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DERIVATIVE OF THE NORM OF A LINEAR MAPPING AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

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Summary. In this paper the notion of the derivative of the norm of a linear mapping in a normed vector space is introduced. The fundamental properties of the derivative of the norm are established. Using these properties, linear differential equations in a Banach space are studied and lower and upper estimates of the norms of their solutions are derived.

Keywords: Normed space, Derivative of the norm of the linear mapping, Solution of the differential equation.

AMS classification: 34A30

Let $(P_1, \|\cdot\|_1)$, $(P_2, \|\cdot\|_2)$ be normed vector spaces over the field of complex numbers. We denote the set of all bounded linear mappings of the space P_1 to the space P_2 by the symbol $L(P_1, P_2)$. We introduce the structure of the linear space on the set $L(P_1, P_2)$ in the standard way. We define the norm of a bounded linear mapping by the relation

$$\|A\| = \sup\{\|Ay\|_2 \colon \|y\|_1 \leqslant 1\} \quad \text{for} \quad A \in L(P_1, P_2).$$

Further we define the function $f_1 \colon L(P_1, P_2) \times L(P_1, P_2) \times \mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+ = (0, +\infty)$, $\mathbf{R} = (-\infty, +\infty)$, by the relation

$$f_1(X, A, t) = \frac{\|X + tA\| - \|X\|}{t}. \tag{1}$$

Now for any $\vartheta \in (0,1)$ and for any $(X, A, t) \in L(P_1, P_2) \times L(P_1, P_2) \times \mathbf{R}^+$ we have

$$\begin{aligned} \vartheta t f_1(X, A, \vartheta t) &= \|X + \vartheta t A\| - \|X\| = \|\vartheta(X + tA) + (1-\vartheta)X\| - \|X\| \\ &\leqslant \vartheta(\|X + tA\| - \|X\|) = \vartheta t f_1(X, A, t), \quad \text{i.e.} \end{aligned} \tag{2}$$

$$f_1(X, A, \vartheta t) \leqslant f_1(X, A, t).$$

For any $t \in \mathbf{R}^+$ we have

$$tf_1(X,A,t) = \|X+tA\| - \|X\| \geqslant \|X\| - t\|A\| - \|X\| = -t\|A\|, \tag{3}$$

and so $f_1(X,A,t) \geqslant -\|A\|$.

It follows from the relations (2), (3) that $f_1(X,A,t)$ is a function of the variable t which has a finite limit $\lim f_1(X,A,t)$ for $t \to 0+$, for any $X, A \in L(P_1,P_2)$. We denote this limit by the symbol $f_X(A)$.

Definition. The mapping $f_X : L(P_1,P_2) \to \mathbf{R}$, where $X \in L(P_1,P_2)$, defined by the relation

$$f_X(A) = \lim_{t\to 0+} \frac{\|X+tA\| - \|X\|}{t}, \tag{4}$$

is called the *derivative of the norm of the mapping* X.

Theorem 1. *If f_X is the derivative of the norm of the mapping $X \in L(P_1,P_2)$, then*

1° $f_X(X) = \|X\|$, $f_X(-X) = -\|X\|$ *for any* $X \in L(P_1,P_2)$, $f_X(O) = 0$, *where* O *is the zero mapping of the space* $L(P_1,P_2)$;

2° $-\|A\| \leqslant -f_X(-A) \leqslant f_X(A) \leqslant \|A\|$ *for any* $X, A \in L(P_1,P_2)$;

3° $f_X(\alpha A) = \alpha f_X(A)$ *for any* $\alpha \in \mathbf{R}^+$ *and for any* $X, A \in L(P_1,P_2)$;

4° $f_X(A+\alpha X) = f_X(A) + \alpha\|X\|$ *for any* $\alpha \in \mathbf{R}$ *and for any* $X, A \in L(P_1,P_2)$;

5° $\max\{f_X(A) - f_X(-B), -f_X(-A) + f_X(B)\} \leqslant f_X(A+B) \leqslant f_X(A) + f_X(B)$ *for any* $X, A, B \in L(P_1,P_2)$;

6° f_X *is a convex functional on the vector space* $L(P_1,P_2)$, *i.e.* $f_X\big(\alpha A + (1-\alpha)B\big) \leqslant \alpha f_X(A) + (1-\alpha) f_X(B)$ *for any* $\alpha \in \langle 0,1\rangle$ *and for any* $A, B \in L(P_1,P_2)$.

