrm{ad}}(\Omega)} J(e).$$

(1.10) $$\lim_{n\to\infty} e_n = e_0 \quad \text{in } U(\Omega), \quad e_0 \in U_{\mathrm{ad}}(\Omega).$$

Denoting $u(e_n) := u_n \in \mathscr{K}(e_n, \Omega)$ we obtain the inequality

(1.11) $$\langle A(e_n)u_n, u_n - v\rangle_{V(\Omega)} - \langle L(e_n), u_n - v\rangle_{V(\Omega)} \leqslant \Phi(e_n, v) - \Phi(e_n, u_n)$$
$$\text{for all } v \in \mathscr{K}(e_n, \Omega).$$

In particular, taking a_n for v in (1.11) we obtain

(1.12) $$\langle A(e_n)u_n - L(e_n), u_n - a_n\rangle_{V(\Omega)} + \Phi(e_n, u_n) \leqslant \Phi(e_n, a_n)$$
$$\text{for every } n.$$

Hence the relations (1.1), (1.3) and ((E1),2°) imply that $\{u_n\}_n$ is a bounded sequence. This implies the existence of a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ and an element $u_0 \in V(\Omega)$ such that

(1.13) $$u_{n_k} \rightharpoonup u_0 \quad \text{(weakly) in } V(\Omega).$$

As $u_{n_k} \in \mathscr{K}(e_{n_k}, \Omega)$, the assumption ((H1), 2°) implies

(1.14) $$u_0 \in \mathscr{K}(e_0, \Omega).$$

Then we observe from Lemma 1, ((E1), 1°), (1.1) and (1.2) that

(1.15) $$\begin{aligned}\Phi(e_0, u_0) &\leqslant \liminf_{k\to\infty} \Phi(e_{n_k}, u_{n_k}) \\ &\leqslant \limsup_{k\to\infty} \{\Phi(e_{n_k}, u_{n_k}) - \langle A(e_{n_k})u_{n_k} - L(e_{n_k}), u_{n_k} - a_{n_k}\rangle_{V(\Omega)}\} \\ &< \infty\end{aligned}$$

since by virtue of the monotonicity of $A(e_{n_k})$ one has

$$\begin{aligned}
&|\langle A(e_{n_k})u_{n_k}, a_{n_k} - u_{n_k}\rangle_{V(\Omega)}| \\
&\qquad \leqslant \langle A(e_{n_k})a_{n_k}, a_{n_k}\rangle_{V(\Omega)} + |\langle A(e_{n_k})a_{n_k}, u_{n_k}\rangle_{V(\Omega)}| \\
&\qquad \leqslant 2Mc^2, \quad \text{where } \|u_{n_k}\|_{V(\Omega)},\ \|a_{n_k}\|_{V(\Omega)} \leqslant c.
\end{aligned}$$

On the other hand, by virtue of Lemma 1, ((E1), 1°) and Remark 1 there exists a sequence $\{h_{n_k}\}_k \subset V(\Omega)$ such that $h_{n_k} \to u_0$ (strongly) in $V(\Omega)$ and

$$\lim_{k\to\infty} \Phi(e_{n_k}, h_{n_k}) = \Phi(e_0, u_0). \tag{1.16}$$

Here, note that $h_{n_k} \in \mathscr{K}(e_{n_k}, \Omega)$ for all k, which follows from the assumption ((E1), 1°) and (1.15), (1.16), so that

$$\langle A(e_{n_k})u_{n_k} - L(e_{n_k}), u_{n_k} - h_{n_k}\rangle_{V(\Omega)} \leqslant \Phi(e_{n_k}, h_{n_k}) - \Phi(e_{n_k}, u_{n_k}) \quad \text{for all } k \tag{1.17}$$

Moreover, from ((H1), 3°) and (1.13) we obtain

$$\|A(e_{n_k})u_{n_k}\|_{V^*(\Omega)} \leqslant C \quad \text{for } k = 1, 2, \dots. \tag{1.18}$$

Then there exists an element $\chi \in V^*(\Omega)$ and a subsequence $\{A(e_{n_{k_j}})u_{n_{k_j}}\}_j$ of $\{A(e_{n_k})u_{n_k}\}_k$ such that

$$A(e_{n_{k_j}})u_{n_{k_j}} \rightharpoonup \chi \quad \text{(weakly) in } V^*(\Omega). \tag{1.19}$$

Thus, by passing to the limit in (1.17) we have

$$\begin{aligned}
&\limsup_{j\to\infty} \langle A(e_{n_{k_j}})u_{n_{k_j}}, u_{n_{k_j}} - u_0\rangle \\
&\qquad \leqslant \limsup_{j\to\infty} \langle A(e_{n_{k_j}})u_{n_{k_j}} - L(e_{n_{k_j}}), u_{n_{k_j}} - h_{n_{k_j}}\rangle_{V(\Omega)} \\
&\qquad \leqslant \limsup_{j\to\infty} \Phi(e_{n_{k_j}}, h_{n_{k_j}}) - \liminf_{j\to\infty} \Phi(e_{n_{k_j}}, u_{n_{k_j}}) \leqslant 0 \quad \text{for all } j.
\end{aligned} \tag{1.20}$$

However, combining the relation (1.19) with the inequality (1.20) we arrive at

$$\limsup_{j\to\infty} \langle A(e_{n_{k_j}})u_{n_{k_j}}, u_{n_{k_j}}\rangle_{V(\Omega)} \leqslant \langle \chi, u_0\rangle_{V(\Omega)}. \tag{1.21}$$

Moreover, the monotonicity of $A(e_{n_{k_j}})$ on $V(\Omega)$ $\big(A(e_{n_{k_j}}) \in E(\alpha, M),\ j = 1, 2, \dots\big)$ implies (in view of (1.21))

$$\begin{aligned}
\langle \chi, u_0\rangle_{V(\Omega)} \geqslant \limsup_{j\to\infty} [&\langle A(e_{n_{k_j}})v, u_{n_{k_j}} - v\rangle_{V(\Omega)} \\
&+ \langle A(e_{n_{k_j}})u_{n_{k_j}}, v\rangle_{V(\Omega)}], \quad j = 1, 2, \dots.
\end{aligned} \tag{1.22}$$

Relations (1.10), (1.13), (1.19) and ((H1), 5°), (1.22) enable us to write

$$\langle \chi - A(e_0)v, u_0 - v \rangle_{V(\Omega)} \geqslant 0 \quad \text{for all } v \in V(\Omega).$$

Let $v = u_0 + t(w - u_0)$, $t \in \mathbf{R}^+$ and $w \in V(\Omega)$. Then we get

$$\text{(1.23)} \qquad \langle \chi - A(e_0)[u_0 + t(w - u_0)], u_0 - w \rangle_{V(\Omega)} \geqslant 0 \quad \text{for any } w \in V(\Omega).$$

