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GROUP ANALYSIS METHODS FOR CONSTRUCTION AND INVESTIGATION OF THE BIFURCATION EQUATION

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It is known [1, 2] that the bifurcation equation (BEq) inherits the group symmetry of the corresponding nonlinear problem. Therefore the problem of construction of the general form of BEq by its symmetry group arises. In [1, Ch. IV] this problem was solved on the basis of a known scheme from the invariant theory [3]. However, the methods of group analysis of differential equations [4–6] are more effective for this purpose since they allow to construct the full explicit form of BEq (not only its main part) requiring significantly smaller computing work. In the present paper the general theory is presented while applications considered by the author and his students are indicated in [7–9]. The author is thankful to prof. N. Ch. Ibragimov for his advice concerning the application of group analysis methods. The terminology and notation of the general branching theory of solutions of nonlinear equations are used, see [10].

1. The bifurcation equation $0 = f(\xi, \varepsilon) = \{f_j(\xi, \varepsilon)\}_1^m : \Xi^n \to \Xi^m$ allows the group G (i.e., is invariant with respect to G) if for its certain representations $\mathcal{A}_g$ in Ξ^n and $\mathcal{B}_g$ in Ξ^m we have

$$f(\mathcal{A}_g\xi, \varepsilon) = \mathcal{B}_g f(\xi, \varepsilon). \tag{1}$$

Often we will not indicate the dependence of f on a small parameter ε, which is not essential for the group analysis. The equality (1) means that for the transformation

$$\tilde{\xi} = \mathcal{A}_g\xi, \quad \tilde{f} = \mathcal{B}_g f \tag{2}$$

the manifold $\mathsf{F}\colon f - f(\xi) = 0$ in the vector space Ξ^{n+m} is an invariant manifold. Considering the ℓ-parametric transformation group (2) we shall suppose that F is its nonsingular manifold, i.e. if $\{(X_\nu\,;\,F_\nu)\}_{\nu=1}^{\ell}$ is the basis of the Lie algebra of the infinitesimal operators of the group (2), then the rank $r\{(X_\nu\,;\,F_\nu)\}\big|_F$ of the matrix

$\mathsf{M}\{(X_\nu; F_\nu)\} = [\eta^j_\nu; \xi^k_\zeta]\nu = \overline{1,\ell}; j = \overline{1,n}; k = \overline{n+1, n+m}$ (ν is the number of rows and j, k are the numbers of columns) on the manifold F coincides with its general rank r_*. If now

$$(3) \qquad I_1(\xi, f), \ \ldots, \ I_{n+m-r_*}(\xi, f)$$

is a complete system of functionally independent invariants of the group (2), then according to [4, 5] the manifold F may be represented in the form

$$(4) \qquad \Psi^S\big(I_1(\xi, f), \ldots, I_{n+m-r_*}(\xi, f)\big) = 0, \quad S = 1, \ldots, m,$$

and a necessary condition for the possibility of construction of the general form of BEq is rank $\left[\frac{\partial I_k}{\partial f_j}\right] = m$, which means the independence of the system (3) with respect to the variables f_j. This condition may be replaced by the requirement $r_*(\mathsf{X}, \mathsf{F}) = r_*(\mathsf{X})$ ([4], p. 250). The scheme of construction of the general form of BEq as an invariant manifold of the group (2) presented here leads to the reduction of the order of the BEq with the aid of the complete system of functionally independent invariants ([1], Ch. III).

It should be noted that for the construction of analytic BEq's in high dimensions, the use only of functionally independent invariants in the form of monoms of a minimal degree in ξ reduces generally to omitting certain monomial summands in the expansion of BEq with respect to ξ. In order to take all possible summands into account it is necessary to use additional invariants, which leads to the repetition of monomial summands. This repetition can be removed by factorization on relations between the invariants, which is further denoted by the symbol $[\ldots]^{\text{out}}$.

