Maia Koicheva; Edward Omey

Stability of characterizations of distribution functions using failure rate functions

Aplikace matematiky, Vol. 35 (1990), No. 6, 481–486

Persistent URL: http://dml.cz/dmlcz/104430

STABILITY OF CHARACTERIZATIONS OF DISTRIBUTION FUNCTIONS USING FAILURE RATE FUNCTIONS

MAIA KOICHEVA, EDWARD OMEY

(Received September 20, 1988)

Summary. Let λ denote the failure rate function of the d.f. F and let λ_1 denote the failure rate function of the mean residual life distribution. In this paper we characterize the distribution functions F for which $\lambda_1 = c\lambda$ and we estimate F when it is only known that λ_1/λ or $\lambda_1 - c\lambda$ is bounded.

1. INTRODUCTION

In reliability theory, the failure rate function $\lambda(x)$ associated with a failure rate distribution (d.f.) $F(x)$ is defined by $\lambda(x) := f(x)/\bar{F}(x)$ where $\bar{F}(x) := 1 - F(x)$, $F(0) = 0$ and $f(x)$ is a density of $F(x)$. It is well-known [1] that $\lambda(x)\,\Delta x$ represents the probability that an object of age x will fail in the interval $[x, x + \Delta x]$. If $F(x)$ has a finite mean μ then the mean residual life at time x is defined by $M(x) := \int_x^\infty \bar{F}(t)\,\mathrm{d}t/\bar{F}(x)$. Clearly $\lambda_1(x) := 1/(M(x))$ is the failure rate function of the d.f. $F_1(x) := 1/\mu \int_0^x \bar{F}(t)\,\mathrm{d}t$. It is well-known that each of $\lambda(x)$ and $\lambda_1(x)$ determine the underlying d.f. F. As we will show later, also the ratio $\lambda(x)/\lambda_1(x)$ may be used to characterize $F(x)$.

In our first result we characterize the d.f.'s $F(x)$ for which $\lambda_1(x) = c\,\lambda(x)$. Then we discuss the stability of such a characterization. We discuss bounds for $F(x)$ in the case when it is only known that $\lambda_1(x)/\lambda(x)$ is bounded and in the case when it is known that $|\lambda_1(x) - c\,\lambda(x)|$ is bounded.

2. MAIN RESULTS

In our first result we consider the case when $\lambda_1(x) = c\,\lambda(x)$ holds.

Theorem 2.1. *Let $c > 0$ and suppose $F(x)$ has a density $f(x)$ and a finite mean μ. Assume $\lambda_1(x) = c\,\lambda(x)$ for all $x \geqq 0$ such that $F(x) < 1$.*

(i) *If* $0 < c < 1$, *then* $F(x) = 1 - (1 + ((1 - c)\mu c)\, x)^{1/(c-1)}$, $x \geqq 0$;

(ii) *if* $c = 1$, *then* $F(x) = 1 - \exp(-(1/\mu)\, x)$, $x \geqq 0$;

(iii) *if* $c > 1$, *then* $F(x) = 1 - (1 - ((c - 1)/\mu c)\, x)^{1/(c-1)}$, $0 \leqq x \leqq \mu c/(c - 1)$.

Conversely, for each of the d.f. $F(x)$ *in* (i), (ii) *or* (iii) *we have* $\lambda_1(x) = c\,\lambda(x)$.

Proof. Integrating the relation $\lambda_1(x) = c\,\lambda(x)$ between 0 and y yields $-\log \int_y^\infty \bar{F}(s)\,ds + \log \int_0^\infty \bar{F}(s)\,ds = c(-\log \bar{F}(y) + \log \bar{F}(0))$.

Using $F(0) = 0$ and $\int_0^\infty \bar{F}(s)\,ds = \mu$ it follows that $\int_y^\infty \bar{F}(s)\,ds = \mu\,\bar{F}^c(y)$ and hence that

$$\frac{f(y)}{\bar{F}^{2-c}(y)} = \frac{1}{\mu c}. \tag{2.1}$$

If $c \neq 1$ it follows after integrating (2.1) that $x/\mu c = (1 - \bar{F}^{c-1}(x))/(c - 1)$ and the results (i) and (iii) follow. If $c = 1$, integrating (2.1) yields the result (ii). A simple calculation also yields the converse results. ■

In the next results we examine the stability of the relation $\lambda_1(x) = c\,\lambda(x)$. In Theorem 2.2 below we discuss bounds for $F(x)$ in the case when $\lambda_1(x)/\lambda(x)$ is bounded. In Theorem 2.3 we consider the case when $\lambda_1(x) - c\,\lambda(x)$ is bounded.

