Applications of Mathematics

Michal Fečkan

Bifurcation of heteroclinic orbits for diffeomorphisms

Applications of Mathematics, Vol. 36 (1991), No. 5, 355–367

Persistent URL: http://dml.cz/dmlcz/104472

BIFURCATION OF HETEROCLINIC ORBITS FOR DIFFEOMORPHISMS

MICHAL FEČKAN

(Received April 20, 1989)

Summary. The paper deals with the bifurcation phenomena of heteroclinic orbits for diffeomorphisms. The existence of a Melnikov-like function for the two-dimensional case is shown. Simple possibilities of bifurcation of the set of heteroclinic points are described for higher-dimensional cases.

Keywords: bifurcation phenomena, heteroclinic points, discrete dynamical systems.

AMS classification: 58F14, 58F30

1. INTRODUCTION

In this paper we investigate bifurcation of heteroclinic orbits for diffeomorphisms. The results are obtained by *the Lyapunov-Schmidt method*. This method was used for the study of an analogous problem for ordinary differential equations in [4, 8].

2. TWO-DIMENSIONAL CASE

Let us consider a C^∞-smooth mapping $\Phi: R^2 \to R^2$ with the following properties on the set $M = (-1/2, 3/2) \times (-\infty, \infty)$

i) Φ has the form $\Phi(x, y) = (f(x), g(x, y))$, where $g(x, 0) = 0$ for each $x \in (-1/2, 3/2)$,

ii) the mapping $f: R \to R$ has fixed points 0, 1 such that $f'(0) > 1$, $f'(1) < 1$, $f'(\cdot) > 0$ and $g_y(\cdot, 0) \neq 0$. Further we assume the existence of a sequence $\{x_n\}_{-\infty}^{+\infty} \subset (0, 1)$, $x_{n+1} = f(x_n)$, $x_n \to 1(0)$ as $n \to \infty(-\infty)$.

Thus Φ has the heteroclinic orbit $\Gamma = \{(x_n, 0)\}_{-\infty}^{+\infty}$ from $(0, 0)$ to $(1, 0)$. We note that Φ also has the family of heteroclinic orbits $\mathcal{M} = \{\{(f^n(x), 0)\}_{-\infty}^{+\infty}, x \in (0, 1)\}$ and this family contains Γ. We perturb this mapping and try to find heteroclinic orbits near Γ for the perturbed mapping.

Let us consider the variational equation of Φ around Γ:

$$u_{n+1} - f'(x_n) \,.\, u_n = a_n \,,$$
$$v_{n+1} - g_y(x_n, 0) \,.\, v_n = b_n \,.$$

For the mapping g we have the following four cases:

A. $|g_y(0, 0)| > 1 \,, \quad |g_y(1, 0)| < 1 \,.$

Lemma 2.1. *Let* $X = \{\{(a_n, b_n)\}_{-\infty}^{+\infty}, a_n, b_n \in R, |\{(a_n, b_n)\}| = \sup\{|a_n|, |b_n|\} < \infty\}$ *and consider the linear operator* $L:X \to X$,

$$L(\{(u_n, v_n)\}_{-\infty}^{+\infty}) = \{u_{n+1} - f'(x_n) \,.\, u_n, v_{n+1} - g_y(x_n, 0) \,.\, v_n\}_{-\infty}^{+\infty} \,.$$

Then

$$\dim \operatorname{Ker} L = 2 \,, \quad \operatorname{codim} \operatorname{Im} L = 0 \,.$$

Proof. From the equation

$$u_{n+1} = f'(x_n) \,.\, u_n \,, \quad v_{n+1} = g_y(x_n, 0) \,.\, v_n$$

using $\lim_{n\to\pm\infty} |f'(x_n)| \lessgtr 1$ and $\lim_{n\to\pm\infty} |g_y(x_n, 0)| \lessgtr 1$ we have

$$\operatorname{Ker} L = R\{(\prod_1^n f'(x_n), 0)\}_{-\infty}^{+\infty} \oplus R\{(0, \prod_1^n g_y(x_n, 0))\}_{-\infty}^{+\infty} \,,$$

where

$$\prod_1^n a_n = \begin{cases} a_0 \dots a_{n-1} & , \quad n \geqq 1 \\ 1 & , \quad n = 0 \\ 1/a_{-1} \dots 1/a_n \,, & n < 0 \,. \end{cases}$$

For $\{(a_n, b_n)\}_{-\infty}^{+\infty} \in X$ we solve the equation

(2.2) $$u_{n+1} = f'(x_n)\, u_n + a_n \,,$$
$$v_{n+1} = g_y(x_n, 0)\, v_n + b_n \,.$$

The first (and similarly the second) equation of (2.2) has the general solution

$$u_n = f'(x_{n-1}) \dots f'(x_0) \left(\Sigma_0^{n-1} \frac{a_i}{f'(x_i) \dots f'(x_0)} + K \right), \quad n \geqq 1$$
$$u_0 = K \,, \quad u_{-1} = (-a_{-1} + K)/f'(x_{-1}) \,,$$
$$u_n = \frac{1}{f'(x_n) \dots f'(x_{-1})} (\Sigma_n^{-2} - a_i f'(x_{i+1}) \dots f'(x_{-1}) - a_{-1} + K) \,,$$
$$n \leqq -2 \,.$$

Since $\lim_{n\to\infty} |f'(x_n)| < 1$ we have $\sup_{n\geqq 1} |u_n| < \infty$. The proof of the other cases is similar.