Proof. 1° The property follows directly from the definition of the derivative of the norm f_X.

2° For any $t \in \mathbf{R}^+$ we have

$$-\|A\| = \frac{\|X\| - t\|A\| - \|X\|}{t} \leqslant \frac{\|X+tA\| - \|X\|}{t} \leqslant \frac{\|X\| + t\|A\| - \|X\|}{t} = \|A\|,$$

thus

$$-\|A\| \leqslant \lim_{t\to 0+} \frac{\|X+tA\| - \|X\|}{t} = f_X(A) \leqslant \|A\|. \tag{5}$$

For any $t \in \mathbf{R}^+$ we have

$$-\|A\| = \frac{\|X\| - t\|A\| - \|X\|}{t} = \frac{\|X\| - t\| - A\| - \|X\|}{t} \leqslant -\frac{\|X + t(-A)\| - \|X\|}{t},$$

thus

$$(6) \qquad -\|A\| \leqslant -\lim_{t\to 0+} \frac{\|X+t(-A)\|-\|X\|}{t} = -f_X(-A).$$

Further,

$$0 = \|X+t(A-A)\|-\|X\| \leqslant \frac{\|X+2tA\|-\|X\|+\|X-2tA\|-\|X\|}{2}$$
$$= \frac{\|X+2tA\|-\|X\|}{2t}t + \frac{\|X+2t(-A)\|-\|X\|}{2t}t, \quad \text{thus}$$
$$0 \leqslant \lim_{t\to 0+} \frac{\|X+2tA\|-\|X\|}{2t} + \lim_{t\to 0+} \frac{\|X+2t(-A)\|-\|X\|}{2t}, \quad \text{i.e.}$$

$$(7) \qquad 0 \leqslant f_X(A) + f_X(-A).$$

Now the relations (5), (6) and (7) imply

$$-\|A\| \leqslant -f_X(-A) \leqslant f_X(A) \leqslant \|A\|.$$

3° The proof of this property is evident.

4° According to the definition of the derivative of the norm f_X we have

$$f_X(A+\alpha X) = \lim_{t\to 0+} \frac{\|X+t(A+\alpha X)\|-\|X\|}{t}$$
$$= \lim_{t\to 0+} \frac{(1+\alpha t)\|X+tA/(1+\alpha t)\|-\|X\|}{t}$$
$$= \lim_{t\to 0+} \frac{\|X+tA/(1+\alpha t)\|-\|X\|}{t/(1+\alpha t)} + \alpha\|X\|$$
$$= f_X(A) + \alpha\|X\|.$$

5° For any $t \in \mathbf{R}^+$ we have

$$\frac{\|X+t(A+B)\|-\|X\|}{t} = \frac{\|X+2tA+X+2tB\|-2\|X\|}{2t} \leqslant$$
$$\leqslant \frac{\|X+2tA\|-\|X\|}{2t} + \frac{\|X+2tB\|-\|X\|}{2t}, \quad \text{thus}$$

$$(8) \qquad f_X(A+B) = \lim_{t\to 0+} \frac{\|X+t(A+B)\|-\|X\|}{t} \leqslant$$
$$\leqslant \lim_{t\to 0+} \frac{\|X+2tA\|-\|X\|}{2t} + \lim_{t\to 0+} \frac{\|X+2tB\|-\|X\|}{2t} =$$
$$= f_X(A) + f_X(B).$$

The first inequality in 5° follows from the inequality

$$f_X(A) \leqslant f_X(A+B) + f_X(-B)$$

and from the inequality arising by the interchanging A and B.