In applications it is usual to consider the real BEq, i.e.

$$(5) \qquad f_{2k}(\xi, \varepsilon) = \overline{f_{2k-1}(\xi, \varepsilon)}.$$

For real BEq's it is more convenient to realize the above described scheme in the complex variables

$$\xi_{2k-1} = \frac{1}{\sqrt{2}}(\tau_1 + \mathrm{i}\tau_2), \ \xi_{2k} = \bar{\xi}_{2k-1}, \ \xi = \mathrm{C}_0\tau,$$
$$f_{2k-1} = \frac{1}{\sqrt{2}}(t_{2k-1} + \mathrm{i}t_{2k}), \ f_{2k} = \frac{1}{\sqrt{2}}(t_{2k-1} - \mathrm{i}t_{2k});$$
$$\frac{\partial}{\partial \xi_{2k-1}} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial \tau_{2k-1}} - \mathrm{i}\frac{\partial}{\partial \tau_{2k}}\Big), \ \frac{\partial}{\partial \xi_{2k}} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial \tau_{2k-1}} + \mathrm{i}\frac{\partial}{\partial \tau_{2k}}\Big);$$
$$\mathcal{A}_g = \mathcal{B}_g = \mathcal{A}^\xi_{g(\alpha)} = \mathrm{C}_0 \mathcal{A}^\tau_{g(\alpha)} \mathrm{C}_0^{-1} = \mathrm{diag}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha_k}, & 0 \\ 0, & \mathrm{e}^{-\mathrm{i}\alpha_k} \end{pmatrix}.$$

This realization helps overcome technical difficulties connected with taking account of the invariance (5) of BEq with respect to the complex conjugation.

Lemma. *The 2-dimensional analytic BEq allowing the rotation group* $\mathrm{SO}(2)$ *has the form*

$$f_j(\xi) \equiv \sum_{S=0}^{\infty} C_S e^{i(-1)^{j+1}\alpha_S} \xi_j (\xi_1\xi_2)^S = 0, \quad j = 1, 2. \tag{6}$$

If, in addition, it is invariant with respect to the reflection $J(\xi) = (\xi_2, \xi_1)$, *i.e. allows the group* $\mathrm{O}(2)$, *then in (6) we have* $\alpha_S = 0$ *for all* S.

In fact,

$$X_{\xi,f} = iX_{\tau,t} = -\xi\frac{\partial}{\partial\xi} + \bar{\xi}\frac{\partial}{\partial\bar{\xi}} - f\frac{\partial}{\partial f} + \bar{f}\frac{\partial}{\partial\bar{f}}$$

and the basic system of invariants may be chosen in the form

$$I_1 = (\xi\bar{\xi})^{\frac{1}{2}} = (\xi_1\xi_2)^{\frac{1}{2}}, \; I_2 = \frac{f}{\xi} = \frac{f_1}{\xi_1}, \; I_3 = \frac{\bar{f}}{\bar{\xi}} = \frac{f_2}{\xi_2}.$$

Then the bifurcation system as a nonsingular invariant manifold of the $SO(2)$ group may be described in the form

$$f_1 = \xi_1\left[U(\sqrt{\xi_1\xi_2}) + iV(\sqrt{\xi_1\xi_2})\right] = 0, \; f_2 = \xi_2[U(\sqrt{\xi_1\xi_2}) - iV(\sqrt{\xi_1\xi_2})] = 0$$

or, by virtue of its analyticity, in the form of (6). The additional symmetry J gives $\alpha_k = 0$.

2. The results of sec. 1 make it possible to construct the BEq of periodic solutions of problem invariant with respect to the group of motions of $\mathbf{R}^\ell$ ($\ell \geqslant 2$), when this invariance is replaced by the invariance with respect to the crystallographic group $G^1 = G_1 \times \tilde{G}^1$. For example, for $\ell = 3$ the group G_1 is the discrete group of shifts with basic translations a_j, $j = 1, 2, 3$. $\tilde{G}^1$ is the symmetry group of the cell Π_0 constructed on basic translations, $l = m_1 l^{(1)} + m_2 l^{(2)} + m_3 l^{(3)}$ is an arbitrary vector of the inverse lattice [11, 12], $\langle l^{(k)}, a_j \rangle \doteq 2\pi\delta_{jk}$.