B. $|g_y(0, 0)| > 1 \,, \quad |g_y(1, 0)| > 1 \,.$

Lemma 2.2. *In this case* dim Ker $L = 1$, codim Im $L = 0$.

Proof. The case dim Ker $L = 1$ is clear. In this case the first equation of (2.2) has a bounded solution for each K. The second has a bounded solution iff the corresponding K is

$$K = -\Sigma_0^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)}.$$

This series is convergent and thus codim Im $L = 0$.

C. $|g_y(0, 0)| < 1, \quad |g_y(1, 0)| > 1.$

In this case we obtain the same result as in the case B.

D. $|g_y(0, 0)| < 1, \quad |g_y(1, 0)| > 1.$

Lemma 2.3. *In this case* dim Ker $L = 1$, codim Im $L = 1$.

Proof. We prove the second part of the lemma. The second equation of (2.2) has a bounded solution for $n \to \infty$ iff the corresponding K is

$$K = -\Sigma_0^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)},$$

and for $n \to -\infty$ iff

$$K = \Sigma_{-\infty}^{-2} b_i\, g_y(x_{i+1}, 0) \ldots g_y(x_{-1}, 0) + b_{-1}.$$

Hence

$$d_{-1} = \Sigma_{-\infty}^{-2} b_i\, g_y(x_{i+1}, 0) \ldots g_y(x_{-1}, 0) + b_{-1} + $$
$$+ \Sigma_0^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)} = 0.$$

We see that (2.2) has a bounded solution if and only if $d_{-1} = 0$ and this relation implies codim Im $L = 1$.

We define the projection $P: X \to X$, $P(\{(a_n, b_n)\}) = \{(0, d_n)\}_{-\infty}^{+\infty}$, where $d_n = 0$ for $n \neq -1$ and d_{-1} is defined in the above proof. We see that $\{(a_n, b_n)\}_{-\infty}^{+\infty} \in \operatorname{Im} L$ if and only if $P(\{(a_n, b_n)\}) = 0$. Thus we define the operator $K: (I - P) X \to X$,

$$K(\{(a_n, b_n)\}_{-\infty}^{+\infty}) = \{(u_n, v_n)\}_{-\infty}^{+\infty}, \quad u_0 = 0,$$

where $\{(u_n, v_n)\}_{-\infty}^{+\infty}$ is unique bounded solution of (2.2).

The mapping Φ has hyperbolic fixed points $(0, 0)$ and $(0, 0)$. Hence a perturbed mapping $\Phi_e: R^2 \to R^2$ has fixed points p_e, q_e near them, which are hyperbolic as well. Consider the equation

$$z_{n+1} - \Phi_e(z_n) = 0 \tag{2.3}$$

on the space X. (We assume $\Phi_{\bullet}(\cdot) \in C^{\infty}$.) This equation can be written in the form

$$(2.4) \qquad \begin{aligned} u_{n+1} + x_{n+1} &= f(x_n + u_n) + O(e), \\ v_{n+1} &= g(x_n + u_n, v_n) + O(e). \end{aligned}$$

We seek for a bounded solution of (2.4) with $|u_n| + |v_n| + |e| \ll 1$, i.e. we solve the equation (2.4) in X near $0 \in X$ for e small. It is clear that the linearization of (2.4) at $0 \in X$ for $e = 0$ is precisely the operator L. According to Lemma 2.3 we have for the case D

$$\dim \operatorname{Ker} L = 1 \quad \text{and} \quad \operatorname{codim} \operatorname{Im} L = 1.$$

Hence applying the *Lyapunov-Schmidt method* [1, 10] we derive a bifurcation eguation of the equation (2.4),

$$(2.5) \qquad Q(c, e) = 0, \quad Q: U \times U \to R,$$

where U is a neighbourhood of $0 \in R$. Since for $e = 0$ the equation (2.4) has the solution $u_n = f^n(x) - x_n$, $v_n = 0$ for each $x \in (0, 1)$, we obtain that $Q(c, 0) = 0$. We note that each small solution of (2.4) yields a heteroclinic orbit of Φ_e near Γ.

Theorem 2.4. *For the case D we obtain the above bifurcation equation* (2.5).

Now we investigate the remaining cases. For these cases we have also the equation (2.4), but according to Lemmas 2.1, 2.2 the linearization of (2.4), which is the operator L, satisfies $\operatorname{codim} \operatorname{Im} L = 0$, i.e. L is surjective and applying the implicit function theorem we have for e small

Theorem 2.5. *In the case A there is a three-parametric family of heteroclinic orbits near Γ, where one parameter is e and the other corresponds to the parameter x from the above mentioned family $\mathscr{M}$* of heteroclinic orbits of Φ.

