Michal Horváth

AR models with uniformly distributed noise

Aplikace matematiky, Vol. 34 (1989), No. 5, 396–401

34 (1989) APLIKACE MATEMATIKY No. 5, 396—401

AR MODELS WITH UNIFORMLY DISTRIBUTED NOISE

Michal Horváth

(Received May 17, 1988)

Summary. AR models are frequently used but usually with normally distributed white noise. In this paper AR model with uniformly distributed white noise are introduced. The maximum likelihood estimation of unknown parameters is treated, iterative method for the calculation of estimates is presented. A numerical example of this procedure and simulation results are also given.

Keywords: AR model, parameter estimation.

AMS Classification: 62M10, 65U05.

INTRODUCTION

Autoregressive (AR) models are currently used in a number of various applications. They have the following form:

$$y_t + \sum_{i=1}^{n} a_i y_{t-i} = w_t .$$

The random sequence $\{y_t\}$ represents the process that is modelled, $a_1, \ldots, a_n$ are real parameters ($a_n \neq 0$), w_t is a white noise and n is the order of the model. The white noise is most frequently regarded as normally distributed. In some cases it appears more suitable to suppose a different type of the distribution. Such models are treated in papers [2], [3], [4] but none of them is devoted to the inovations w_t with the uniform distribution. Uniformly distributed white noise may be encountered if one observes a sequence of rounding errors as well as in some other situations.

The application of AR models for modelling real processes requires to estimate the parameters $a_1, \ldots, a_n$ and the variance of the white noise w_t. If the white noise is normally distributed, the known estimators obtained by the conditional maximum likelihood method are utilized and they coincide with the least squares estimators. How can the parameters of AR models with the uniform distribution of the white noise be estimated? This problem is treated in the next sections.

Let us have a realization of a process y:

$$\mathcal{Y} = \{y(1), y(2), \ldots, y(N)\} .$$

We want to estimate the vector of unknown parameters $\varphi = (a_1, \ldots, a_n, h)$ by using the set $\mathcal{Y}$ if the white noise w_t has the uniform distribution over $\langle -h, h,\rangle$ $h > 0$. For the construction of the estimator, we apply the conditional maximum likelihood method, i.e.

$$\hat{\varphi} = \arg\left[\max_{\varphi} L(\mathcal{Y}_1 \mid \mathcal{Y}_2)\right] \tag{1}$$

where

$$\mathcal{Y}_1 = \{y(n+1), y(n+2), \ldots, y(N)\}$$
$$\mathcal{Y}_2 = \{y(1), y(2), \ldots, y(n)\} .$$

The likelihood function L may be easily rewritten in a more suitable form

$$L(\mathcal{Y}_1 \mid \mathcal{Y}_2) = \prod_{i=n+1}^{N} p_{y_i}(y(i) \mid y(i-1), \ldots, y(i-n))$$

where $p_{y_i}(\cdot \mid y(i-1), \ldots, y(i-n))$ is the conditional probability density of the random variable y_i.

Since

$$y_t = w_t - \sum_{i=1}^{n} a_i y_{t-i} ,$$

$p_{y_i}(\cdot \mid y(i-1), \ldots, y(i-n))$ is the density of the uniform distribution over

$$\langle -h - \sum_{j=1}^{n} a_j y_{i-j} , \quad h - \sum_{j=1}^{n} a_j y_{i-j}\rangle$$

and we can affirm that

$$L(\mathcal{Y}_1 \mid \mathcal{Y}_2) = \begin{cases} 0 & \text{if } \textit{Condition } C \text{ is not satisfied ,} \\ \dfrac{1}{(2h)^{N-n}} & \text{if } \textit{Condition } C \text{ is satisfied .} \end{cases}$$

Condition C:

$$\left|y(i) + \sum_{j=1}^{n} a_j\, y(i-j)\right| \leqq h \quad \text{for all} \quad i \in \{n+1, \ldots, N\} .$$

According to equation (1), the estimators of unknown parameters have the following form:

$\hat{h}$ (estimator of h): choose the minimal value of h, for which there exists an estimate of $a_1, \ldots, a_n$.

$\hat{a}_1, \ldots, \hat{a}_n$ (estimator of $a_1, \ldots, a_n$): choose such values of the parameters $a_1, \ldots, a_n$ for which *Condition C* is fulfilled for the above value $\hat{h}$ of h.

It is obvious that the estimators cannot be represented explicitly.