We consider the scalar case when in the subspace $N(B)$ of zeros of the linearized at the bifurcation point nonlinear operator a basis is chosen in the form $\varphi_j = \exp(i \langle l_j, q \rangle)$, $j = 1, \ldots, n$, $q = (x, y, z)$. Let us agree to enumerate the elements φ_j in such a way that if an odd number corresponds to a vector l even number corresponds to the vector $-l$. Thus the problem of construction of the general form of the BEq allowing the group $\mathcal{A}_{g(\alpha)} = \mathrm{diag}\{\exp i \langle l_1, \alpha \rangle, \ldots, \exp i \langle l_n, \alpha \rangle\}$ induced in

N(B) by the 3-dimensional shifts $\mathsf{L}_\alpha U(x,y,z) = U(x+\alpha_1, y+\alpha_2, z+\alpha_3)$ and the symmetry group $\tilde{\mathsf{G}}^1$ of elementary cell Π_0 is considered here.

For the case of a simple cubic lattice investigated in [1, 7], $n = \dim \mathsf{N}(\mathsf{B})$ is equal to the number of presentations of an integer $S = |l_j| = m_1^2 + m_2^2 + m_3^2$ in the form of sums of three squares, i.e. the possible values of n_s are the divisors 6, 8, 12, 24, and 48 of the order $|\mathsf{O}_h| = 48$ of the group O_h, or certain sums of these divisors. Since $J \in \mathsf{O}_h$, the invariance of the real BEq with respect to J implies that the coefficients of this BEq are real.

We will carry out the construction of the BEq for the case $S = 3$, $n_S = 8$. Here the BEq is invariant with respect to rotations (the symbol $[2k-1, 2k] \rightarrow \beta_k$ denotes the rotation transformation with the matrix $\mathrm{diag}(e^{i\beta_k}, e^{-i\beta_k})$),

$$
\begin{aligned}
&[1,2] \rightarrow (\alpha_1 + \alpha_2 + \alpha_3), \quad [3,4] \rightarrow (-\alpha_1 + \alpha_2 + \alpha_3),\\
&[5,6] \rightarrow (\alpha_1 - \alpha_2 + \alpha_3), \quad [7,8] \rightarrow (\alpha_1 + \alpha_2 - \alpha_3),
\end{aligned}
$$

and to the substitutions of indexes of variables (the standard notation [13] for substitution elements of O_h is used)

$$
\begin{aligned}
&C_4^{(1)} \cong (1,5,4,7)(2,6,3,8), \quad C_4^{(2)} \cong (1,7,6,3)(2,8,5,4),\\
&C_4^{(3)} \cong (1,3,8,5)(7,6,2,4), \quad J \cong (1,2)(3,4)\dots(7,8),\dots
\end{aligned} \tag{7}
$$

Writing down the basis of the Lie algebra of the group (2) corresponding to the above indicated rotations, i.e.

$$
\begin{aligned}
\mathsf{X}_1 &= (-\xi_1, \xi_2, \xi_3, -\xi_4 - \xi_5, \xi_6, -\xi_7, \xi_8),\\
\mathsf{X}_2 &= (-\xi_1, \xi_2, -\xi_3, \xi_4, \xi_5, -\xi_6, -\xi_7, \xi_8),\\
\mathsf{X}_3 &= (-\xi_1, \xi_2, -\xi_3, \xi_4, -\xi_5, \xi_6, \xi_7, -\xi_8)
\end{aligned}
$$

(F_i can be written analogously), we obtain the system of invariants

$$
\begin{gathered}
\mathsf{I}_k = (\xi_{2k-1}\xi_{2k})^{\frac{1}{2}}, \quad k = 1,\dots,4, \quad \mathsf{I}_{4+S} = \frac{f_S}{\xi_S}, \quad S = 1,\dots,8,\\
\mathsf{I}_{13} = \xi_1\xi_4\xi_6\xi_8, \quad \mathsf{I}_{14} = \xi_2\xi_3\xi_5\xi_7,
\end{gathered}
$$

where $l_{13}l_{14} = l_1 l_2 l_3 l_4$. Hence the BEq has the form (the symbol $[\ldots]^{\text{out}}$ means the factorization of the expression in brackets by the relation $l_{13}l_{14} = l_1 l_2 l_3 l_4$)