Theorem 2.6. *In the cases B, C we have a two-parametric family of heteroclinic orbits near Γ, where one parameter is e and the other corresponds to the parameter x from the above mentioned family $\mathscr{M}$.*

3. GENERAL CASE

Definition 3.1 (*see* [6]). *Let X be a Banach space and $\{T_n\}_{n \in I} \in \mathscr{L}(X)$. We say that $\{T_n\}_{n \in I}$ has a discrete dichotomy on $I = (Z, Z_+ = N \cup \{0\}, Z_- = -Z_+)$ if there exist positive numbers $M, \theta < 1$ and a sequence of projections $\{P_n\}_{n \in I}$ such that*

i) $T_n P_n = P_{n+1} T_n$,

ii) $T_n/\operatorname{Im} P_n$ *is an isomorphism from* $\operatorname{Im} P_n$ into $\operatorname{Im} P_{n+1}$.

iii) *if* $T_{n,m} = T_{n-1} \dots T_{m+1} T_m$ *for* $n > m$, $T_{n,n} = Identity$, *then*

$$|T_{n,m}(I - P_m)\,x| \leqq M\theta^{n-m}|x| \quad \text{for} \quad n \geqq m\,,$$

$$|T_{n,m}P_m x| \qquad \leqq M\theta^{m-n}|x| \quad \text{for} \quad n < m\,,$$

where $T_{n,m}P_m x = y$ *iff* $P_m x = T_{m,n}\,y$ *for the case* $m > n$.

Remark 3.2. If T_n is a sequence of isomorphisms then the above definition is equivalent to the property that there is a projection $P \in \mathscr{L}(X)$ such that

$$|T(m)\,P\,T^{-1}(s) \leqq M\theta_m^{\ -s}\,, \qquad m \geqq s\,,$$

$$|T(m)\,(I - P)\,T^{-1}(s)| \leqq M\theta^{s-m}\,, \quad s \geqq m\,,$$

where $T(n) = T_{-1} \dots T_0$ for $n \geqq 1$, $T(n) = T_n^{-1} \dots T_{-1}^{-1}$ for $n < 0$, $T(0) = I$.

Theorem 3.3. *Let* $\{A_n\}_{n \in Z}$ *be a sequence of invertible matrices* $A_n \in \mathscr{L}(R^m, R^m)$ *with bounded* $|A_n|$, $|A_n^{-1}|$ *on* Z. *We assume that* $\{A_n\}$ *has a discrete dichotomy both on* Z_+ *and* Z_-. *Define the operator*

$$L\colon X \to X = \{\{a_n\}_{-\infty}^{+\infty},\ \sup\,|a_n| < \infty,\ a_n \in R^m\}\,,$$

$$L(\{a_n\})_n = a_{n+1} - A_n a_n\,.$$

Then L *is a Fredholm operator and* $\{f_n\} \in \operatorname{Im} L$ *iff* $\Sigma_{-\infty}^{+\infty}\, c_n^* f_n = 0$ *for each bounded solution* $\{c_n\}$ *of the equation*

$$c_n = (A_n^*)^{-1}\, c_{n-1} \quad (\text{* means the transpose})\,. \tag{3.1}$$

Proof. We consider the equation

$$x_{n+1} = A_n x_n\,. \tag{3.2}$$

By assumption this equation has a discrete dichotomy on $Z_{+(-)}$ with projection P, Q. (3.2) has the fundamental solution on Z_+

$$T(n) = A_{n-1} \dots A_0\,, \quad n \geqq 1\,, \quad T(0) = I\,.$$

The equation (3.1) on the set $I_1 = \{-1, 0, 1, \dots\}$ has the fundamental solution

$$S(n) = (A_n^*)^{-1} \dots I = (T(n+1)^*)^{-1}\,.$$

We see that (3.1) has a discrete dichotomy on I_1 with the projection $I - P^*$. Indeed, by Remark 3.2 and using the fact $|A| = |A^*|$ we have

$$|(T(s+1)^*)^{-1}\,P^*\,T(m+1)^*| \leqq M\theta^{m-s} \qquad m \geqq s\,,$$

$$|(T(s+1)^*)^{-1}\,(I - P^*)\,T(m+1)^*| \leqq M\theta^{s-m} \quad s \geqq m\,,$$

i.e.

$$|S(s)\,P^*\,S^{-1}(m)| \leqq M\theta^{m-s} \qquad m \geqq s\,,$$

$$|S(s)\,(I - P^*)\,S^{-1}(m)| \leqq M\theta^{s-m} \qquad s \geqq m\,.$$

Similarly, on the set $I_2 = \{\ldots, -2, -1\}$ the equation (3.1) has a discrete dichotomy with the projection $I - Q^*$. It is clear that $\operatorname{Ker} L \cong V \cap W$, where $V = \operatorname{Im} P$ and $W = \operatorname{Ker} Q$. Hence $\dim \operatorname{Ker} L = \dim V \cap W$. For (3.1) we have $\dim \operatorname{Ker} L^* = \dim V^\perp \cap W^\perp$, where $V^\perp$ is the orthogonal complement of V and $L^*: X \to X$ has the form

$$(L^*(\{c_n\}_{-\infty}^{+\infty}))_n = c_n - (A_n^*)^{-1} c_{n-1} .$$

Using the fact $\dim \operatorname{Ker} L = \dim V^\perp \cap W^\perp$ we see that $\{c_n\}$ is a bounded solution of (3.1) iff $c_0 \in V^\perp \cap W^\perp$, and since $\{A_n^*\}$ has a discrete dichotomy on I_1 and I_2 we obtain that for each such solution $\{c_n\}$, c_n tends geometrically to zero as $n \to \pm\infty$. Hence $\Sigma c_n^* f_n$ is convergent for $\{f_n\}_{-\infty}^{+\infty}$ bounded.