For $v = u_0 - t(w - u_0)$ we can analogously write

$$\text{(1.24)} \qquad \langle \chi - A(e_0)[u_0 - t(w - u_0)], w - u_0 \rangle_{V(\Omega)} \geqslant 0.$$

Then combining (1.23) with (1.24) for $t \to 0$ we see that

$$\langle \chi - A(e_0)u_0, u_0 - w \rangle_{V(\Omega)} = 0 \quad \text{for any } w \in V(\Omega).$$

This means that

$$\text{(1.25)} \qquad \chi = A(e_0)u_0.$$

$$\text{(1.26)} \qquad A(e_{n_{k_j}})u_{n_{k_j}} \rightharpoonup A(e_0)u_0 \quad \text{(weakly) in } V^*(\Omega)$$

Using again the monotonicity of $A(e_{n_{k_j}})$ we have

$$\langle A(e_{n_{k_j}})u_{n_{k_j}}, u_{n_{k_j}} - u_0 \rangle_{V(\Omega)} \leqslant \langle A(e_{n_{k_j}})u_0, u_{n_{k_j}} - u_0 \rangle_{V(\Omega)} \quad j = 1, 2, \ldots$$

Next, by the convergences (1.10) and (1.13), by assumption ((H1), 5°) and by the last inequality we obtain

$$\lim_{j \to \infty} \inf \langle A(e_{n_{k_j}})u_{n_{k_j}}, u_{n_{k_j}} - u_0 \rangle_{V(\Omega)} \geqslant 0,$$

which compared with (1.20) leads to

$$\text{(1.27)} \qquad \lim_{j \to \infty} \langle A(e_{n_{k_j}})u_{n_{k_j}} u_{n_{k_j}} - u_0 \rangle_{V(\Omega)} = 0.$$

Clearly (by virtue of (1.26) and (1.27))

$$\text{(1.28)} \qquad \lim_{j \to \infty} \langle A(e_{n_{k_j}})u_{n_{k_j}}, u_{n_{k_j}} \rangle_{V(\Omega)} = \langle A(e_0)u_0, u_0 \rangle_{V(\Omega)}.$$

We shall show that

$$(1.29)\qquad \langle A(e_0)u_0 - L(e_0), u_0 - v\rangle_{V(\Omega)} \leqslant \Phi(e_0, v) - \Phi(e_0, u_0) \quad \text{for all } v \in \mathscr{K}(e_0, \Omega).$$

Let v be any element of $\mathscr{K}(e_0, \Omega)$. If $\Phi(e_0, v) = +\infty$, then (1.29) is trivial. Thus, assume $\Phi(e_0, v) < \infty$. According to Lemma 1 and ((E1), 1°) again, there is a sequence $\{\omega_{n_{k_j}}\}_j$ with $\omega_{n_{k_j}} \in \mathscr{K}(e_{n_{k_j}}, \Omega)$ for all j strongly convergent to v such that

$$(1.30)\qquad \lim_{j\to\infty} \Phi(e_{n_{k_j}}, \omega_{n_{k_j}}) = \Phi(e_0, v).$$

Since $L(e_{n_{k_j}}) \to L(e_0)$ (strongly) in $V^*(\Omega)$ as $j \to \infty$ and

$$\langle A(e_{n_{k_j}})u_{n_{k_j}} - L(e_{n_{k_j}}), u_{n_{k_j}} - \omega_{n_{k_j}}\rangle_{V(\Omega)} \leqslant \Phi(e_{n_{k_j}}, \omega_{n_{k_j}}) - \Phi(e_{n_{k_j}}, u_{n_{k_j}})$$

for all j, we obtain (1.29) by letting $j \to \infty$ and using (1.15), (1.28) and (1.30). As the element $v \in \mathscr{K}(e_0, \Omega)$ is chosen arbitrary we get $u_0 \equiv u(e_0)$ and

$$u(e_n)(= u_n) \rightharpoonup u(e_0)(= u_0) \quad \text{(weakly) in } V(\Omega).$$

Then (E2), (1.9) yield

$$\mathscr{L}(e_0, u(e_0)) \leqslant \lim_{n\to\infty} \inf \mathscr{L}(e_n, u(e_n)) = \inf_{e\in U_{ad}(\Omega)} \mathscr{L}(e, u(e)),$$

hence

$$\mathscr{L}(e_0, u(e_0)) = \inf_{e\in U_{ad}(\Omega)} \mathscr{L}(e, u(e)).$$

which completes the proof. □

Due to $A(e_0)$, $A(e_n) \in E(\alpha, M)$ for $n = 1, 2, \ldots$, the strong convergence will follow from the relation

$$\begin{aligned} &\alpha \lim_{n\to\infty} \sup \|u(e_n) - u(e_0)\|^2_{V(\Omega)} \\ &\qquad\leqslant \lim_{n\to\infty} \langle A(e_n)(u(e_n) - u(e_0)), u(e_n) - u(e_0)\rangle_{V(\Omega)} \\ &\qquad= \lim_{n\to\infty} \{\langle A(e_n)u(e_n), u(e_n)\rangle_{V(\Omega)} + \langle A(e_n)u(e_0), u(e_0)\rangle_{V(\Omega)} \\ &\qquad\quad - 2\langle A(e_n)u(e_0), u(e_n)\rangle_{V(\Omega)}\} = 0 \end{aligned}$$

(by virtue of ((H1), 4°, 5°) and (1.13), (1.28)).

In the case of conventional materials (as e.g. isotropic metals) linear-elastic Kirchhoff plate problems usually split into two different topics, the one of pure inplane (extensional) deformations and the other of transversal (bending) deformations. In the case of new materials as laminated fiber-reinforced composites, however, coupling between bending and extension is possible and requires an appropriate theoretical consideration for engineering purposes.