$$
\begin{aligned}
f_1(\xi) &\equiv C_0\xi_1 + \sum_{P_\alpha, q_\beta} C_{P_\alpha, q_\beta}(\xi_1\xi_2)^{P_1}\ldots(\xi_1\xi_8)^{P_4}[\xi_1(\xi_1\xi_4\xi_6\xi_8)^{q_1}\cdot(\xi_2\xi_3\xi_5\xi_7)^{q_2}]^{\text{out}} \\
&= \xi_1 \sum_{|P_k|\geqslant 0} C_{P_k 0}(\xi_1\xi_2)^{P_1}\ldots(\xi_7\xi_8)^{P_4} \\
&\quad + \sum_{P_\alpha, k>0} C_{P_\alpha k}(\xi_1\xi_2)^{P_1}\ldots(\xi_7\xi_8)^{P_4}\xi_2^{k-1}(\xi_3\xi_5\xi_7)^k \qquad (8)\\
&\quad + \sum_{P_\alpha, k>0} C^i_{P_\alpha k}(\xi_i\xi_2)^{P_1}\ldots(\xi_7\xi_8)^{P_4}\xi_1^{k+1}(\xi_4\xi_6\xi_8)^k = 0 \\
f_j(\xi) &\equiv \mathsf{P}_{j-1}f_1(\xi) = f_1(\mathsf{P}_{j-1}(\xi)) = 0, \quad j = 2, \ldots, 8,
\end{aligned}
$$

where

$$
\mathsf{P}_1 = U_{45} = C_4^{(3)^3} \circ \mathsf{C}_4^{(1)^2}, \quad \mathsf{P}_2 = \mathsf{C}_4^{(3)}, \quad \mathsf{P}_3 = \mathsf{C}_4^{(1)^2}, \quad \mathsf{P}_4 = \mathsf{C}_4^{(3)^3},
$$
$$
\mathsf{P}_5 = \mathsf{C}_4^{(2)^2}, \quad \mathsf{P}_6 = J\mathsf{C}_4^{(3)^2}, \quad \mathsf{P}_7 = JU_{17} = J\mathsf{C}_4^{(3)} \circ \mathsf{C}_4^{(1)^2}.
$$

The construction of the general form of the BEq (8) invariant with respect to a given group reduces by several orders the quantity of computing work for finding the coefficients $C_{P_\alpha q_\beta}$. In particular, the substitution (7) preserving the number j of j-th equation in the bifurcation system gives the symmetry relations for the function $f_j(\xi)$ with respect to the arguments ξ_k.

R e m a r k . A vector case of the construction of the general form of BEq allowing a given group was considered in [8] where a three dimensional problem about capillary-gravitational waves in the liquid layer over a flat bottom was studied, where the elements $\varphi_j \in \mathsf{N}(\mathsf{B})$ were 2-componental.

3. In [9] the general form of the BEq was constructed for the Monge-Ampere equation on a 2-dimensional (for simplicity of presentation) torus $\mathbf{T}^2$. Let $(\mathsf{V}_{\ell,g})$, $g = \|g_{ij}(x)\|_{i,j=1}^{\ell}$, be a C^∞ compact Riemannian manifold without edge and Θ the set of twice continuously differentiable realvalued functions φ on V_ℓ such that the matrix $g'_\varphi = g + \nabla^2\varphi$ is positive definite at every point of V_ℓ (∇ is the covariant derivative in the metric g). On Θ the equation

$$
\mathsf{M}(\varphi) \equiv \frac{\det g'_\varphi}{\det g} = \mathrm{e}^{-\lambda\varphi} \quad \big(M(0) = 1\big) \qquad (9)
$$

is considered. For $\lambda < 0$ this equation has only the trivial solution $\varphi_\lambda = 0$, for $\lambda = 0$ all solutions are constants and for $\lambda > 0$ the bifurcation of solutions is possible. Since on $\mathbf{T}^\ell$ the covariant derivatives are the usual partial derivatives [14], [15], we have on $\mathbf{T}^\ell$

$$\mathsf{M}(\varphi) = 1 + \Delta\varphi + \sum_{k=2}^{\ell} \frac{(-1)^k}{k!} \det \|\partial_{\alpha_{j_1}} \partial_{\alpha_{j_2}} \varphi\|_{j_1, j_2=1}^{k}, \quad \partial_{\alpha_j} = \frac{\partial}{\partial x_{\alpha_j}}.$$