6° The proof follows from the properties 3°, 5°. □

Theorem 2. *Let $I \in L(P_1, P_1)$ be the identical mapping of the space P_1 to the space P_1 and let λ be an eigenvalue of the linear mapping $A \in L(P_1, P_1)$. Then*

(i) $-f_I(-A) \leqslant \operatorname{Re}\lambda \leqslant f_I(A)$,

(ii) $-f_I(-A)\|x\|_1 \leqslant \|Ax\|_1$, $-f_I(A)\|x\|_1 \leqslant \|Ax\|_1$

for any $x \in P_1$.

Proof. Ad (i). Let $v \in P_1$ be a vector, where $Av = \lambda v$ and $\|v\|_1 = 1$. For any $t \in \mathbf{R}^+$ we have $\|I + t(-A)\| \geqslant \|Iv - tAv\|_1 = \|v - t\lambda v\|_1$, and so

$$-\frac{\|I + t(-A)\| - 1}{t} \leqslant -\frac{\|v - t\lambda v\|_1 - 1}{t} = -\frac{|1 - t\lambda| - 1}{t}.$$

Consequently,

$$-f_I(-A) = -\lim_{t \to 0+} \frac{\|I + t(-A)\| - 1}{t} \leqslant -\lim_{t \to 0+} \frac{|1 - t\lambda| - 1}{t} = \operatorname{Re}\lambda.$$

Further,

$$\frac{\|I + tA\| - 1}{t} \geqslant \frac{\|v + t\lambda v\|_1 - 1}{t} = \frac{|1 + t\lambda| - 1}{t}, \quad \text{thus}$$

$$f_I(A) = \lim_{t \to 0+} \frac{\|I + tA\| - 1}{t} \geqslant \lim_{t \to 0+} \frac{|1 + t\lambda| - 1}{t} = \operatorname{Re}\lambda.$$

Ad (ii). For any $t \in \mathbf{R}^+$ and for any vector $x \in P_1$ we have

$$\|Ax\|_1 = \frac{\|x - (x - tAx)\|_1}{t} \geqslant$$
$$\geqslant \frac{\|x\|_1 - \|I - tA\| \, \|x\|_1}{t} = -\frac{\|I + t(-A)\| - 1}{t}\|x\|_1, \quad \text{thus}$$
$$\|Ax\|_1 \geqslant -\lim_{t \to 0+} \frac{\|I + t(-A)\| - 1}{t}\|x\|_1 = -f_I(-A)\|x\|_1.$$

Replacing the mapping A by the mapping $-A$ in this relation we get the second relation.

The theorem is proved. □

In what follows we consider a differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = A(s)x \tag{11}$$

in a Banach space $(P_1, \|.\|_1)$, where $A \in L(P_1, P_1)$ is a continuous mapping on an open, unbounded from above interval $J \subset \mathbf{R}$, and $A(t)$ is a continuous mapping of J into $L(P_1, P_1)$. It has been shown in the theory of differential equations (see [2], p. 353, Theorem 10.8.4) that there exists just one continuous mapping $x(., s_0, x_0)\colon \{s \in$

$J: s \geqslant s_0\} \to P_1$ for any $(s_0, x_0) \in J \times P_1$ such that $\mathrm{d}x(s)/\mathrm{d}s = A(s)x(s)$ for any $s \geqslant s_0$ and $x(s_0) = x_0$. This mapping is called the solution of the differential equation (11). Besides, for any s, where $s \geqslant s_0 \in J$, there exists a mapping $F(s) \in L(P_1, P_1)$—the so-called fundamental mapping of the equation (11)—and there exists its inverse mapping $F^{-1}(s) \in L(P_1, P_1)$ such that $x(s, s_0, x_0) = F(s) \circ F^{-1}(s_0)x_0$.

