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LINEAR TRANSFORMATIONS OF LOCALLY STATIONARY PROCESSES

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Summary. The paper deals with linear transformations of harmonizable locally stationary random processes. Necessary and sufficient conditions under which a linear transformation defines again a locally stationary process are given.

Keywords: harmonizable process, locally stationary process, covariance function.
AMS subject classification: 60G.

The notion of a weakly locally stationary process was introduced by Silverman in [1]. Let $\{x(t), t \in \mathbb{R}_1\}$ be a second order random process with a vanishing expected value and with a covariance function $R(\cdot,\cdot)$ defined on $\mathbb{R}_1 \times \mathbb{R}_1$. If for every pair s, t of reals one can write

$$R_x(s, t) = R_x^{(1)}\left(\frac{s + t}{2}\right) R_x^{(2)}(s - t),$$

where $R_x^{(1)} \geqq 0$ and $R_x^{(2)}$ is a stationary covariance, then, in accordance with [1], $R_x(\cdot, \cdot)$ is a locally stationary covariance function. A process possessing such a covariance function is called weakly locally stationary, too. Further, we shall need some facts about the harmonic analysis of nonstationary random processes. Following [2] we say that a random process $\{x(t), t \in \mathbb{R}_1\}$ is harmonizable if it can be written in the form of a stochastic integral understood in the quadratic mean sense

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$$

where $\{\xi(\lambda), \lambda \in \mathbb{R}_1\}$ is a second order random process with zero mean and a covariance function $\gamma(\cdot,\cdot)$ of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. A random process is harmonizable if and only if its covariance function $R_x(\cdot, \cdot)$ is harmonizable, i.e.

$$R_x(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, dd\gamma(\lambda, \mu).$$

Let us suppose that the process $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary and harmonizable. In the theory of weakly stationary processes linear transformations of these processes play a very important role. If $\{x(t), t \in \mathbb{R}_1\}$ is a weakly stationary process having a spectral decomposition $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$, and $\phi(\cdot) \in \mathscr{L}_2(\mathbb{R}_1, \gamma(\cdot))$ where $\gamma(\cdot)$ is the corresponding spectral measure, then the process

$$(1) \qquad y(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, \phi(\lambda) \, d\xi(\lambda) \,, \quad t \in \mathbb{R}_1$$

is weakly stationary, too. In the case of a locally stationary process the situation is not clear. We shall formulate the following problem: if $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary and harmonizable, under which conditions put on a function $\phi(\cdot)$ the process (1) will be locally stationary as well.

First, we immediately see that the process $\{y(t), t \in \mathbb{R}_1\}$ must be of the second order, i.e. for every $s, t \in \mathbb{R}_1$ the integral

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, \phi(\lambda) \, \bar{\phi}(\mu) \, dd\gamma(\lambda, \mu)$$

must exist. The process $\{y(t), t \in \mathbb{R}_1\}$ will be locally stationary if its covariance function $R_y(\cdot,\cdot)$ is a product of $R_y^{(1)}$ and $R_y^{(2)}$,

$$R_y(s, t) = R_y^{(1)} \left(\frac{s + t}{2} \right) R_y^{(2)}(s - t)$$

with $R_y^{(1)} \geqq 0$ and $R_y^{(2)}(\cdot)$ being a stationary covariance function. Let us consider the transformation

$$T: \frac{\lambda + \mu}{2} = u \,, \quad \lambda - \mu = v$$

which, under the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$, makes it possible to express $R_y(\cdot,\cdot)$ in the form

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iu(s-t)} \, e^{iv[(s+t)/2]} \, \phi \left(u + \frac{v}{2} \right) \bar{\phi} \left(u - \frac{v}{2} \right) dF_1(u) \, dF_2(v)$$

where

$$(2) \qquad \iint_{E \times F} dF_1(u) \, dF_2(v) = \iint_{T^{-1}(E \times F)} dd\gamma(\lambda, \mu)$$

($E \times F$ is a measurable rectangle in $\mathbb{R}_1 \times \mathbb{R}_1$). This relation is in more detail explained in [3]. Because $R_x^{(2)}(y) = \int_{-\infty}^{+\infty} e^{iyu} \, dF_1(u)$ is a stationary covariance function, $F_1(u)$ must be a distribution function of a nonnegative measure of finite variation; because $R_x^{(1)}(\cdot) \geqq 0$, the Fourier image of $F_2(\cdot)$ must be nonnegative.