Therefore for $\ell = 2$ the equation (9) may be rewritten in the form $\mathsf{B}\varphi = \mathsf{R}(\varphi, \lambda)$, $\mathsf{R}(0, \lambda) = 0$, $\mathsf{B}\colon \mathsf{C}^{2+\alpha}(\mathbf{T}^2) \to \mathsf{C}^{\alpha}(\mathbf{T}^2)$,

$$(10) \qquad (\Delta + \lambda)\varphi = -\frac{1}{2} \begin{pmatrix} \varphi''_{x_1 x_2}, & \varphi''_{x_1 x_2} \\ \varphi''_{x_2 x_1}, & \varphi''_{x_2 x_2} \end{pmatrix} + \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \lambda^j \varphi^j, \quad \Delta\varphi = \sum_{\alpha=i}^{2} \frac{\partial^2 \varphi}{\partial x_\alpha^2}$$

with periodicity conditions.

On the 2-dimensional torus $\mathbf{T}^2 \cong \mathbf{R}^2/2\pi\mathbf{Z}^2$ the shift transformations $\mathsf{L}_a\varphi(x) = \varphi(x + a)$, $a = (a_1, a_2) \in \mathbf{T}^2$ and the reflection $\mathsf{S}\varphi(x) = \varphi(-x)$ preserve the matrix g and, consequently, (9) is invariant with respect to L_a and S [14]. It si not difficult to verify that the equation (9) allows also the group of a square

$$\mathsf{P}_1(x) = (-x_2, x_1),\ \mathsf{P}_2(x) = (-x_1, -x_2),\ \mathsf{P}_3(x) = (x_2, -x_1),$$
$$\mathsf{P}_4(x) = (x_1, -x_2),\ \mathsf{P}_5(x) = (-x_1, x_2),\ \mathsf{P}_6(x) = (x_2, x_1),\ \mathsf{P}_7(x) = (-x_2, -x_1).$$

By the method of separation of variables we obtain that $\lambda = \lambda_n = S = |n|^2 = n_1^2 + n_2^2$ are the bifurcation points for (10), and the corresponding eigenfunctions have the form $\varphi_n = \exp(\mathrm{i}\,\langle n, x\rangle)$. The dimension of $\mathsf{N}(\mathsf{B}_S)$ ($\mathsf{B}_S = \Delta + \lambda_n$, $\varphi(x_1 + 2\pi, x_2) = \varphi(x_1, x_2 + 2\pi) = \varphi(x)$) is equal to the number r_S of representations of the integer $S = |n|^2$ in the form of sums of two squares. If $n_1 = n_2$ then the multiplicity of the eigenvalue $\lambda_n = 2n_1^2$ is at least 4; if $n_1 \neq n_2$ then the multiplicity of λ_n is at least 8. The general formula for r_S may be found in [16]. We agree on the previous condition about enumeration of the eigenelements φ_n.

Let $\dim \mathsf{N}(\mathsf{B}_S) = 8$, $n_1 \neq_2$. The zero space $\mathsf{N}(\mathsf{B}_S)$ has the form $\varphi_j = \exp(\mathrm{i}\,\langle l_j, x\rangle)$, where $l_1 = (n_1, n_2)$, $l_3 = (-n_1, n_2)$, $l_5 = (n_2, n_1)$, $l_7 = (-n_2, n_1)$, $l_{2k} = -l_{2k-1}$. The group of a square induces in Ξ^8 the representation expressing the following substitutions of the indexes of ξ_k:

$$(11) \qquad \begin{aligned} &\mathsf{P}_1 = (12)(34)(56)(78), \quad \mathsf{P}_2 = (13)(24)(57)(68), \\ &\mathsf{P}_3 = (14)(23)(58)(67), \quad \mathsf{P}_4 = (15)(26)(38)(47), \\ &\mathsf{P}_5 = (16)(25)(37)(48), \quad \mathsf{P}_6 = (1728)(3645), \\ &\mathsf{P}_7 = (1827)(3546). \end{aligned}$$