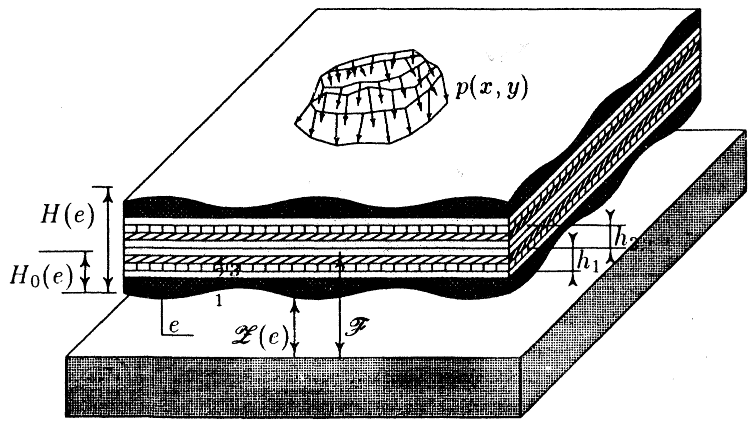

Fig. 1

The laminate plate considered is supposed to be thin enough so that deformations are in accordance with the familiar Kirchhoff hypothesis. Thus, a line, originally straight and perpendicular to the middle surface of the plate (XY-plane), remains straight and perpendicular to the middle surface when the plate is stretched and bent. In addition, the normals are presumed to have constant length. This implies that the inplane deflections u_* and v_* and the transversal deflection w_* can be represented by the midplane deflections u, v, w in the following way:

$$(2.1) \qquad \begin{cases} u_*(x,y,z) = u(x,y) - \dfrac{z\partial w(x,y)}{\partial x}, \\[2ex] v_*(x,y,z) = v(x,y) - \dfrac{z\partial w(x,y)}{\partial y}, \\[2ex] w_*(x,y,z) = w(x,y). \end{cases}$$

Then the non-vanishing (infinitesimal) strain components are

$$\varepsilon_x^*(x,y,z)=\frac{\partial u_*(x,y,z)}{\partial x},\quad \varepsilon_y^*(x,y,z)=\frac{\partial v_*(x,y,z)}{\partial y},$$
$$\gamma_{xy}^*(x,y,z)=\frac{\partial u_*(x,y,z)}{\partial y}+\frac{\partial v_*(x,y,z)}{\partial x}.$$

The strain-displacement relations take the form

$$\begin{aligned}(2.2)\qquad &\varepsilon_x^*(x,y,z)=\varepsilon_x(x,y)+zk_x(x,y),\quad \varepsilon_y^*(x,y,z)=\varepsilon_y(x,y)+zk_y(x,y),\\ &\varepsilon_z^*(x,y,z)=0,\quad \gamma_{yz}^*(x,y,z)=0,\quad \gamma_{zx}^*(x,y,z)=0,\\ &\gamma_{xy}^*(x,y,z)=\gamma_{xy}(x,y)+zk_{xy}(x,y),\end{aligned}$$

where the quantities ε_x, ε_y, γ_{xy} represent midsurface strains while k_x, k_y, k_{xy} are simple curvatures.

We shall need the following system of strain operators:

$$(2.3)\qquad \begin{cases}\mathscr{N}_x(\mathbf{v})=\dfrac{\partial\xi}{\partial x}, & \mathscr{N}_y(\mathbf{v})=\dfrac{\partial\eta}{\partial y}, \quad \mathscr{N}_{xy}(\mathbf{v})=\dfrac{\partial\xi}{\partial y}+\dfrac{\partial\eta}{\partial x},\\ \mathscr{N}_x^*(\mathbf{v})=\dfrac{\partial^2\theta}{\partial x^2}, & \mathscr{N}_y^*(\mathbf{v})=\dfrac{\partial^2\theta}{\partial y^2}, \quad \mathscr{N}_{xy}^*(\mathbf{v})=\dfrac{\partial^2\theta}{\partial x\partial y}\end{cases}$$

where $\mathbf{v}=\langle\xi,\eta,\theta\rangle$.

(We have the system of six deformation operators—the strain-displacement relations for the small strains theory of a plate.) The stress tensor have the form

$$(2.4)\qquad \sigma_i=C_{ij}\varepsilon_j,\quad i,j=1,2,\dots,6$$

where $\sigma_x=\sigma_1$, $\sigma_y=\sigma_2$, $\sigma_z=\sigma_3$, $\tau_{yz}=\sigma_4$, $\tau_{xz}=\sigma_5$, $\tau_{xy}=\sigma_6$, and $\varepsilon_x=\varepsilon_1$, $\varepsilon_y=\varepsilon_2$, $\varepsilon_z=\varepsilon_3$, $2\gamma_{yz}=\varepsilon_4$, $2\gamma_{zx}=\varepsilon_5$, $2\gamma_{xy}=\varepsilon_6$, C_{ij} is a symmetric matrix, confirming that there are at most 21 independent elastic constants.

Moreover, we apply the constitutive equations in the form

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix}=[Q]\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}.$$

(the stress-strain relations are valid for an arbitrary coordinate system (X,Y,Z) which is rotated by an angle Φ (in the XY-plane) from the (X^*,Y^*,Z) coordinate system—the principal material directions), where $i,j=1,2,4,5,6$,

$$Q_{ij}=C_{ij}-(C_{13}C_{j3}/C_{33})\quad(\sigma_3=0),\qquad Q_{ij}=C_{ij}\quad(\sigma_3\neq 0)$$

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}.$$

The angle definition for the transformed stiffness $Q_{ij}^{(k)}$ for the lamina k is as shown in Fig. 2.

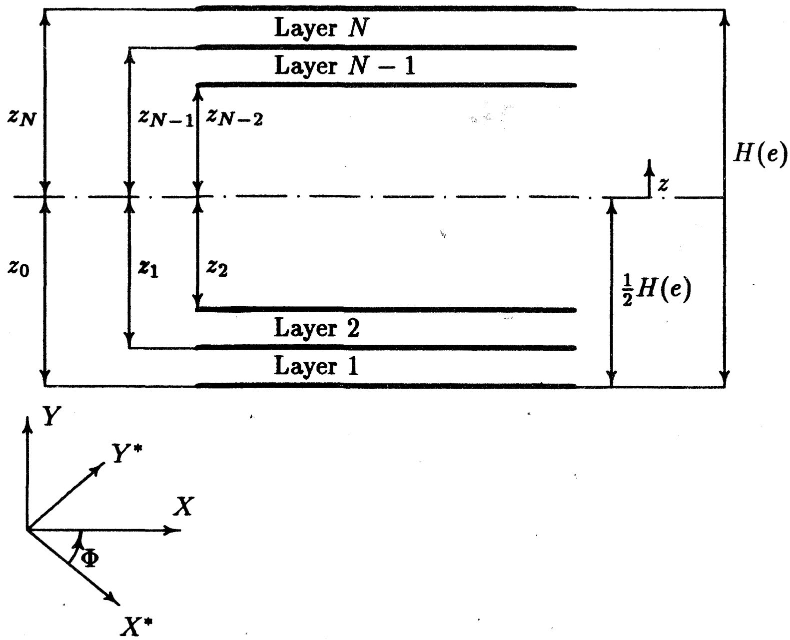

Fig. 2

We assume that $C_{ij}([x,y],t)$ is defined on $\Omega \times [\min H(e), \max H(e)]$ ($H(e)$ is the total laminate thickness) and satisfies the following conditions:

$$(2.5) \quad \begin{cases} C_{ij}(.,t) \text{ is a measurable function on } \Omega \text{ for every } t \in [\min H(e), \\ \max H(e)] \text{ and } C_{ij}([x,y],.) \text{ is a continuous function on } [\min H(e), \\ \max H(e)] \ (e \in U_{\mathrm{ad}}(\Omega)) \text{ for almost every } (x,y) \in \Omega \text{ (Caratheo-} \\ \text{dory-conditions);} \\ \text{there exists a real positive constant } M_* \text{ with } |C_{ij}([x,y],t)| \leqslant M_* \\ \text{for any } t \in [\min H(e), \max H(e)], \text{ a.e. } (x,y) \in \Omega \end{cases}$$

where $U_{\mathrm{ad}}(\Omega)$ is the set of admissible control functions (thickness functions and their properties will be specified below).