Theorem 1. *If $N > 2n$, then there exists the unique solution $\hat{a}_1, \ldots, \hat{a}_n, \hat{h}$.*

Proof. The problem of finding $\hat{a}_1, \ldots, \hat{a}_n, \hat{h}$ can be considered as the linear programming problem with the restrictions

$$|b_0\, y(i) + \sum_{j=1}^{n} b_j\, y(i-j)| \leqq 1 \quad i \in \{n+1, \ldots, N\}$$

and with the objective function b_0 to be maximized where

$$b_0 = \frac{1}{h} \quad \text{and} \quad b_j = \frac{a_j}{h} \quad \text{for} \quad j = 1, \ldots, n\,.$$

It can be shown that the white noise attains such values for which arbitrary n vectors $(y(i-1), \ldots, y(i-n))$ are linearly independent if $n \geqq 2$ and $y(i) \neq 0$ if $n \geqq 1$, where $i \in \{n+1, \ldots, N\}$ with probability 1. If furthermore $N > 2n$, then the set of admissible values creates a convex polyhedron in $n+1$-dimensional Euclidean space E_{n+1} and that yields the existence of the solution of the problem.

Let us suppose that the solution is not unique. Then the set of solutions creates convex polyhedron in E_n which contains at least one boundary interval. The points of this boundary interval must fulfill the equation

$$\left| \frac{1}{\hat{h}}\, y(i) + \sum_{j=1}^{n} b_j\, y(i-j) \right| = 1$$

at least for n different $i \in \{n+1, \ldots, N\}$. Becouse N is finite and there exists just one solution of each system consisting of n equations that is in contradiction to our assumption and from that then exists just one solution $\hat{a}_1, \ldots, \hat{a}_n, \hat{h}$.

ITERATIVE ESTIMATION METHOD

The following method appears convenient for numerical calculations. It consists of two parts: the calculation of the estimate $\hat{h}$ and the calculation of the estimate $\hat{a}_1, \ldots, \hat{a}_n$.

The calculation of $\hat{a}_1, \ldots, \hat{a}_n$ requires the use of a suitable criterion function which facilitates the iterative calculation of $\hat{a}_1, \ldots, \hat{a}_n$. We can use the following function J:

$$J(a_1, \ldots, a_n) = \sum_{i=n+1}^{N} \delta_i [|y(i) + \sum_{j=1}^{n} a_j\, y(i-j)| - h]^2$$

where $\delta_i = 0 \quad \text{if} \quad |y(i) + \sum_{j=1}^{n} a_j\, y(i-j)| \leqq h$

$$\delta_i = 1 \quad \text{if} \quad |y(i) + \sum_{j=1}^{n} a_j\, y(i-j)| > h\,.$$

Condition C is fulfilled if and only if $J(a_1, \ldots, a_n) = 0$.
The calculation of $\hat{a}_1, \ldots, \hat{a}_n$:

A: choose the initial values of $a_1, \ldots, a_n$ and the step d.

B: $J_0 \leftarrow J(a_1, \ldots, a_n)$

C:
$$\begin{aligned}
J_{-1} &\leftarrow J(a_1 - d, a_2, \ldots, a_n)\,;\\
J_1 &\leftarrow J(a_1 + d, a_2, \ldots, a_n)\,;\\
J_{-2} &\leftarrow J(a_1, a_2 - d, \ldots, a_n)\,;\\
J_2 &\leftarrow J(a_1, a_2 + d, \ldots, a_n)\,;\\
&\vdots\\
J_{-n} &\leftarrow J(a_1, \ldots, a_{n-1}, a_n - d)\,;\\
J_n &\leftarrow J(a_1, \ldots, a_{n-1}, a_n + d)\,.
\end{aligned}$$

D: $J_0 \leftarrow \min_i (J_i)$

E: if J_0 has been changed go to G.

F: $d \leftarrow d/2$; if d is greater than the given accuracy go to C else end, the solution has not been found.

G: execute the correction of a_i; if $J_0 > 0$ go to C else you have got the desired solution.

The calculation of $\hat{h}$ utilizes the fact that

$$\min_{a_1,\ldots,a_n} J(a_1, \ldots, a_n, h) \begin{cases} >0 & \text{if } h < \hat{h} \\ =0 & \text{if } h \geqq \hat{h}\,. \end{cases}$$

The estimate $\hat{h}$ is calculated by the halving method.
The calculation of $\hat{h}$:

A: choose the initial values of h and the step k.

B: try to find the values of $a_1, \ldots, a_n$ fulfilling *Condition C*. If the attempt is not successful go to D.

C: $h \leftarrow h - k$; $k \leftarrow k/2$; go to B.

D: $h \leftarrow h + k$; $k \leftarrow k/2$; if k is greater than the given accuracy go to B else and, the calculated value is the desired estimate h.

Finally, let us discuss the choice of the initial values.
The advantageous choice of initial values of $a_1, \ldots, a_n$ is that of the values obtained by the last squares method. The initial value of h may be obtained from the expression for the variance of uniform distribution:

$$h_0 = \sqrt{(3\sigma)}$$

where σ is the estimate of variance obtained by the least squares method. The initial step d should be proportional to the initial values of $a_1, \ldots, a_n$ and the initial step k should be proportional to h_0.