Two-dimensional shifts L_a generate in Ξ^8 the representation

$$\mathcal{A}(a) = \operatorname{diag}\{\exp(\mathrm{i}\,\langle l_1, a\rangle), \dots, \exp(\mathrm{i}\,\langle l_8, a\rangle)\}.$$

Basic infinitesimal operators X_1, X_2 to the group $\mathcal{A}(a)$ have the form

$$X_1 = (n_1\xi_1, -n_1\xi_2, -n_1\xi_3, n_1\xi_4, n_2\xi_5, -n_2\xi_6, -n_2\xi_7, n_2\xi_8)\,\frac{\partial}{\partial\xi},$$
$$X_2 = (n_2\xi_1, -n_2\xi_2, n_2\xi_3, -n_2\xi_4, n_1\xi_5, -n_1\xi_6, n_1\xi_7, -n_1\xi_8)\,\frac{\partial}{\partial\xi}.$$

The full system of functionally independent invariants is defined by the differential equations $(X_i + F_i)I = 0$, $i = 1, 2$, with general rank $r_* = 2$. Therefore we have $16 - 2 = 14$ functionally independent invariants. Among them there are four invariants of the form $I_j(\xi) = \xi_{2j-1}\xi_{2j}$, $j = 1, \dots, 4$, eight invariants $I_{4+j}(\xi, f) = f_j/\xi_j$, $j = 1, \dots, 8$. The other invariants must be written in the form of monoms of a possibly minimal degree. They are found as solutions of the system $X_i I(\xi) = 0$, $i = 1, 2$:

(12) $$I_{13}(\xi) = (\xi_1\xi_4)^{N/n_1}(\xi_6\xi_7)^{N/n_2}, \quad I_{14}(\xi) = (\xi_1\xi_3)^{N/n_2}(\xi_6\xi_8)^{N/n_2},$$
$$I_{15}(\xi) = (\xi_2\xi_3)^{N/n_1}(\xi_5\xi_8)^{N/n_2}, \quad I_{16}(\xi) = (\xi_2\xi_4)^{N/n_2}(\xi_5\xi_7)^{N/n_1},$$

where $N = \frac{n_1 n_2}{(n_1, n_2)}$. The invariants (12) satisfy the following relations ($I_{15}(\xi) = \overline{I_{13}(\xi)}$, $I_{16}(\xi) = \overline{I_{14}(\xi)}$):

(13) $$I_{13}(\xi)I_{15}(\xi) = (I_1 I_2)^{N/n_1}(I_3 I_3)^{N/n_2}, \quad I_{14}(\xi)I_{16}(\xi) = (I_1 I_2)^{N/n_2}(I_3, I_4)^{N/n_1}$$

According to the theorem on the representation of a nonsingular invariant manifold of the group (2) [4, 5], the first equation of the bifurcation system can be written in the form

(14) $$f_1(\xi, \varepsilon) \equiv \sum_P a_P^{(1)}(\varepsilon) I_1^{P_1}(\xi) I_2^{P_2}(\xi) I_3^{P_3}(\xi) I_4^{P_4}(\xi),$$
$$[\xi_1 I_{13}^{P_5}(\xi) I_{14}^{P_6}(\xi) I_{15}^{P_7}(\xi) I_{16}^{P_8}(\xi)]^{\mathrm{out}} = 0,$$

where the symbol $[\dots]^{\mathrm{out}}$ means the factorization of the expression $[\dots]$ by the relations (13). The other seven equations of the bifurcation system are found from the condition of its invariance with respect to the substitutions (11):

(15) $$f_k(\xi, \varepsilon) \equiv P_{k-1} f_1(\xi, \varepsilon) = f_1(P_{k-1}(\xi), \varepsilon) = 0, \quad k = 2, \dots, 8.$$

The construction of asymptotics of small solutions of (14), (15) requires the specification of the values n_1 and n_2.

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