Of course, we have also to assume the ellipticity condition

$$(2.6)\qquad \left(C_{ij}([x,y],t)\xi,\xi\right)_{\mathbf{R}^6} \geqslant \alpha_c|\xi|^2_{\mathbf{R}^6}$$

for any $\xi \in \mathbf{R}^6$, for any $t \in [\min H(e), \max H(e)]$ ($e \in U_{\mathrm{ad}}(\Omega)$), $\alpha_c = \text{const.} > 0$, where

$$(\mathbf{a},\mathbf{b})_{\mathbf{R}^6} = \sum_{i=1}^{6} a_i b_i \quad \text{for any } \mathbf{a},\mathbf{b} \in \mathbf{R}^6,\ |\mathbf{a}|_{\mathbf{R}^6} = \Big(\sum_{i=1}^{6} a_i^2\Big)^{1/2}.$$

Next we will consider a laminate plate constructed of a finite number of nonhomogeneous layers of an orthotropic material. It is assumed that all the layers in the plate remain elastic during the deformation and that no slip occurs between any two layers.

The strains and curvatures (the deformation operators (2.3)) give rise to the resultant forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy}. The constitutive relations can be given by the matrix equation

$$(2.7)\qquad \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11}(e) & A_{12}(e) & A_{16}(e) & B_{11}(e) & B_{12}(e) & B_{16}(e) \\ A_{12}(e) & A_{22}(e) & A_{26}(e) & B_{12}(e) & B_{22}(e) & B_{26}(e) \\ A_{16}(e) & A_{26}(e) & A_{66}(e) & B_{16}(e) & B_{26}(e) & B_{66}(e) \\ B_{11}(e) & B_{12}(e) & B_{16}(e) & D_{11}(e) & D_{12}(e) & D_{16}(e) \\ B_{12}(e) & B_{22}(e) & B_{26}(e) & D_{12}(e) & D_{22}(e) & D_{26}(e) \\ B_{16}(e) & B_{26}(e) & B_{66}(e) & D_{16}(e) & D_{26}(e) & D_{66}(e) \end{bmatrix} \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_y \\ \mathcal{N}_{xy} \\ \mathcal{N}_x^* \\ \mathcal{N}_y^* \\ \mathcal{N}_{xy}^* \end{bmatrix}.$$

The matrix in (2.7) is the $[ABD(e)]$ matrix known from the classical lamination theory [20]. It consists of the extensional stiffnesses $A_{ij}(e)$, the coupling stiffnesses $B_{ij}(e)$ and the bending stiffnesses $D_{ij}(e)$, and which are defined by the elements of the matrix $[Q]$. Here $A_{ij}(e) = A_{ij}(.,e(.))$, $B_{ij}(.,e(.))$, $D_{ij}(.,e(.))$ and

$$A_{ij}(e) = \int_{-H(e)/2}^{H(e)/2} Q_{ij}(x,y,z)\,\mathrm{d}z, \quad B_{ij}(e) = \int_{-H(e)/2}^{H(e)/2} Q_{ij}(x,y,z)z\,\mathrm{d}z,$$

$$D_{ij}(e) = \int_{-H(e)/2}^{H(e)/2} Q_{ij}(x,y,z)z^2\,\mathrm{d}z, \quad i,j = 1,2,6.$$

Although our terminology is taken from laminate analysis, in principle, the underlying plate does not necessarily have to be a laminate. For the present purpose, the only important feature is that the constitutive behavior is given in the form of the relationship (2.7).

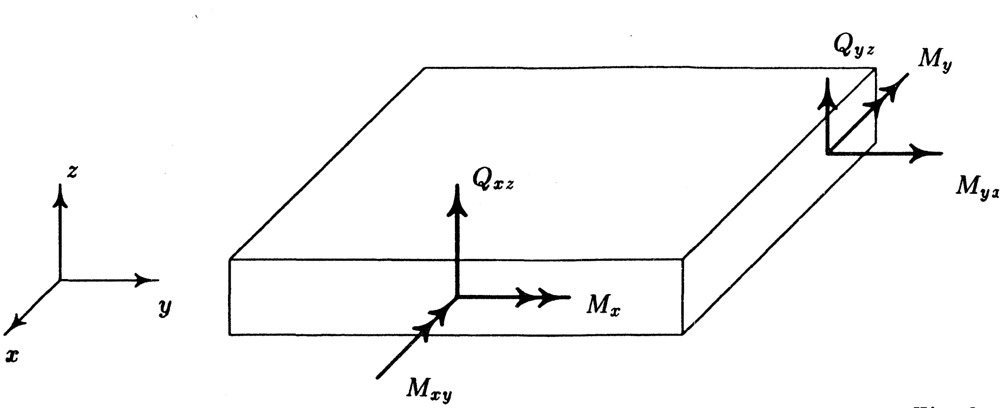

Fig. 3

We consider a laminate plate constructed of a finite number $(N-2)$ of nonhomogeneous uniform thickness layers of an orthotropic material and two (external) variable thickness layers, where the variable thickness is equal to $e(x,y)$ (for $k=1,N$). Thus we have: $H(e)$ (total thickness of plate) $= 2e + \sum_{k=2}^{N-1} h_k$ (h_k = uniform thickness of k-layers).

We assume that the laminate plate is clamped at a part $\partial\Omega_u$ of the boundary $\partial\Omega$ and free at the remaining part $\partial\Omega_F$. Thus one has $\partial\Omega = \partial\Omega_u \cup \partial\Omega_F$, meas $\partial\Omega_u > 0$, $\partial\Omega_u \cap \partial\Omega_F = \emptyset$.