For $\{f_n\} \in \operatorname{Im} L$ and a bounded solution $\{c_n\}$ of (3.1) we have

$$a_{n+1} = A_n a_n + f_n .$$

Hence

$$\Sigma_{-\infty}^{+\infty} c_n^* a_{n+1} = \Sigma_{-\infty}^{+\infty}(c_n^* A_n a_n + c_n^* f_n) .$$

Thus

$$\Sigma_{-\infty}^{+\infty} a_n^* c_{n-1} = \Sigma_{-\infty}^{+\infty} a_n^* A_n^* c_n + \Sigma_{-\infty}^{+\infty} c_n^* f_n$$

and

$$0 = \Sigma_{-\infty}^{+\infty} a_n^*(c_{n-1} - A_n^* c_n) = \Sigma_{-\infty}^{+\infty} c_n^* f_n .$$

Conversely, if $\Sigma_{-\infty}^{+\infty} c_n^* f_n = 0$ for each bounded solution $\{c_n\}$ of (3.1) then we see that for each $d \in R^m$ satisfying $d^*(P - (I - Q)) = 0$ and putting $T_j = T(j)$ for $j \geqq 0$, $T_j = T(j) = A_j^{-1} \ldots A_{-1}^{-1}$ for $j < 0$, the sequence

$$(3.3) \qquad \begin{aligned} c_n &= (T_{n+1}^*)^{-1} (I - P^*)\, d\,, & n &\geqq -1 \\ c_n &= (T_{n+1}^*)^{-1}\, Q^* d\,, & n &\leqq -1 \end{aligned}$$

is the solution of (3.1) and hence

$$d^*(\Sigma_{-\infty}^{-1} Q(T_{n+1})^{-1} f_n + \Sigma_0^{+\infty}(I - P)(T_{n+1})^{-1} f_n) = 0 .$$

Thus the following matrix equation has a solution g:

$$(P - (I - Q))\, g = \Sigma_{-\infty}^{-1} Q(T_{n+1})^{-1} f_n + \Sigma_0^{+\infty}(I - P)\, T_{n+1}^{-1} f_n .$$

Let us define the sequence $\{x_n\}$ by

$$\begin{aligned} x_n &= T_n P g + \Sigma_0^{n-1} T_n P T_{s+1}^{-1} f_s - \Sigma_n^{+\infty} T_n(I - P)\, T_{s+1}^{-1} f_s\,, & n &\geqq 0\,, \\ x_n &= T_n(I - Q) g + \Sigma_{-\infty}^{n-1} T_n Q T_{s+1}^{-1} f_s - \Sigma_n^{-1} T_n(I - Q) T_{s+1}^{-1}\,, & n &\leqq 0\,, \end{aligned}$$

where we consider $\Sigma_p^q \ldots = 0$ for $p > q$. The sequende $\{x_n\}$ is well-defined since g satisfies the above matrix equation. It is not difficult to see that $\{x_n\}$ is a solution of $Lx = f$. Now we proceed in the same way as in [4] and hence we obtain that $\operatorname{codim} \operatorname{Im} L = \dim V^\perp \cap W^\perp$ and index $L = \dim V + \dim W - m$. This completes the proof.

Lemma 3.4. *Let $\{A_n\}_{n\geq 0}$ have a discrete dichotomy on Z_+, A_n being invertible, bounded on Z_+, $A_n \in \mathscr{L}(R^m)$. Let $|B_n| \to 0$, $B_n \in \mathscr{L}(R^m)$, as $n \to +\infty$. Further we assume that $\{A_n + B_n\}$ are invertible. Then $\{A_n + B_n\}_{n\geq 0}$ has a discrete dichotomy on Z_+ and, moreover, if P, P' are projections of dichotomies for $\{A_n\}$, $\{A_n + B_n\}$ (see Remark* 3.2), *then* $\dim \operatorname{Im} P = \dim \operatorname{Im} P'$.

Proof. For $e > 0$ sufficiently small there is $j \in N$ such that for each $n \geqq j$ we have $|B_n| < e$. Hence by [6] the sequence $\{A_n \times B_n\}_{n\geqq j}$ has a discrete dichotomy with projections $\{P'_n\}_{n\geqq j}$. If $\{P_n\}_{n\geqq j}$ are projections for $\{A_n\}_{n\geqq 1}$ then by [6] we also have

$$(3.4) \qquad |P_n - P'_n| < eM_1\,, \quad n \geqq j\,.$$

Since $A_n + B_n$ are invertible we can construct back projections $P'_0, P'_1, \ldots, P'_{j-1}$ such that $\{A_n + B_n\}_{n\geqq 0}$ has a discrete dichotomy on Z_+ with the projections $\{P'_n\}_{n\geqq 0}$. It is clear that

$$\dim \operatorname{Im} P_n = \dim \operatorname{Im} P_{n+1} = \dim \operatorname{Im} P\,,$$

$$\dim \operatorname{Im} P'_n = \dim \operatorname{Im} P'_{n+1} = \dim \operatorname{Im} P'\,.$$