Now, if the following *separation* of the variables u, v

$$\phi \left(u + \frac{v}{2} \right) \bar{\phi} \left(u - \frac{v}{2} \right) = f(u) \, g(v)$$

is possible then

$$R_y(s, t) = \int_{-\infty}^{+\infty} e^{iu(s-t)} f(u) \, dF_1(u) \int_{-\infty}^{+\infty} e^{iv\,(s+t)/2} \, g(v) \, dF_2(v) \,.$$

Further, if $\int_{-\infty}^{+\infty} e^{iu(s-t)} f(u) \, dF_1(u)$ is a stationary covariance function and, simultaneously, if

$$\int_{-\infty}^{+\infty} e^{iv[(s+t)/2]}\, g(v)\, dF_2(v) \geqq 0$$

for every s, $t \in \mathbb{R}_1$ then $R_y(\cdot,\cdot)$ will be locally stationary. The following theorem gives necessary and sufficient conditions on $\phi(\cdot)$ in order that the process $\{y(t), t \in \mathbb{R}_1\}$ may be locally stationary.

Theorem 1. *Let $\{x(t), t \in \mathbb{R}_1\}$ be a harmonizable locally stationary random process,*

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, d\xi(\lambda)\,.$$

Then the process $\{y(t), t \in \mathbb{R}_1\}$ where $y(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, \phi(\lambda)\, d\xi(\lambda)$ is locally stationary if and only if there exist functions $f(\cdot)$, $g(\cdot)$ such that

1° $\quad \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u)\, g(v)$ a.e. $[F_1 \times F_2]$,

2° $\quad \int_{-\infty}^{+\infty} e^{itu} f(u)\, dF_1(u)$ is a stationary covariance function,

3° $\quad \int_{-\infty}^{+\infty} e^{isv} g(v)\, dF_2(v) \geqq 0$ for every $s \in \mathbb{R}_1$,

where $F_1(\cdot)$, $F_2(\cdot)$ are induced by the transformation T described above under the local stationary of $\{x(t), t \in \mathbb{R}_1\}$.

Proof. Let us suppose that both $\{x(t), t \in \mathbb{R}_1\}$ and $\{y(t), t \in \mathbb{R}_1\}$ are locally stationary. Then the covariance function $R_y(\cdot,\cdot)$ of $\{y(t), t \in \mathbb{R}_1\}$ can be written as the product

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)}\, \phi(\lambda)\, \bar{\phi}(\mu)\, dd\gamma(\lambda, \mu) = R_y^{(1)}\left(\frac{s+t}{2}\right) R_y^{(2)}(s - t)$$

where $R_y^{(1)}(\cdot) \geqq 0$ and $R_y^{(2)}(\cdot)$ is a stationary covariance. By means of transformation T (described above) we can express

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]}\, e^{iu(s-t)}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)$$

where $F_1(\cdot)$ is a probability distribution function (without loss of generality we can put $R_x(0, 0) = 1$) and the Fourier image of $F_2(\cdot)$ is nonnegative. We immediately see that

$$R_y(s, s) = R_y^{(1)}(s)\, R_y^{(2)}(0)\,, \quad R_y\left(\frac{t}{2}, -\frac{t}{2}\right) = R_y^{(2)}(t)\, R_y^{(1)}(0)$$

and hence

$$R_y^{(2)}(0)\, R_y^{(1)}(s) = \iint_{-\infty}^{+\infty} e^{isv}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)\,,$$

$$R_y^{(1)}(0)\, R_y^{(2)}(t) = \iint_{-\infty}^{+\infty} e^{itu}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)\,.$$

In this way we obtain the relation

$$R_y(0,0)\iint_{-\infty}^{+\infty} e^{isv}\, e^{itu}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v) =$$

$$= \iint_{-\infty}^{+\infty} e^{isv}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v)\ \times$$

$$\times \iint_{-\infty}^{+\infty} e^{itu}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v)$$

holding for every pair $(s, t) \in \mathbb{R}_2$. Properties of the two-dimensional Fourier transform imply that

$$R_y(0,0)\iint_{-\infty}^{uv} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_1(x)\, dF_2(y) =$$

$$= \int_{-\infty}^{u}\int_{-\infty}^{+\infty} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_2(y)\, dF_1(x)\ \times$$

$$\times \int_{-\infty}^{v}\int_{-\infty}^{+\infty} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_1(x)\, dF_2(y)$$

for every $u, v \in \mathbb{R}_1$. This fact proves that

$$(3)\qquad \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) = \frac{1}{R_y(0,0)}\int_{-\infty}^{+\infty} \phi\left(x+\frac{v}{2}\right)\bar{\phi}\left(x-\frac{v}{2}\right) dF_2(v)\ \times$$

$$\times \int_{-\infty}^{+\infty} \phi\left(u+\frac{y}{2}\right)\bar{\phi}\left(u-\frac{y}{2}\right) dF_1(u) = f(x)\, g(y)$$

a.e. $[F_1 \times F_2]$.