If $x(.)$ is a solution of the differential equation (11) then there exists $y(s,t) \in P_1$ for any $(s,t) \in J \times \mathbb{R}^+$ with $\lim \|y(s,t)\|_1 = 0$ for $t \to 0+$ and such that

$$x(s+t) = x(s) + \frac{\mathrm{d}x(s)}{\mathrm{d}s}t + ty(s,t) = x(s) + tA(s)x(s) + ty(s,t).$$

This implies

$$\lim_{t\to 0+} \frac{\|x(s+t)\|_1 - \|x(s)\|_1}{t} = \lim_{t\to 0+} \frac{\|x(s) + tA(s)x(s)\|_1 - \|x(s)\|_1}{t}. \tag{12}$$

Theorem 3. *If $(P_1, \|.\|_1)$ is a Banach space and $x(., s_0, x_0)$ is a solution of the differential equation (11), $F \in L(P_1, P_1)$ is its fundamental mapping, f_I is the derivative of the norm of the identical mapping $I \in L(P_1, P_1)$, then for any $s \geqslant s_0 \in J$ the estimates*

$$\|x_0\|_1 \exp\Big[-\int_{s_0}^{s} f_I(-A(\sigma))\,\mathrm{d}\sigma\Big] \leqslant \|x(s)\|_1 \leqslant \|x_0\|_1 \exp\Big[\int_{s_0}^{s} f_I(A(\sigma))\,\mathrm{d}\sigma\Big], \tag{13}$$

$$\exp\Big[-\int_{s_0}^{s} f_I(-A(\sigma))\,\mathrm{d}\sigma\Big] \leqslant \|F(s) \circ F^{-1}(s_0)\| \leqslant \exp\Big[\int_{s_0}^{s} f_I(A(\sigma))\,\mathrm{d}\sigma\Big], \tag{14}$$

hold whenever the integrals involved are defined.

Proof. For any $x \in P_1$, $t \in \mathbb{R}^+$ we have

$$2\|x\|_1 = \|(I+tA)x + (I-tA)x\|_1 \leqslant \|(I+tA)\| \cdot \|x\|_1 + \|I + t(-A)\| \cdot \|x\|_1,$$

and this implies

$$-\frac{\|I+t(-A)\| - 1}{t}\|x\|_1 \leqslant \frac{\|x + tAx\|_1 - \|x\|_1}{t} \leqslant \frac{\|I+tA\| - 1}{t}\|x\|_1. \tag{15}$$

According to (12) the relation (15) implies the inequality

$$-f_I(-A(s))\|x(s)\|_1 \leqslant \lim_{t\to 0+} \frac{\|x(s+t)\|_1 - \|x(s)\|_1}{t} \leqslant f_I(A(s))\|x(s)\|_1. \tag{16}$$

If $x_0 = o \in P_1$, the relation (13) is obviously true. Thus, let us suppose $x_0 \neq o$. Then also $\|x(s, s_0, x_0)\|_1 > 0$ for any $s \geqslant s_0 \in J$. From the inequality (16) the relation

$$-f_I(-A(\sigma)) \leqslant \lim_{t\to 0+} \frac{\|x(\sigma+t)\|_1 - \|x(\sigma)\|_1}{t\|x(\sigma)\|_1} \leqslant f_I(A(\sigma)),$$

follows. By its integration we obtain

$$-\int_{s_0}^{s} f_I(-A(\sigma))\,\mathrm{d}\sigma \leqslant \ln\frac{\|x(s)\|_1}{\|x_0\|_1} \leqslant \int_{s_0}^{s} f_I(A(\sigma))\,\mathrm{d}\sigma$$

provided $\langle s_0, s\rangle \subset J$.