By (3.4) we have

$$\dim \operatorname{Im} P' = \dim \operatorname{Im} P\,.$$

Now we consider a C^1-mapping $G\colon U \to R^m$, U being an open subset of R^m. We assume that G has two fixed points y_1, y_2 which are hyperbolic and there is a subsequence $\{x_n\}_{-\infty}^{+\infty} \subset U$ such that

$$\lim_{n\to-\infty} x_n = y_1\,, \quad \lim_{n\to+\infty} x_n = y_2\,, \quad x_{n+1} = G(x_n)\,, \quad \det D\,G(x_n) \neq 0\,.$$

Then we can solve the same problem as in the previous section: we put G into a smooth family $G_e\colon R^m \to R^m$ of mappings, $G_0 = G$. We want to find heteroclinic orbits of G_e for e small near $\Gamma = \{x_n\}_{-\infty}^{+\infty}$. To this end we consider the equation $H_e(\cdot) = 0$, $H_e\colon X \to X$,

$$H_e(\{z_n\}_{-\infty}^{+\infty})_n = z_{n+1} - G_e(z_n)\,.$$

We see that $H_0(\Gamma) = 0$ and $(D\,H_0(\Gamma)\,\{z_n\}_{-\infty}^{+\infty})_n = z_{n+1} - D\,G(x_n)\,z_n$, and if we put $L = D\,H_0(\Gamma)$, by Theorem 3.3 L is a Fredholm operator. Since $x_n \to y_{1(2)}$ as $n \to \infty(-\infty)$, applying Lemma 3.4 we have

$$\operatorname{index} L = m_1 + m_2 - m\,,$$

where $m_{1(2)}$ is the number (counting multiplicities) of the eigenvalues of $D\,G(y_{2(1)})$ with absolute values smaller (greater) than 1. Hence we can reduce the equation $H_e(z) = 0$ near $z = \Gamma$ by using the *Lyapunov-Schmidt method* to the bifurcation equation

$$Q(c, e) = 0\,,$$

where $Q: U_1 \times U_2 \to R^{\dim \operatorname{Ker} L^*}$, U_1, U_2 are open neighbourhoods of $O \in R^{\dim \operatorname{Ker} L}$, R respectively, and $Q(0, 0) = 0$. Note that dim Ker L^* = dim codim Im L. Finally, we can investigate the equation $Q(c, e) = 0$ near $c = 0$, $e = 0$ by applying the *theory of singularities of finite-dimensional mappings* [5, 9]. We note that each solution of $H_e(\cdot) = 0$ near Γ for e small yields a heteroclinic orbit of G_e near $\{x_n\}_{-\infty}^{+\infty}$.

We will follow the above mentioned procedure for special cases of G in the next section.

4. APPLICATIONS

We generalize the problem from *Section* 2. Consider a mapping $f: R \to R$ with the same properties as in *Section* 2. Further, we consider a C^3-mapping G

$$G: \begin{aligned} x_1 &= f(x) + o(|y|) \\ y_1 &= A(x)\, y + o(|y|)\,, \end{aligned}$$

where $y \in R^{m-1}$. We assume that $A(\cdot) \in \mathscr{L}(R^{m-1})$, $\det A(\cdot)/\langle 0, 1\rangle \neq 0$ and $A(0)$, $A(1)$ are hyperbolic, i.e. they have no eigenvalues on the unit circle. Then G has the trajectory $\Gamma = \{(x_n, 0)\}_{-\infty}^{+\infty}$ and $(0, 0)$, $(1, 0)$ are hyperbolic fixed points. Consider a perturbed mapping $G_e: R^m \to R^m$, $e \in R$, $G_0 = G$, $G_{\cdot}(\cdot) \in C^3$. Now we apply the above mentioned procedure from the end of Section 3, and the relevant operator L has the index

$$\text{index } L = 2 \dim \operatorname{Ker} L + m_1 + m_2 - m\,, \tag{4.1}$$

where dim Ker $L + m_{1(2)} - 1$ is the number of the eigenvalues of $A(1, (0))$ with absolute values smaller (greater) than 1.

We shall investigate two cases:

A. dim Ker $L = 1$, index $L = 0$.

In this case the bifurcation equation (see the end of Section 3) has the form

$$Q: U_1 \times U_2 \in R\,,$$

where $U_{1(2)}$ are neighbourhoods of $0 \in R$ and $Q(c, 0) = 0$, since $G_0 = G$ has the family of heteroclinic orbits $\mathscr{M} = \{\{(f^n(x), 0)\}_{-\infty}^{+\infty}, x \in (0, 1)\}$. Hence $Q(c, e) = e\, H(c, e)$. Thus a necessary condition for the bifurcation is $H(0, 0) = 0$. Moreover, if $H(0, 0) = 0$ and $H_c(0, 0) \neq 0$ then by the *implicit function theorem* we have near $(0, 0)$

$$e \neq 0 \quad \text{and} \quad Q(c, e) = 0 \quad \text{iff} \quad c = c(e)\,, \quad c(0) = 0\,.$$

Summing up we have proved the following theorem:

Theorem 4.1. *If $H(0, 0) = 0$ and $H_c(0, 0) \neq 0$ then in a neighbourhood of Γ there is a unique trajectory Γ_e of G_e for $e \neq 0$ small. From (4.1) we have $m \geqq 2$.*