Theorem 2.2. *Suppose* $F(x)$ *has a density* $f(x)$ *and a finite mean* μ. *If there are constants* c *and* d $(0 < c \leqq d < 1)$ *such that* $d\,\lambda(x) \leqq \lambda_1(x) \leqq c\,\lambda(x)$ *holds for* $x \geqq 0$, *then for all* $x \geqq 0$,

$$\left(1 + \frac{1 - c}{\mu c}\, x\right)^{c/d(c-1)} \leqq \bar{F}(x) \leqq \left(1 + \frac{1 - d}{\mu d}\, x\right)^{d/c(d-1)}. \tag{2.2}$$

Proof. Using $F(0) = 0$, $\int_0^\infty \bar{F}(s)\,ds = \mu$ and $\lambda_1(x) \leqq c\,\lambda(x)$, we obtain after integration that $\mu\,\bar{F}^c(x) \leqq \int_x^\infty \bar{F}(t)\,dt$. Now define $F_1(x) := 1/\mu \int_0^x \bar{F}(s)\,ds$; we have $F_1(0) = 0$, $F_1'(x) = \bar{F}(x)/\mu$ and

$$\mu\,F_1'(x) \leqq (1 - F_1(x))^{1/c} \tag{2.3}$$

or equivalently

$$\mu\,F_1'(x)\,(1 - F_1(x))^{-1/c} \leqq 1.$$

Integrating this relation yields

$$1 - F_1(x) \geqq \left(1 + \frac{1 - c}{\mu c}\, x\right)^{c/(c-1)}. \tag{2.4}$$

In a similar way, from $d\,\lambda(x) \leqq \lambda_1(x)$ we obtain

$$\mu\,F_1'(x) \geqq (1 - F_1(x))^{1/d} \tag{2.5}$$

and

$$(2.6) \qquad 1 - F_1(x) \leqq \left(1 + \frac{1-d}{\mu d}\, x\right)^{d/(d-1)}.$$

Now we use $F_1'(x) = 1/\mu\, \bar{F}(x))$ and (2.3) – (2·6) to obtain (2.2).

Remark. If $\lim_{x\to\infty} \lambda_1(x)/\lambda(x) = c$, $0 < c < 1$, it follows from the results of de Haan [5 p. 100] that $\bar{F}(x)$ is regularly varying with index $1/(c-1)$, i.e. $\lim_{t\to\infty} (\bar{F}(tx)/\bar{F}(t) = x^{1/(c-1)}$ for each $x > 0$. If so, it is well-known that for each $\varepsilon > 0$ there exist constants A, B, C such that

$$Bx^{-\varepsilon} \leqq \bar{F}(x)\, x^{1/(1-c)} \leqq Ax^{\varepsilon}, \quad \forall x \geqq C.$$

In our final result we estimate $F(x)$ in the case when $\varrho := \sup_{x\geqq 0} |\lambda_1(x) - c\,\lambda(x)| < \infty$.

Theorem 2.3. *Suppose $F(x)$ has a density $f(x)$ and a finite mean μ and suppose $F(x) < 1$ for all $x \in \mathbb{R}$.*

Suppose that for some constant c $(0 < c \leqq 1)$,

$$\varrho := \sup_{x\geqq 0} |\lambda_1(x) - c\,\lambda(x)| < \infty.$$

Then

(i) *if* $c < 1$, $$\left|\bar{F}(x) - \left(1 + \frac{1-c}{\mu c}\, x\right)^{1/(c-1)}\right| \leqq \mu\varrho(1 + c);$$

(ii) *if* $c = 1$, $$\left|\bar{F}(x) - \exp\left(-\frac{1}{\mu}\, x\right)\right| \leqq 2\varrho\mu.$$

Proof. For further use we define $\Psi(x) := \int_x^\infty \bar{F}(t)\,dt$ and $\varphi(x) := \bar{F}(x) - A/(1 + Bx)\,\Psi(x)$ where $A = 1/\mu$ and $B = A((1/c) - 1)$ ($B = 0$ if $c = 1$). Crucial in the proof of the theorem is the following

Proposition $\sup_{x\geqq 0} |\varphi(x)| \leqq \mu\varrho$.