This yields the inequality (13) as well as the relation (14). □

Theorem 4. *Let $(P_1, \|.\|_1)$ be a Banach space, in which a differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = (A + B(s))x \tag{17}$$

is given, where $A \in L(P_1, P_1)$ is a constant mapping and $B(s) \in L(P_1, P_1)$ is a continuous mapping on an open, unbounded from above interval $J \subset \mathbb{R}$. Let $x(., s_0, x_0)$ be a solution of the equation (17) and let f_I be the derivative of the norm of the identical mapping $I \in L(P_1, P_1)$. Then the following implications hold:

(i) $f_I(A) = 0$, $\int_{s_0}^{+\infty} f_I(B(s))\,\mathrm{d}s < +\infty \Rightarrow$ *the solution $x(.)$ is bounded;*

(ii) $f_I(A) < 0$, $\int_{s_0}^{+\infty} f_I(B(s))\,\mathrm{d}s < +\infty \Rightarrow \lim_{s\to+\infty} \|x(s)\|_1 = 0$;

(iii) $-f_I(-A) > 0$, $-\int_{s_0}^{+\infty} f_I(-B(s))\,\mathrm{d}s > -\infty$, $x_0 \neq o \Rightarrow \lim_{s\to+\infty} \|x(s)\|_1 = +\infty$;

(iv) $-f_I(-A) = 0$, $-\int_{s_0}^{+\infty} f_I(-B(s))\,\mathrm{d}s = +\infty$, $x_0 \neq o \Rightarrow \lim_{s\to+\infty} \|x(s)\|_1 = +\infty$.

Proof. The inequality (13) and the property 5° from Theorem 1 imply

$$\begin{aligned}
&\|x_0\|_1 \exp\Big[-\int_{s_0}^{s} f_I(-A)\,\mathrm{d}\sigma - \int_{s_0}^{s} f_I(-B(\sigma))\,\mathrm{d}\sigma\Big] \\
&\leqslant \|x_0\|_1 \exp\Big[-\int_{s_0}^{s} f_I(-A - B(\sigma))\,\mathrm{d}\sigma\Big] \leqslant \|x(s)\|_1 \\
&\leqslant \|x_0\|_1 \exp\Big[\int_{s_0}^{s} f_I(A + B(\sigma))\,\mathrm{d}\sigma\Big] \\
&\leqslant \|x_0\|_1 \exp\Big[\int_{s_0}^{s} f_I(A)\,\mathrm{d}\sigma + \int_{s_0}^{s} f_I(B(\sigma))\,\mathrm{d}\sigma\Big],
\end{aligned}$$

i.e.

$$\|x_0\|_1 \exp[-f_I(-A)(s-s_0)] \exp\Big[-\int_{s_0}^{s} f_I(-B(\sigma))\,\mathrm{d}\sigma\Big]$$

$$\leqslant \|x(s)\|_1 \leqslant \|x_0\|_1 \exp[f_I(A)(s-s_0)]\cdot\exp\Big[\int_{s_0}^{s} f_I(B(\sigma))\,\mathrm{d}\sigma\Big],$$

from which the validity of the implications (i), (ii), (iii), (iv) is evident. □

References

[1] *A. N. Kolmogorov, S. V. Fomin*: Foundations of the theory of functions and functional analysis (Russian). Czech translation, Praha, 1975.

[2] *J. Dieudonné*: Foundations of modern analysis, Academic Press, New York-London, 1960; Russian translation, Mir, Moscow.

Souhrn

DERIVACE NORMY LINEÁRNÍHO ZOBRAZENÍ A JEJÍ APLIKACE V DIFERENCIÁLNÍCH ROVNICÍCH

FRANTIŠEK TUMAJER

V článku je zaveden pojem derivace normy lineárního zobrazení v normovaném vektorovém prostoru. Odvozují se základní vlastnosti derivace normy. Užitím těchto vlastností jsou studovány lineární diferenciální rovnice v Banachově prostoru a jsou odvozeny dolní i horní odhady pro normu jejich řešení.

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