B. dim Ker $L \geqq 2$, codim Im $L = 1$.

From (4.1) we have $m \geqq \dim \operatorname{Ker} L + 1$. In this case the bifurcation equation has the form

$$Q\colon U_1 \times U_3 \times U_2 \to R\,,$$

where $U_{1(2)}$ are neighbourhoods of $0 \in R$, U_3 is a neighbourhood of $0 \in R^{\dim \operatorname{Ker} L - 1}$, $e \in U_2$. The variable $c \in U_1$ corresponds to the family $\mathcal{M}$. Hence $Q(c, 0, 0) = 0$ and since Q is the bifurcation equation we have $D_x Q(0, 0, 0) = 0$, $x \in U_3$. We assume that $D_x^2 Q(0, 0, 0)$ is a nondegenerate matrix. Then using the *splitting lemma* [9] we obtain that $Q(\cdot, \cdot, \cdot)$ is strongly right equivalent to

$$Q(c, 0, e) + \langle D_x^2 Q(0, 0, 0)\, x, x\rangle\, (1/2)\,,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $R^{\dim \operatorname{Ker} L - 1}$. Since $Q(c, 0, 0) = 0$, we obtain

$$Q(c, 0, e) = e\, H(c, e)\,.$$

If we assume that $H(0, 0) \neq 0$, then the following theorem holds:

Theorem 4.2. *Under the above conditions in a neighbourhood of Γ for e small either there are infinitely many trajectories of G_e or*

i) *there is no heteroclinic point near $(x_0, 0) \in R^m$ for $e < 0 (>0)$,*

ii) *the set of heteroclinic points of G_e near $(x_0, 0)$ lies on $(0, 1) \times \{0\} \subset R \times R^{m-1}$ and is homeomorphic to $(0, 1)$ for $e = 0$,*

iii) *the set of heteroclinic points of G_e near $(x_0, 0)$ is homeomorphic to $S^{\dim \operatorname{Ker} L - 2} \times (0, 1)$ for $e > 0 (<0)$.*

(*We note that a heteroclinic point is a point which lies on a heteroclinic orbit and S^k is the k-dimensional sphere.*)

Proof. Near $(0, 0)$ we must solve in (c, x) the equation

$$e\, H(c, e) + \langle D_x^2 Q(0, 0, 0)\, x, x\rangle\, (1/2) = 0$$

for e small. Since $H(0, 0) \neq 0$ and $D_x^2 Q(0, 0, 0)$ is nondegenerate the structure of solutions near $(0, 0)$ depends mainly on the matrix $D_x^2 Q(0, 0, 0)$. According as this matrix is indefinite or not we obtain either the first or the second assertion.

Remark 4.3. The conditions of regularity from the above theorems 4.1 and 4.2 can be expressed explicitly.

Remark 4.4. Using *the Morse critical point theory* [5] we obtain a precise picture of the set of heteroclinic points of G_e near $(x_0, 0)$ for e small in Theorem 4.2. For instance, in the second part of this theorem the sphere, which is homeomorphic to $S^{\dim \operatorname{Ker} L - 2}$, in the case iii) shrinks to the point 0 as $e \to 0$.

We see that we can use this method for the investigation of local intersections of stable and unstable manifolds. For instance, let $f\colon R^m \to R^m$ be a C^3-diffeomorphism with hyperbolic fixed points y_1, y_2 and let us assume that $m_1 = 1$, $m_2 = m - 1$

(see the end of Section 3). The point y_2 has a one-dimensional stable manifold S_0 and y_1 has an $(m - 1)$-dimensional unstable manifold R_0. If $R_0 \cap S_0 \ni x_0$ then for the orbit $\{f^n(x_0)\}_{-\infty}^{+\infty}$ we have an operator L from Section 3 and index $L = 0$, dim Ker $L \leqq 1$. If dim Ker $L = 0$ then L is inverible and for a perturbed smooth mapping f_e: $R^m \to R^m$, R_e and S_e have a transversal intersection near x_0 for e small, where R_e, S_e, are the stable and unstable manifolds of f_e near R_0, S_0, respectively. This follows from the fact that in this case the operator $H_e(\cdot)$. (see the end of Section 3) is invertible in $\{f^n(x_0)\}_{-\infty}^{+\infty}$. If dim Ker $L = 1$ then for f_e we obtain the bifurcation equation $Q(c, e) = 0$, $Q: U \times U \to R$, where U is a neighbourhood of $0 \in R$ and $Q(0, 0) = 0$, $Q_c(0, 0) = 0$. The generic conditions are $Q_{cc}(0, 0) \neq 0$ and $Q_e(0, 0) \neq 0$. Under these conditions R_0 is tangent to S_0 at x_0, since R_e, S_e have no intersections near x_0 for small $e > 0$ $(e < 0)$, and have precisely a two-point transversal intersection near x_0 for small $e < 0 (> 0)$. This last assertion follows from the fact that our assumptions for Q imply that $Q = 0$ is equivalent to $c^2 \pm e = 0$.