We define the space $\mathbf{W}(\Omega) = [H^1(\Omega)]^2 \times H^2(\Omega)$. Let $\partial\Omega_u$ be an open part of $\partial\Omega$ and let the length of $\partial\Omega_u$ be positive. We define

$$\begin{aligned}
V_0(\Omega) &= \{v \in H^1(\Omega)\colon v = 0 \ \text{ on } \ \partial\Omega_u\},\\
W_0(\Omega) &= \{z \in C^\infty(\overline{\Omega})\colon z = \partial z/\partial n = 0 \ \text{ on } \ \partial\Omega_u\};\\
H^2_{\partial\Omega_u}(\Omega) &\quad \text{is the closure of } W_0(\Omega) \quad \text{in } H^2(\Omega),\\
\mathbf{V}(\Omega) &= [V_0(\Omega)]^2 \times H^2_{\partial\Omega_u}(\Omega) \subset \mathbf{W}(\Omega).
\end{aligned}$$

We denote by $L_2(\Omega)$ the space of all measurable square integrable functions with respect to the Lebesgue measure $\mathrm{d}\Omega = \mathrm{d}x\mathrm{d}y$. $H^k(\Omega)$ is a Hilbert space with the scalar product

$$(v,z)_{H^k(\Omega)} = \sum_{|\alpha|\leqslant k} \int_\Omega D^\alpha v D^\alpha z \,\mathrm{d}\Omega, \quad |\alpha| = \alpha_1 + \alpha_2, \ D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$$

and the norm $(v,v)^{1/2}_{H^k(\Omega)} = \|v\|_{H^k(\Omega)}$, $k = 1,2$.

Further we denote

$$H^k_0(\Omega) = \{v \in H^k(\Omega)\colon v = D^\alpha v = 0 \ \text{ on } \ \partial\Omega \ \text{ for } \ |\alpha| \leqslant k-1\}.$$

It is well known that $H_0^k(\Omega)$ is the Hilbert space with the scalar product

$$(v,z)_{H_0^k(\Omega)} = \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha z \, d\Omega.$$

Moreover, we write $\mathscr{D}(\Omega) = \{v \in \mathscr{E}(\overline{\Omega})$ (the space of functions having derivatives of all orders continuous on Ω and continuously extentable to $\overline{\Omega}$): $\operatorname{supp} v \subset \Omega\}$ (the space of functions with compact supports). We denote by $L_\infty(\Omega)$ the vector space consisting of all functions that are essentially bounded on Ω, with the norm

$$\|v\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{(x,y)\in\overline{\Omega}} |v(x,y)|.$$

Moreover, we note that the system of the deformation operators from the system (2.3) belong to $L(V(\Omega), L_2(\Omega))$-the space of linear continuous operators from $V(\Omega)$ to $L_2(\Omega)$.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary. We set $U(\Omega) = H^2(\Omega)$—a reflexive Banach space with the norm $\|\cdot\|_{U(\Omega)} = \|\cdot\|_{H^2(\Omega)}$. Let us introduce the set of admissible control thickness functions

$$U_{\mathrm{ad}}(\Omega) = \Big\{ e \in H^3(\Omega) \colon 0 < e_{\min} \leqslant e(x,y) \leqslant e_{\max} \text{ for all } (x,y) \in \Omega,$$
$$\|e\|_{H^3(\Omega)} \leqslant c_1, \ \int_\Omega e \, d\Omega = c_2, \ H(e)\big|_{\partial\Omega_u} = \varphi_0, \ \frac{\partial H(e)}{\partial n}\Big|_{\partial\Omega_u} = \varphi_1 \Big\}$$

where φ_0 and φ_1 are given functions, $\varphi_0 \in C(\partial\Omega_u)$, $\varphi_1 \in C(\partial\Omega_u)$, and positive constants c_1, c_2, $e_{\min}$, $e_{\max}$ are chosen in such a way that $U_{\mathrm{ad}}(\Omega) \neq \emptyset$.

It results from the compact imbedding $H^3(\Omega) \hookrightarrow H^2(\Omega)$ that the set $U_{\mathrm{ad}}(\Omega)$ is compact in $U(\Omega)$.

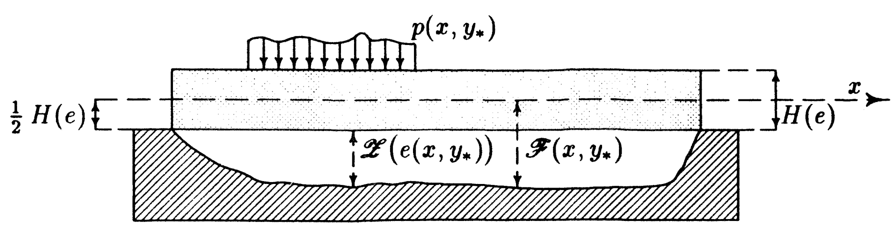

Fig. 4

We further suppose the laminate plate to be forced to lie over a "shallow" obstacle represented by a function $\mathscr{F}(x,y) \colon \Omega \to \mathbf{R}$. Hence the function $\theta(x,y)$ describing the deflection of the plate satisfies (we will consider physical situation such as those in Fig. 4, in which the transverse displacement of a thin anisotropic laminate plate is constrained by the presence of the foundation—a rigid frictionless surface located at a distance $\mathscr{F}(x,y)$ under the middle plane of the plate) $\theta(x,y) \geqslant \mathscr{Z}(e(x,y))$ (admissible transverse displacement) where

$$
(2.8) \quad \begin{cases}
\mathscr{Z}\big(e(x,y)\big) = \mathscr{F}(x,y) + H\big(e(x,y)\big)/2 \ (\leqslant 0), \\
H(e) = 2e + \sum\limits_{k=2}^{N-1} h_k, \\
\text{the function } \mathscr{F}\colon \Omega \to \mathbf{R} \text{ (representing the obstacle lying under the} \\
\text{plate) has to satisfy the conditions} \\
\mathscr{F} \in C^\infty(\overline{\Omega}),\ \mathscr{F}(s) = -\varphi_0(s)/2,\ \partial\mathscr{F}(s)/\partial n = -\varphi_1(s)/2 \text{ for all } s \in \\
\partial\Omega_u,\ \mathscr{Z}(e) \leqslant 0 \text{ (for all } e \in U_{\mathrm{ad}}(\Omega)) \text{ on } \overline{\Omega} \Leftrightarrow H(e)_{\max} \leqslant -2\mathscr{F}(x,y) \\
\text{for any } (x,y) \in \overline{\Omega}.
\end{cases}
$$

Physically, if $\theta(x,y) > \mathscr{Z}\big(e(x,y)\big)$, then the laminate plate does not come in contact with the rigid frictionless surface (shallow obstacle) and no reactive force is developed on the rigid surface. On the other hand, if $\theta(x,y) = \mathscr{Z}\big(e(x,y)\big)$ at a point $(x,y) \in \Omega$, then the laminated plate is in contact with the rigid surface and a transverse reactive force $r^c(e)$ is developed on the plate. Further, we introduce the set of kinematically admissible displacements by

$$
(2.9) \qquad \mathscr{K}(e,\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega) \colon \theta - \mathscr{Z}(e) \geqslant 0 \ \text{ on } \ \Omega\} \quad \text{and} \quad \mathbf{v} = \langle \xi, \eta, \theta \rangle .
$$