As $R_y^{(1)}(\cdot) \geqq 0$ then

$$\int_{-\infty}^{+\infty} e^{ivs}\left\{\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\right\} dF_2(v) \geqq 0$$

must be nonnegative for every $s \in \mathbb{R}_1$. Similarly, as $R_y^{(2)}(\cdot)$ is a stationary covariance function then

$$\int_{-\infty}^{+\infty} e^{iut}\left\{\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_2(v)\right\} dF_1(u)$$

must be a stationary covariance function, too. Since $F_1(\cdot)$ is a probability distribution function $R_y^{(2)}(\cdot)$ will be a covariance function if and on if

$$\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_2(v) \geqq 0 \quad \text{a.e.} \quad [F_1]\,.$$

On the contrary, let the conditions 1°, 2°, 3° of Theorem 1 hold. The covariance function $R_y(\cdot, \cdot)$ can be expressed as

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]} e^{iu(s-t)} \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)$$

because $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary. As $\phi(u + v/2)\, \bar{\phi}(u - v/2) = f(u)\, g(v)$ a.e. $[F_1 \times F_2]$ then

$$\begin{aligned} R_y(s, t) &= \textstyle\iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]} g(v)\, e^{iu(s-t)} f(u)\, dF_1(u)\, dF_2(v) = \\ &= \textstyle\int_{-\infty}^{+\infty} e^{iv[(s+t)/2]} g(v)\, dF_2(v) \int_{-\infty}^{+\infty} e^{iu(s-t)} f(u)\, dF_1(u) = \\ &= R_y^{(1)}\left(\frac{s+t}{2}\right) R_y^{(2)}(s - t) \end{aligned}$$

where $R_y^{(1)}(\cdot) \geqq 0$ and $R_y^{(2)}(\cdot)$ is a stationary covariance. We have proved that the process $\{y(t), t \in \mathbb{R}_1\}$ is locally stationary. Q.E.D.

In Theorem 1 we met an interesting relation concerning the function $\phi(\cdot)$, namely

$$\phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u)\, g(v)\, [F_1 \times F_2] \quad \text{a.s.}$$

Let us now suppose a somewhat stronger condition, namely

$$\phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u)\, g(v)$$

for every $u, v \in \mathbb{R}_1$. Then for $v = 0$ we get

$$|\phi(u)|^2 = f(u)\, g(0) \geqq 0 \tag{4}$$

and similarly for $u = 0$

$$\phi\left(\frac{v}{2}\right) \bar{\phi}\left(-\frac{v}{2}\right) = f(0)\, g(v)\,.$$

Both the relations together give that (provided $f(0) \neq 0$, $g(0) \neq 0$)

$$f(u)\, g(v) = \frac{|\phi(u)|^2\, \phi\left(\frac{v}{2}\right) \bar{\phi}\left(-\frac{v}{2}\right)}{f(0)\, g(0)}$$

and hence

$$\phi(\lambda)\, \bar{\phi}(\mu) = K \,.\, \left|\phi\left(\frac{\lambda + \mu}{2}\right)\right|^2 \phi\left(\frac{\lambda - \mu}{2}\right) \bar{\phi}\left(-\frac{\lambda - \mu}{2}\right)$$

where $K = f(0)\, g(0)$, $u = (\lambda + \mu)/2$, $v = \lambda - \mu$.

As $g(v) = \int_{-\infty}^{+\infty} \phi(u + v/2)\, \bar{\phi}(u - v/2)\, dF_1(u)$ (see Theorem 1), thus $g(0) = \int_{-\infty}^{+\infty} |\phi(u)|^2\, dF_1(u) \geqq 0$ and hence the assumption $g(0) > 0$ is quite natural.

This fact together with (4) yields that $f(u) \geqq 0$ for every $u \in \mathbb{R}_1$, hence also $K > 0$. In the sequel, for simplicity, we will assume $K = 1$. In this way we have obtained the following functional equation for the function $\phi(\cdot)$

$$(5) \qquad \phi(\lambda)\,\bar{\phi}(\mu) = \left|\phi\left(\frac{\lambda+\mu}{2}\right)\right|^2 \phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(-\frac{\lambda-\mu}{2}\right), \quad \lambda, \mu \in \mathbb{R}_1,$$
$$\phi(0) = 1$$

which is very close to the local stationarity. If the function $g(v) = \phi(v/2)\,\bar{\phi}(-v/2)$ is a characteristic function then the covariance function $\phi(\cdot)\,\bar{\phi}(\cdot)$ will be locally stationary because

$$\left|\phi\left(\frac{\lambda+\mu}{2}\right)\right|^2 \geqq 0$$

and

$$\phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(-\frac{\lambda-\mu}{2}\right)$$

is a stationary covariance. We see that the linear transformation between two locally stationary random processes determined by the function $\phi(\cdot)$ is closely connected with the question which covariances of the type $\phi(\cdot)\,\bar{\phi}(\cdot)$ are locally stationary.