Proof of the Proposition. Clearly $\varphi(x)$ is continuous, differentiable and bounded. Also $\varphi(\infty) = 0$ and $\varphi(0) = 1 - A\mu = 0$ by the choice of A. Let x_0 denote a point at which $|\varphi(x)|$ attains its maximum. Clearly $\varphi'(x_0) = 0$ and $\sup_{x\geqq 0} |\varphi(x)| = |\varphi(x_0)|$. Straightforward calculation yields

$$(2.7) \qquad \varphi(x) = \Psi(x)\left(\lambda_1(x) - \frac{A}{1 + Bx}\right)$$

and

$$(2.8) \qquad \varphi'(x) = \bar{F}(x)\left\{-\lambda(x) + \frac{A}{c(1 + Bx)} + \frac{B\left(-\lambda_1(x) + \dfrac{A}{1 + Bx}\right)}{(1 + Bx)\,\lambda_1(x)}\right\}.$$

Replacing x by x_0 in (2.8), we obtain

$$\text{(2.9)} \qquad \lambda(x_0) - \frac{A}{c(1 + Bx_0)} = \frac{B\left(-\lambda_1(x_0) + \dfrac{A}{(1 + Bx_0)}\right)}{(1 + Bx_0)\,\lambda_1(x_0)}\,.$$

Now by assumption $-\varrho \leqq \lambda_1(x) - c\,\lambda(x) \leqq +\varrho$ and hence also

$$\text{(2.10)} \qquad -\varrho \leqq \lambda_1(x) - \frac{A}{1 + Bx} + c\left(\frac{A}{c(1 + Bx)} - \lambda(x)\right) \leqq +\varrho\,.$$

Now replace x by x_0 in (2.10) and use (2.9) to obtain

$$-\varrho \leqq \left(\lambda_1(x_0) - \frac{A}{1 + Bx_0}\right)\left(1 + \frac{cB}{(1 + Bx_0)\,\lambda_1(x_0)}\right) \leqq +\varrho\,.$$

It follows that

$$\text{(2.11)} \qquad \left|\lambda_1(x_0) - \frac{A}{1 + Bx_0}\right| \leqq \varrho\,.$$

Now use (2.7) and (2.11) to obtain

$$|\varphi(x_0)| \leqq \int_{x_0}^{\infty} \bar{F}(s)\,ds\,.\,\varrho = \mu\varrho.$$

Proof of the Theorem. The remainder of the proof of the theorem now follows easily. From the definition of Ψ it follows that $\Psi'(x) = \bar{F}(x)$ and then it follows that

$$\Psi'(x) + \frac{A}{1 + Bx}\,\Psi(x) = -\varphi(x)\,.$$

First consider the case $c < 1$.

Since $\Psi(0) = \mu$, the solution to this differential equation is given by

$$\Psi(x) = \mu(1 + Bx)^{-A/B} - (1 + Bx)^{-A/B} \int_0^x \varphi(t)\,(1 + Bt)^{A/B}\,dt\,.$$

Hence

$$\bar{F}(x) - \frac{A}{(1 + Bx)^{1+A/B}} = \varphi(x) - \frac{A}{(1 + Bx)^{1+A/B}} \int_0^x \varphi(t)\,(1 + Bt)^{A/B}\,dt\,.$$

Using the proposition it follows that

$$\left|\bar{F}(x) - \frac{A\mu}{(1 + Bx)^{1+A/B}}\right| \leqq$$

$$\leqq \sup_{x \geqq 0} |\varphi(x)| \left\{1 + \frac{A}{(1 + Bx)^{1+A/B}} \int_0^x (1 + Bt)^{A/B}\,dt\right\} \leqq$$

$$\leqq \mu\varrho\left\{1 + \frac{A}{A + B}\right\} = \mu\varrho(1 + c)\,.$$

This proves the result (i).