Lemma 2. *The set $\mathscr{K}(e,\Omega)$ is nonempty, convex and closed in $\mathbf{V}(\Omega)$. Moreover, the system $\{\mathscr{K}(e,\Omega)\}$ (for any $e \in U_{\mathrm{ad}}(\Omega)$) fulfils the condition ((H1), 1°).*

Proof. $\mathscr{K}(e,\Omega)$ is nonempty (for any $e \in U_{\mathrm{ad}}(\Omega)$). We have $H(e)_{\max} \leqslant -2\mathscr{F}(x,y)$ for any $(x,y) \in \overline{\Omega}$. Hence one has $\mathscr{Z}(e) \leqslant 0$. This implies that $\langle 0,0,0\rangle \in \mathscr{K}(e,\Omega)$ for any $e \in U_{\mathrm{ad}}(\Omega)$. Hence $\mathscr{K}(e,\Omega)$ is nonempty. This means that the condition ((H1),1°) holds. $\mathscr{K}(e,\Omega)$ is closed (for any $e \in U_{\mathrm{ad}}(\Omega)$). Let $\mathbf{v}_n \to \mathbf{v}$ (strongly) in $\mathbf{V}(\Omega)$, where $\mathbf{v}_n \in \mathscr{K}(e,\Omega)$ and $\mathbf{v} \in \mathbf{V}(\Omega)$ ($e \in U_{\mathrm{ad}}(\Omega)$). Hence due to the imbedding theorem for the space $H^2(\Omega)$ we get $\theta_n(x,y) \to \theta(x,y)$ (strongly) in $C(\overline{\Omega})$. Thus, as $\theta_n(x,y) - \mathscr{Z}\big(e(x,y)\big) \geqslant 0$ on Ω, we obtain $\theta(x,y) - \mathscr{Z}\big(e(x,y)\big)$ on Ω and hence $v \in \mathscr{K}(e,\Omega)$ as claimed.

The convexity of $\mathscr{K}(e,\Omega)$ is trivial. □

Lemma 3. *The system of convex closed sets $\{\mathscr{K}(e,\Omega)\}$ defined by (2.9) fulfils the condition ((H1), 1°).*

Proof. Let $e_n \to e$ (strongly) in $U(\Omega)(=H^2(\Omega))$, e_n, $e \in U_{\mathrm{ad}}(\Omega)$. Then there exists a subsequence $\{e_{n_k}\}_k$ of $\{e_n\}_n$ weakly convergent in $H^3(\Omega)$ to the element $e \in U_{\mathrm{ad}}(\Omega)$. Let $\{\langle\xi_n,\eta_n,\theta_n\rangle\}_n \rightharpoonup \langle\xi,\eta,\theta\rangle$ ($\langle\xi_n,\eta_n,\theta_n\rangle \in \mathscr{K}(e_n,\Omega)$, $\langle\xi,\eta,\theta\rangle \in \mathbf{V}(\Omega)$) be weakly converegent in $V(\Omega)$. We then have $\theta_n(x,y) - \mathscr{Z}\big(e_n(x,y)\big) \geqslant 0$ for all $(x,y) \in \Omega$, which by virtue of the compact imbedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ implies that $\theta(x,y) - \mathscr{Z}\big(e(x,y)\big) \geqslant 0$ for all $(x,y) \in \Omega$ and hence $\langle\xi,\eta,\theta\rangle \in \mathscr{K}(e,\Omega)$. This means that the condition ((1°) in Definition 1) holds. Next, let $\langle\xi,\eta,\theta\rangle \in \mathscr{K}(e,\Omega)$, then we put $\langle\xi_n,\eta_n,\theta_n\rangle = \langle\xi,\eta,\theta\rangle + \langle 0,0,(e_n-e)\rangle$. The elements $\{\langle\xi_n,\eta_n,\theta_n\rangle\}_n$ satisfy the conditions $\langle\xi_n,\eta_n,\theta_n\rangle \in \mathscr{K}(e_n,\Omega)$ and $\lim\limits_{n\to\infty} \langle\xi_n,\eta_n,\theta_n\rangle = \langle\xi,\eta,\theta\rangle$ (strongly) in $V(\Omega)$. Thus the condition ((2°) in Definition 1) holds. □

On the open set Ω we now define a bilinear form $\mathbf{a}(e,.,..)\colon \mathbf{V}(\Omega)\times\mathbf{V}(\Omega) \to \mathbf{R}$ (for all $e \in U_{\mathrm{ad}}(\Omega)$) (introducing the energy bilinear form-virtual work equation) by

$$\mathbf{a}(e,\mathbf{v},\mathbf{z}) = \int_\Omega \Big\{ \big\langle \mathscr{N}_x(\mathbf{v}), \mathscr{N}_y(\mathbf{v}), \mathscr{N}_{xy}(\mathbf{v}), \mathscr{N}_x^*(\mathbf{v}), \mathscr{N}_y^*(\mathbf{v}), \mathscr{N}_{xy}^*(\mathbf{v}) \big\rangle$$

$$[ABD(e)] \big\langle \mathscr{N}_x(\mathbf{z}), \mathscr{N}_y(\mathbf{z}), \mathscr{N}_{xy}(\mathbf{z}), \mathscr{N}_x^*(\mathbf{z}), \mathscr{N}_y^*(\mathbf{z}), \mathscr{N}_{xy}^*(\mathbf{z}) \big\rangle^T \Big\} \partial\Omega \tag{2.10}$$

$$\text{for all} \quad \mathbf{v},\mathbf{z} \in \mathbf{V}(\Omega).$$

Moreover, we define a linear functional $\mathbf{L}(e) \in \big(\mathbf{V}(\Omega)\big)^*$ (the load space) by

$$\langle \mathbf{L}(e), \mathbf{v}\rangle_{\mathbf{V}(\Omega)} = \int_\Omega \Big[p - \Big(k_1 e + \sum_{c=2}^{N-1} k_i h_i + k_N e\Big)\Big]\theta \, \mathrm{d}\Omega, \tag{2.11}$$

where $k_i(x,y) \in L_2(\Omega)$ $(. > 0)$, $i = 1, 2, \ldots, N$, $p \in L_2(\Omega)$ (of external loads).