Now we return to the case D from Section 2. It is a particular case of the case A of this section and we are going to derive the bifurcation equation Q from the end of Section 3. Thus we consider the mapping

$$(4.2) \qquad \begin{aligned} z_{n+1} &= f(z_n) + e\, h(z_n, y_n)\,, \\ y_{n+1} &= g(z_n, y_n) + e\, t(z_n, y_n)\,, \end{aligned}$$

where f, g have the properties from Section 2, $h, t \in C^3$. We put $v_n = y_n$, $z_n = x_n + ce_n + u_n$, where $\Gamma = \{x_n\}_{-\infty}^{+\infty}$, $\{e_n\}_{-\infty}^{+\infty} \in$ Ker L, $u_0 = 0$. Then

$$\begin{aligned} u_{n+1} &= f(x_n + c, e_n + u_n) - f(x_n) - ce_{n+1} + eh(\cdot, \cdot) \\ v_{n+1} &= g(x_n + ce_n + u_n, v_n) + e.\, t(\cdot, \cdot)\,. \end{aligned}$$

Using the projection P from Section 2 we have

$$\begin{aligned} & u_{n+1} = f(x_n + ce_n + u_n) - f(x_n) - ce_{n+1} + eh(\cdot, \cdot) \\ & (I - P)\{v_{n+1} - g(x_n + ce_n + u_n, v_n) - et(\cdot, \cdot)\} = 0 \\ & P\{v_{n+1} - g(x_n + ce_n + u_n, v_n) - et(\cdot, \cdot)\} = 0\,, \end{aligned}$$

where by *the implicit function theorem* we can solve the first two equations and inserting this solution in the last equation we obtain the bifurcation equation

$$Q(c, e) = P\{v_{n+1}(c, e) - g(x_n + ce_n + u_n(c, e), v_n(c, e)) - et(\cdot, \cdot)\} = 0\,.$$

As a matter of fact, we have just carried out the *Lyapunov-Schmidt procedure* for our case.

We see that

$$Q_e(0, 0) = P\{t(x_n, 0)\}\,.$$

Further, using $v_n(c, 0) = 0$, $u_n^c(0, 0) = (d/dc)\, u_n(c, 0)/_{c=0} = 0$ we obtain

$$Q_{ce}(0, 0) = P\{-t_x(x_n, 0)\, e_n - v_n^e(0, 0)\, g_{yx}(x_n, 0)\, e_n\}\,,$$

where the sequence $\{v_n^e(0,0)\}$ satisfies

$$(4.3)\qquad \{v_{n+1}^e(0,0) - v_n^e(0,0)\, g_y(x_n, 0)\} = (I - P)\{t(x_n, 0)\}\,.$$

Taking the system $\{x_n(s)\}_{-\infty}^{+\infty}$, $s \in (-\delta, \delta)$, $x_n(s) = f^n(s + x_0)$ we repeat the above procedure and the equation (4.3) assumes the form

$$\{v_{n+1}^e(s,0,0) - v_n^e(s,0,0)\, g_y(x_n(s), 0)\} = (I - P(s))\{t(x_n(s), 0)\}\,,$$

where $P(s)$ is the projection from Section 2 corresponding to $\{x_n(s)\}_{-\infty}^{\infty}$. Differentiating the above equation by s we find

$$(4.4)\qquad \{v_{n+1}^{es}(0,0,0) - v_n^{es}(0,0,0)\, g_y(x_n, 0) - v_n^e(0,0,0)\, g_{yx}(x_n, 0)\, x_n^s(0)\} =$$
$$= (I - P(0))\{t_x(x_n, 0)\, x_n^s(0)\} - P^s(0)\{t(x_n, 0)\}\,.$$

Note that $x_n(s) = x_n + se_n + u_n(s, 0)$ for small s, hence

$$x_n^s(0) = e_n\,.$$

Finally, we put

$$\bar{r}(s) = P(s)\{t(x_n(s), 0)\}\,,$$

then

$$\bar{r}(0) = Q_e(0,0)\,.$$

From (4.4) we have

$$Q_{ce}(0,0) = P\{-t_x(x_n, 0)\, e_n - v_n^e(0,0)\, g_{yx}(x_n, 0)\, e_n\} =$$
$$= P\{-P(0)\{t_x(x_n, 0)\, e_n\} - P^s(0)\{t(x_n, 0)\}\} =$$
$$= -P(0)\{t_x(x_n, 0)\, e_n\} - P^s(0)\{t(x_n, 0)\} = -\bar{r}'(0)\,.$$

Hence the conditions $Q_e(0,0) = 0$, $Q_{ce}(0,0) \neq 0$ are equivalent to $r(x_0) = 0$, $r'(x_0) \neq 0$ and r has the explicit form

$$(4.5)\qquad r(s) = \Sigma_{-\infty}^{-2} t(f^i(s), 0)\, g_y(f^{i+1}(s), 0) \ldots g_y(f^{-1}(s), 0) + t(f^{-1}(s), 0) +$$
$$+ \Sigma_0^{+\infty} \frac{t(f^i(s), 0)}{g_y(f^i(s), 0) \ldots g_y(s, 0)}\,.$$

Summing up we have proved

Theorem 4.4. *For the mapping (4.2) the function (4.5) $r\colon (0,1) \to R$ has the following properties: If there is $s \in (0,1)$ such that $r(s) = 0$ and $r'(s) \neq 0$ then the mapping (4.2) has for e small an orbit Γ_e near $\Gamma = \{(f^n(s), 0)\}_{-\infty}^{+\infty}$. Moreover, for $e \neq 0$, Γ_e is a transversal heteroclinic orbit. Hence the function r plays the same role as the Melnikov function for ordinary differential equations*