NUMERICAL EXAMPLE

We apply the procedure from the previous section to estimate the parameters. We have 150 data generated by AR model of the third order with uniformly distributed white noise over $(-0{\cdot}5, 0{\cdot}5)$. The parameters are $a_1 = 0{\cdot}7$, $a_2 = 1$, $a_3 = 0{\cdot}3$, the accuracy is 0·0001. The values of the parameters obtained by the least squares method are

$$a_1 = 0{\cdot}7208 \quad a_2 = 0{\cdot}8770 \quad a_3 = 0{\cdot}2615 .$$

The algorithm from the previous section yields the following results:

Step	a_1	a_2	a_3	h
1	0·6762	0·8954	0·2318	0·5445
2	0·6674	0·9572	0·2596	0·5105
3	0·6749	0·9722	0·2596	0·4935
4	0·6686	0·9579	0·2512	0·4913
5	0·6695	0·9577	0·2512	0·4908
6	0·6695	0·9584	0·2512	0·4907

SIMULATION RESULTS

Here we will describe results from simulations which were carried out to compare the maximum likelihood estimates and least squares estimates. The simulations include two sets of 50 realizations ($N = 200$ and $N = 100$). The data were generated from AR models of the third order with uniformly distributed white noise. The accuracy of the calculations was 0·001. Tables 1–2 show the results.

Method		$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{h}$
ML	mean	0·699	1·004	0·301	0·493
LS		0·678	0·987	0·270	0·495
ML	mean square	0·644	0·168	0·557	0·061
LS	error. 10^3	3·622	1·189	3·648	0·249

Tab. 1. $a_1 = 0{\cdot}7$, $a_2 = 1$, $a_3 = 0{\cdot}3$, $h = 0{\cdot}5$, $N = 200$

Method		$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{h}$
ML	mean	−1·888	1·178	−0·184	0·098
LS		−1·847	1·131	−0·167	0·098
ML	mean square	3·282	9·608	2·865	0·006
LS	error. 10^3	12·14	30·69	11·26	0·020

Tab. 2. $a_1 = -1{\cdot}9$, $a_2 = 1{\cdot}2$, $a_3 = -0{\cdot}2$, $h = 0{\cdot}1$, $N = 100$

CONCLUSION

The estimators of unknown parameters of AR model with uniformly distributed white noise have been derived. To this end the maximum likelihood method was used which yields generally better results than the least squares method. Simulations confirm this fact. The estimates can be calculated by the iterative method,

References

[1] *G. E. P. Box, G. M. Jenkins:* Time Series Analysis: Forecasting and Control. Holden-Day, San Francisco 1970.

[2] *N. Davies, T. Spedding, W. Watson:* Autoregressive moving average process with non-normal residueals, J. Time Series Anal. 2 (1980), 155—171.

[3] *A. J. Lawrence, P. A. W. Lewis:* The exponential autoregressive moving — average EARMA (p, q) process. J. Roy. Statist. Soc., B 42 (1980), 150—161.

[4] *R. D. Martin, V. J. Yohai:* Robustness in Time Series and Estimating ARMA Models. Proc. Handbook of Statistics 5 Time Series in the Time Domain, Elsevier, Amsterdam 1985.

[5] *R. L. Kashyap, A. R. Rao:* Dynamic Stochastic Models from Empirical Data, Academic Press, New York 1976.

Súhrn

AR MODELY S ROVNOMERNÝM ROZDELENÍM ŠUMU

MICHAL HORVÁTH

AR modely sa často používajú pri modelovaní dynamických dejov, ale vo väčšine prípadov obsahujú biely šum s normálnym rozdelením. Článok pojednáva o AR modeloch s rovnomerne rozdeleným bielym šumom. Možno tu nájsť odhady parametrov odvodené metódou maximálnej vierohodnosti, a tiež iteračnú metódu určenú na ich numerické výpočty. Uvedený je aj číselný príklad, ktorý ozrejmuje iteračný postup.

Резюме

МОДЕЛИ АР С РАВНОМЕРНЫМ РАСПРЕДЕЛЕНИЕМ ШУМА

MICHAL HORVÁTH

Модели АР часто применяются при моделировании динамических процессов, но в большинстве случаев они содержат белый шум с нормальным распределением. Статья посвящена проблемам моделей АР с равномерным распределением белого шума. В статье также приведены оценки параметров, полученные методом максимального правдоподобия, и итерационый метод, предназначенный для их приближённого вычисления. В состав статьи входит и численный пример, иллюстрирующий итерационный процесс.

Author's address: Ing. *Michal Horváth,* Ústav merania a meracej techniky CEFV SAV, Dúbravská cesta 9, 842 19 Bratislava.