In the case when $c = 1$, in a similar way it follows that $\bar{F}(x) - \mu A \exp(-Ax) = \varphi(x) - A/\exp(Ax) \int_0^x \varphi(t) \exp(At) \, dt$ where $A = 1/\mu$. Using the proposition we obtain

$$|\bar{F}(x) - \mu A \exp(-Ax)| \leqq \mu \varrho \, 2 .$$

This proves case (ii) and the theorem. ∎

3. CONCLUDING REMARKS

3.1 In [4] the length biased d.f. is defined by its density $g(x) := 1/\mu x f(x)$. The failure rate function associated with it is given by $\lambda_g(x) = x f(x) / \int_x^\infty t f(t) \, dt$. It is easily seen that $\lambda_g(x) = [x \lambda(x) \lambda_1(x)]/[x \lambda_1(x) + 1]$. Obviously λ_g uniquely determines the d.f. $F(x)$. Since λ_1 uniquely determines $F(x)$, also $\lambda_g(x)/\lambda(x)$ uniquely determines $F(x)$. In [4] such characterizations are carried out.

3.2 The problem of characterizing the exponential d.f. and its stability has been studied by many authors (see e.g. [3], [6]). In [2] the authors characterize the gamma d.f. via exponential mixtures. Let $F_t(x) = 1 - \exp(-tx)$ $(x \geqq 0)$ denote the family of exponential d.f. with a parameter $t > 0$. If t has d.f. G then the mixture F_G of F_t with the mixing d.f. G is given by

$$F_G(x) := \int_0^\infty (1 - \exp(-tx)) \, dG(t) \quad (x \geqq 0) . \tag{3.1}$$

Clearly $\bar{F}_G(x)$ is the Laplace-Stieltjes transform of G and therefore uniquely determines G. In the case when G is gamma $\gamma(\alpha, \beta)$ with parameters $\alpha > 1$ and $\beta > 0$ (i.e. $dG(t) = [(\beta^\alpha \exp(-\beta t) \, t^{\alpha-1})/\Gamma(\alpha)] \, dt$), (3.1) reduces to $\bar{F}_G(x) = (1 + (1/\beta x))^{-\alpha}$ $(x \geqq 0)$ so that F_G is Pareto distributed. For the d.f. $F_{G'}$ let λ and λ_1 be defined as in Section 1. From Theorem 2.1 we obtain the following characterization of the gamma d.f.

Corollary. *Let* $\alpha > 1$ *and let* F_G *and* G *be related by* (3.1). *Suppose* $\mu := \int_0^\infty t^{-1} \, . \, dG(t) < \infty$. *Then* $G = \gamma(\alpha, \beta)$ *if and only if* $\lambda_1(x) = (\alpha - 1/\alpha) \lambda(x)$ *where* β, α *and* μ *are related by* $\beta = (\alpha - 1) \mu$. ∎

References

[1] *R. Barlow, F. Proshan:* Mathematical Theory of Reliability. John Wiley & Sons, New York, 1965.

[2] *Manish C. Bhattarcharjee:* On a characterization of a gamma distribution via exponential mixtures. J. Appl. Prob. 17 (2) 574—576, 1980.

[3] *J. Galambos, S. Kotz:* Characterization of probability distributions. Lecture Notes in Mathematics, 675, 1978.

[4] *R. C. Gupta, J. P. Keating:* Relations for reliability measures under length biased sampling. Scand. J. Statist. 13, 49—56, 1986.
[5] *L. de Haan:* On regular variation and its application to the weak convergence of sample extremes. Math. Centre Tracts 32, Amsterdam, 1970.
[6] *V. V. Kalashnikov, S. T. Rachev:* Characterization problems in queueing and their stability. Adv. Appl. Prob. 17, 868—886, 1985.

Souhrn

STABILITY CHARAKTERIZACÍ DISTRIBUČNÍCH FUNKCÍ POUŽÍVAJÍCÍCH FUNKCE INTENZIT PORUCH

MAIA KOICHEVA, EDWARD OMEY

Nechť F je distribuční funkce doba do poruchy a $M(x)$ příslušná podmíněná střední hodnota za podmínky, že doba do poruchy je rovna alespoň x. Označme λ funkci intenzita poruch odpovídající distribuční funkci F a $\lambda_1(x) = 1/(M(x))$ pro všechna reálná x. V článku jsou charakterizovány distribuční funkce F, pro které platí $\lambda_1 = c\lambda$, a je odhadnuto F, když je známo pouze, že λ_1/λ nebo $\lambda_1 - c\lambda$ je omezené.

Authors' addresses: Prof. *Maia Koicheva*, Institute of Mathematics, P.B.O. 373, 1090 Sofia, Bulgaria; Prof. *Edward Omey*, E.H.S.A.L., Stormstraat 2, 1000 Brussel, Belgium.