Finally, we consider *the quasi-linear mappings*

$$f(x) = \begin{cases} ax\,, & x \leqq 1/2\,, \quad a > 1\,, \quad a < 2 \\ (2-a)\,x - 1 + a\,, & x \geqq 1/2 \end{cases}$$

$$g(x, y) = \begin{cases} yp, & x \leqq 1/2, \quad 0 < p < 1 \\ y\,v(x), & 1/2 \leqq x = a/2 \\ y/d, & x \geqq a/2, \quad 0 < d < 1, \end{cases}$$

where $v \in C^3$ is increasing on $\langle 1/2, a/2 \rangle$ and $v = p$ for $x \leqq 1/2$, $v = 1/d$ for $x \geqq a/2$,

$$t(x, 0) = \begin{cases} t_1, & x \leqq 1/2 \\ w(x), & 1/2 \leqq x \leqq a/2 \\ t_2, & x \geqq a/2, \end{cases}$$

where $t \in C^3$, a, p, d, t_1, t_2 are constants. We will apply Theorem 4.4. In this case the sequence $\{x_n\}_{-\infty}^{+\infty}$ has the form

$$x_j = a^j z, \quad j < 0$$
$$x_0 = z, \qquad z \in (1/2, a/2)$$
$$x_j = (2 - a)^j (z - 1) + 1, \quad j > 0,$$

and

$$r(z) = \Sigma_{-\infty}^{-1} p^{|j|-1} t_1 + \frac{w(z)}{v(z)} + \Sigma_1^{+\infty} \frac{t_2 d^j}{v(z)} =$$
$$= t_1 \frac{1}{1 - p} + \left(w(z) + t_2 \frac{d}{1 - d} \right) \frac{1}{v(z)}.$$

Further, if

$$\text{(4.6)} \qquad \begin{aligned} r(1/2) &= \frac{t_1}{1 - p} + \left(t_1 + t_2 \frac{d}{1 - d} \right) \frac{1}{p} > 0 \\ r(a/2) &= \frac{t_1}{1 - p} + \left(t_2 + t_2 \frac{d}{1 - d} \right) d < 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} &< 0 \\ &> 0 \end{aligned}$$

then we obtain the following theorem.

Theorem 4.5. *If f, v, t have the above properties, $h \in C^3(R \times R, R)$, the numbers t_1, t_2, p, d satisfy the condition (4.6) and $r'(\cdot) \neq 0$ on $(1/2, a/2)$, then the mapping*

$$x_1 = f(x) + e\,h(x, y),$$
$$y_1 = y\,v(x) + e\,t(x, y)$$

has at least one transversal heteroclinic orbit for $e \neq 0$ small near the set $(0, 1) \times \times \{0\}$.

Note that for a general t the function r has the form

$$r(z) = \Sigma_{-\infty}^{-1}\; t(a^j z,\; 0)\; p^{|j|-1} + \frac{t(z, 0)}{v(z)} +$$
$$+ \Sigma_1^{+\infty}\, t((2 - a)^j (z - 1) + 1, 0) \frac{d^j}{v(z)}, \qquad z \in (1/2, a/2).$$

References

[1] *M. Medveď:* Dynamical Systems (Slovak). Veda, Bratislava 1988.

[2] *S. Smale:* Differentiable dynamical systems. Bull. Amer. Math. Soc. V. 73 (1967), 747—817.

[3] *V. K. Melnikov:* On the stability of the center for the time periodic solutions. Trans. Moscow Math. Soc. V. 12 (1963), 3—56.

[4] *K. J. Palmer:* Exponential dichotomies and transversal homoclinic points. J. Diff. Equations V. 55 (1984), 225—256.

[5] *M. Golubitsky, V. Guillemin:* Stable Mappings and their Singularities. Springer-Verlag, New York, Heidelberg, Berlin, 1973, Mir Moskva, 1977.

[6] *D. Henry:* Geometric Theory of Semilinear Parabolic Equations. LNM 840, Springer-Verlag, New York, Berlin, 1981.

[7] *Z. Nitecki:* Differentiable Dynamics. The MIT Press, Cambridge, Massachusetts, London, 1971, Mir, Moskva, 1975.

[8] *S. N. Chow, J. K. Hale, J. Mallet-Paret:* An example of bifurcation to homoclinic orbits. J. Differ. Equations V. 37 (1980), 351—373.

[9] *Th. Bröcker, L. Lander:* Differentiable Germs and Catastrophes. Cambridge Univ. Press, Cambridge, 1975, Mir. Moskva, 1977.

[10] *S. N. Chow, J. K. Hale:* Methods of Bifurcation Theory. Springer-Verlag, New York, Berlin, Heidelberg, 1982.

Súhrn

BIFURKÁCIA HETEROKLINICKÝCH TRAJEKTÓRIÍ DIFEOMEORFIZMOV

MICHAL FEČKAN

V článku sa študujú bifurkácie heteroklinických trajektórií difeomorfizmov. Hlavnou metódou je Lyapunovova-Schmidtova redukcia. Pre dvojrozmerný prípad je odvodená funkcia, ktorá hrá tú istú úlohu pre bifurkácie ako Melnikova funkcia pre diferenciálne rovnice.

Author's address: Michal Fečkan, Matematický ústav SAV, Štefanikova 49, 814 73 Bratislava.