The formula (2.11) defines the virtual work of external loads. (The operator B is continuous and corresponds to the loading caused e.g. by the own weight of the laminated anisotropic plate.) On the other hand, we set

$$\langle \mathbf{A}(e), \mathbf{v}\rangle_{\mathbf{V}(\Omega)} = \mathbf{a}(e,\mathbf{v},\mathbf{z}) \quad \text{for any } \mathbf{v},\mathbf{z} \in \mathbf{V}(\Omega),\ e \in U_{\mathrm{ad}}(\Omega); \tag{2.12}$$

then $\mathbf{A}(e) \in L\big(\mathbf{V}(\Omega), \mathbf{V}^*(\Omega)\big)$ (the operator for the anisotropic elastic laminated plate), $\mathbf{a}(e,.,.)\colon \mathbf{V}(\Omega)\times\mathbf{V}(\Omega) \to \mathbf{R}$ is the Dirichlet form associated with $\mathbf{A}(e)$, $\mathbf{A}(e)\mathbf{v} \in \mathbf{V}^*(\Omega)$ is the canonical isometric operator (by $\mathbf{a}(e.,.,.)$).

The subspace $\mathbf{R}(\Omega) \subset \mathbf{V}(\Omega)$ is the set of the rigid body motion (representing virtual displacements of a rigid laminate plate) given by

$$\mathbf{R}(\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega)\colon (\|\mathscr{N}_x(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_y(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}(\mathbf{v})\|^2_{L_2(\Omega)}$$
$$\|\mathscr{N}_x^*(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_y^*(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}^*(\mathbf{v})\|^2_{L_2(\Omega)}) = 0\}$$

where the system of six deformation operators is defined in (2.3).

Lemma 4. *We have* $\mathbf{R}(\Omega) = \{0\}$.

Proof. Parallel to that of Lemma 2.1 [10]. $\square$

Moreover, in our case we have $\Phi(e, \mathbf{v}) = 0$.

The desired thickness of the laminate plate is given by the distribution $z_d(x, y)$ of the deflection, and we look for a control parameter subject to constrainst, i.e., $e \in U_{\mathrm{ad}}(\Omega)$ such that the system response $w(e)$ has a minimum deviation of $z_d(x, y)$ in any definite sense. This means, that the cost functional is given by the formula

$$\mathscr{L}(e, v) = \int_\Omega [\theta(e) - z_d]^2 \mathrm{d}\Omega. \tag{2.13}$$

Lemma 5. *The family* $\{\mathbf{A}(e)\}$, $e \in U_{\mathrm{ad}}(\Omega)$ *of operators defined by* (2.10) *and* (2.12) *satisfies the assumptions ((H1),* $3°$, $4°$, $5°$*).*

Proof. We define

$$|[ABD(e)](x, y)| = \sup_{\xi \in \mathbf{R}^6 - \{0\}} |[ABD(e)]\xi|/|\xi|$$

for all $e \in U_{\mathrm{ad}}(\Omega)$, (from (2.5) and (2.6) we clearly find that the function $(x, y) \to |[ABD(e)](x, y)|$ belongs to $L_\infty(\Omega)$ by $\|[ABD(e)]\|_{L_\infty(\Omega)}$ we denote the $L_\infty(\Omega)$-norm of the above function), and we then have, from the assumption (2.5) and from the Schwarz inequality

(2.14)

$$\begin{aligned}
|\langle \mathbf{A}(e)\mathbf{v}, \mathbf{z}\rangle_{\mathbf{V}(\Omega)}| \leqslant c\,\|[ABD(e)]\|_{L_\infty(\Omega)} \big(&\|\mathscr{N}_x(\mathbf{v})\|^2_{L_2(\Omega)} \\
&+ \|\mathscr{N}_y(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_x^*(\mathbf{v})\|^2_{L_2(\Omega)} \\
&+ \|\mathscr{N}_y^*(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}^*(\mathbf{v})\|^2_{L_2(\Omega)}\big) \\
\times \big(\|\mathscr{N}_x(\mathbf{z})\|^2_{L_2(\Omega)} &+ \|\mathscr{N}_y(\mathbf{z})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}(\mathbf{z})\|^2_{L_2(\Omega)} + \|\mathscr{N}_x^*(\mathbf{z})\|^2_{L_2(\Omega)} \\
&+ \|\mathscr{N}_y^*(\mathbf{z})\|^2_{L_2(\Omega)} + \|\mathscr{N}_{xy}^*(\mathbf{z})\|^2_{L_2(\Omega)}\big) \leqslant c_{[ABD]}\|\mathbf{v}\|_{\mathbf{V}(\Omega)}\|\mathbf{z}\|_{\mathbf{V}(\Omega)} \\
&\text{for any } \mathbf{v}, \mathbf{z} \in \mathbf{V}(\Omega),\ e \in U_{\mathrm{ad}}(\Omega)
\end{aligned}$$

where the positive constant $c_{[ABD]}$ is independent of $[e, v, z]$. Relation (2.12) implies the continuity of $\langle \mathbf{A}(e).,.\rangle_{\mathbf{V}(\Omega)}$ for all $e \in U_{\mathrm{ad}}(\Omega)$.

Moreover, due to the assumption (2.6) we have (realizing that either $[ABD(e)]$ is uniformly positive definite with respect to $e \in U_{\mathrm{ad}}(\Omega)$ of the energy of the laminate plate is positive definite)

$$\langle \mathbf{A}(e)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}(\Omega)} \geqslant \alpha^*_{[ABD]}\left(\sum_{l=1}^{6} \|\mathcal{N}_l(\mathbf{v})\|^2_{L_2(\Omega)}\right) \tag{2.15}$$

where the positive constant $\alpha^*_{[ABD]}$ is independent of $[e, \mathbf{v}]$ and $\mathcal{N}_1 = \mathcal{N}_x$, $\mathcal{N}_2 = \mathcal{N}_y$, $\mathcal{N}_3 = \mathcal{N}_{xy}$, $\mathcal{N}_4 = \mathcal{N}^*_x$, $\mathcal{N}_5 = \mathcal{N}^*_y$, $\mathcal{N}_6 = \mathcal{N}^*_{xy}$. In our further consideration we shall use results of [7] about the inequalities of Korn's type, employing the same notation as in [7]. The components of a vector displacement $\mathbf{v}$ are denoted by $\xi = v_1$, $\eta = v_2$, $\theta = v_3$. The operators $\mathcal{N}_l$ have the form

$$\mathcal{N}_l v = \sum_{s=1}^{3} \sum_{|\alpha| \leqslant k_s} n_{l_{s\alpha}} D^\alpha v_s, \quad l = 1, 2, \dots, 6.$$

We define the components of the matrix $\left[\mathcal{N}_{l_s}((x, y), (\xi_1, \xi_2))\right]^T_{3\times 6}$ (we see that in the given case $n_{ls\alpha} =$ const. which means that the matrix $\left[\mathcal{N}_{l_s}((x, y), (\xi_1, \xi_2))\right]$ does not depend on (x, y) and on the control variable e) in the following way:

$$\mathcal{N}_{l_s}(\xi_1, \xi_2) \sum_{|\alpha| \leqslant k_s} n_{ls\alpha} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \quad |\alpha| = \alpha_1 + \alpha_2.$$

In our case we have

$$\left[N_{l_s}((x, y), (\xi_1, \xi_2))\right]^T = \begin{bmatrix} \xi_1 & 0 & \xi_2 & 0 & 0 & 0 \\ 0 & \xi_2 & \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1^2 & \xi_2^2 & 2\xi_1\xi_2 \end{bmatrix}_{(3\times 6)}$$

If $[\xi_1, \xi_2] \in C^2$, $[\xi_1, \xi_2] \neq [0, 0]$ then the rank of this matrix is 3 under the conditions imposed above. Then by Theorem 3.1 from [7] we obtain that the system $\{\mathcal{N}_l\}$ is coercive on $\mathbf{V}(\Omega)$, i.e.

$$\sum_{l=1}^{6} \|\mathcal{N}_l(\mathbf{v})\|^2_{L_2(\Omega)} + \|\mathbf{v}\|^2_{L_2(\Omega)} \geqslant \hat{c}_{[ABD]} \|\mathbf{v}\|^2_{V(\Omega)}, \quad \hat{c}_{[ABD]} > 0. \tag{2.16}$$

Then by virtue of (2.15), Theorem 2.3 [7], and Lemma 4 we obtain

$$\langle \mathbf{A}(e)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}(\Omega)} \geqslant \alpha_{[ABD]} \|\mathbf{v}\|^2_{\mathbf{V}(\Omega)} \tag{2.17}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, $e \in U_{\mathrm{ad}}(\Omega)$, with $\alpha_{[ABD]} > 0$ independent of $[e, \mathbf{v}]$.

Now ((H1), 3°, 4°) is an immediate consequence of (2.14) and (2.17). Let $e_n \in U_{\mathrm{ad}}(\Omega)$ be such that $e_n \to e$ (strongly) in $U(\Omega)$. Then

$$
\begin{aligned}
&\left|\langle(\mathbf{A}(e_n)-\mathbf{A}(e))\mathbf{v},\mathbf{z}\rangle_{\mathbf{V}(\Omega)}\right| \\
&\quad = \int_\Omega \Big\{ \big| \langle \mathcal{N}_x(\mathbf{v}), \mathcal{N}_y(\mathbf{v}), \mathcal{N}_{xy}(\mathbf{v}), \mathcal{N}_x^*(\mathbf{v}), \mathcal{N}_y^*(\mathbf{v}), \mathcal{N}_{xy}^*(\mathbf{v})\rangle \\
&\qquad [ABD(e_n-e)]\, \langle \mathcal{N}_x(\mathbf{z}), \mathcal{N}_y(\mathbf{z}), \mathcal{N}_{xy}(\mathbf{z}), \mathcal{N}_x^*(\mathbf{z}), \mathcal{N}_y^*(\mathbf{z}), \mathcal{N}_{xy}^*(\mathbf{z})\rangle^T \big| \Big\}\, d\Omega \\
&\quad \leqslant c^*_{[ABD]}(\|e_n-e\|_{L_\infty(\Omega)} + \|e_n^2-e^2\|_{L_\infty(\Omega)} + \|e_n^3-e^3\|_{L_\infty(\Omega)}) \\
&\qquad \|\mathbf{v}\|_{V(\Omega)}\|\mathbf{z}\|_{V(\Omega)} \to 0 \quad \text{for every } \mathbf{v},\mathbf{z}\in\mathbf{V}(\Omega),\ c^*_{[ABD]}>0.
\end{aligned}
\tag{2.18}
$$

This means that

$$
\begin{aligned}
\|\mathbf{A}(e_n)-\mathbf{A}(e)\|_{L(\mathbf{V}(\Omega),\mathbf{V}^*(\Omega))} &= \sup_{\mathbf{v}\in\mathbf{V}(\Omega),\|\mathbf{v}\|_{\mathbf{V}(\Omega)}=1} \|(\mathbf{A}(e_n)-\mathbf{A}(e))\mathbf{v}\|_{\mathbf{V}^*(\Omega))} \\
&= \sup_{\substack{\mathbf{v}\in\mathbf{V}(\Omega),\\ \|\mathbf{v}\|_{\mathbf{V}(\Omega)}=1}} \ \sup_{\substack{\mathbf{z}\in\mathbf{V}(\Omega),\\ \|\mathbf{z}\|_{\mathbf{V}(\Omega)}=1}} \left|\langle(\mathbf{A}(e_n)-\mathbf{A}(e))\mathbf{v},\mathbf{z}\rangle_{\mathbf{V}(\Omega)}\right| \to 0
\end{aligned}
$$

(due to (2.18)). Consequently, the condition ((H1), 5°) is verified. On the other hand, by (2.11) for $e_n \to e$ (strongly) in $U(\Omega)$ we may write

$$
\begin{aligned}
&|\langle \mathbf{L}(e)-\mathbf{L}(e_n),\mathbf{v}\rangle_{\mathbf{V}(\Omega)}| = |\langle (k_1+k_N)(e_n-e),\theta\rangle_{H^2(\Omega)}| \leqslant \text{const.} \\
&\|e_n-e\|_{C(\overline{\Omega})}\|\mathbf{v}\|_{H^2(\Omega)} \to 0. \quad \text{Hence ((E1), 2°) is satisfied.}
\end{aligned}
\tag{2.19}
$$

Let us verify (E2). For $e \in U_{\mathrm{ad}}(\Omega)$ the functional $\mathcal{L}(e,.)\colon \mathbf{V}(\Omega) \to \mathbf{R}$ is weakly lower semicontinuous. Consequently, we may immediately write

$$
\liminf_{n\to\infty} \mathcal{L}(e_n,\mathbf{v}_n) = \liminf_{n\to\infty} \|\theta_n - z_d\|^2_{L^2(\Omega)} \geqslant \|\theta - z_d\|^2_{L_2(\Omega)} = \mathcal{L}(e,\mathbf{v}).
$$

Thus, from Lemma 5 and the above argument we conclude that all the assumptions of Theorem 1 are satisfied. Hence the existence of a solution of the optimization problem ($\mathcal{B}$) follows (optimization of the thickness of a laminate plate): The optimal control problem ($\mathcal{B}$), where the data are defined above, has at least one solution. □

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OPTIMÁLNE RIADENIE LAMINÁTOVEJ DOSKY S PREKÁŽKOU

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Je študovaná úloha riadenia systému lineárnych rovníc a nerovníc pro laminátovú dosku. Funkcie riadenia vystupujú v koeficientoch operátora nerovnice, v pravej strane a v konvexnej množine prípustkových stavov. Dokazuje sa existencia optimálneho riadenia, na úrovni abstrakcie, pre riadenie variačnou nerovnicou pre jeden tvar účelového funkcionálu. Hrúbku vonkajšej vrstvy laminátovej dosky berieme za funkciu riadenia.

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