Let us try to solve the functional equation (5). At the first sight it is evident that $\phi(\cdot) = 1$ is a solution of (5) and thus the set of solutions is nonempty. Similarly, the function $\phi(\cdot)$ equal to 1 at 0 and vanishing otherwise also solves this equation. Hence, there is a discontinuous solution of (5). It is evident as well that the product $\phi_1\phi_2(\cdot)$ solves (5) if $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are solutions of (5). The equation can be easily expressed in an equivalent form

$$\phi(u+v)\,\bar{\phi}(u-v) = |\phi(u)|^2\,\phi(v)\,\bar{\phi}(-v), \quad u, v \in \mathbb{R}_1,$$

$\phi(0) = 1$. First we shall be interested in continuous solutions of the equation (5). Let $\phi(\cdot)$ be a solution of (5) continuous at zero with $\phi(\lambda_0) = 0$, $\lambda_0 \neq 0$. Then

$$0 = \phi(\lambda_0)\,\bar{\phi}(\mu) = \left|\phi\left(\frac{\lambda_0+\mu}{2}\right)\right|^2 \phi\left(\frac{\lambda_0-\mu}{2}\right)\bar{\phi}\left(\frac{\mu-\lambda_0}{2}\right)$$

for every real μ. For $\mu = 0$ we have

$$0 = \left|\phi\left(\frac{\lambda_0}{2}\right)\right|^2 \phi\left(\frac{\lambda_0}{2}\right)\bar{\phi}\left(-\frac{\lambda_0}{2}\right)$$

and hence either $\phi(\lambda_0/2) = 0$ or $\phi(-\lambda_0/2) = 0$. In the case of $\phi(\lambda_0/2) = 0$ we again obtain either $\phi(\lambda_0/4) = 0$ or $\phi(-\lambda_0/4) = 0$. In this way we can construct a sequence $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \to 0$ for $n \to \infty$ with $\phi(\lambda_n) = 0$. This conclusion contradicts the assumption that $\phi(0) = 1$. We can summarize; if there exists a continuous at zero solution $\phi(\cdot)$ of (5) then $\phi(\lambda) \neq 0$ for every $\lambda \in \mathbb{R}_1$. Thus $1/\phi(\cdot)$ is a solution of (5) as well.

We see that all solutions of (5) continuous at zero form a group with respect to multiplication. Let us describe this group explicitly. At the beginning we must realize that if $\phi(\cdot)$ is a solution of (5) then the absolute value $|\phi(\cdot)|$ solves the same equation, hence $\phi(\cdot)/|\phi(\cdot)|$ is a solution of (5) as well. As $|\phi(\cdot)/|\phi(\cdot)|| = 1$ the equation (5) in this case has the form

$$(6) \qquad \phi(\lambda)\,\bar{\phi}(\mu) = \phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in R_1\,, \quad \phi(0) = 1$$

and $|\phi(\lambda)| = 1$ for every $\lambda \in R_1$. Then one can write $\phi(\lambda) = e^{i\alpha(\lambda)}$ where $\alpha(\cdot)$ is a real function, and we have obtained an equivalent transcription of (6)

$$\alpha(\lambda) - \alpha(\mu) = \alpha\left(\frac{\lambda-\mu}{2}\right) - \alpha\left(\frac{\mu-\lambda}{2}\right)$$

or

$$\alpha(u+v) - \alpha(u-v) = \alpha(u) - \alpha(v)\,.$$

We see that $\Delta_h\alpha(\lambda) = \Delta_h\alpha(0)$ for every λ, $h \in R_1$. This implies that

$$\alpha(\lambda) = C_0 + C_1\lambda$$

and hence $\phi(\lambda) = e^{i(C_0 + C_1\lambda)}$. As we demand $\phi(0) = 1$, we have $C_0 = 0$.

The equation (5) for the absolute value $A(\cdot) = |\phi(\cdot)|$ has the form

$$A(\lambda)\,A(\mu) = A^2\left(\frac{\lambda+\mu}{2}\right)A\left(\frac{\lambda-\mu}{2}\right)A\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in R_1\,,$$

$A(0) = 1$ and $A(\lambda) > 0$.

We can write $A(\lambda) = e^{a(\lambda)}$ and arrive at the equation

$$a(\lambda) + a(\mu) = 2a\left(\frac{\lambda+\mu}{2}\right) + a\left(\frac{\lambda-\mu}{2}\right) + a\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in R_1\,.$$

We immediately obtain that $a(0) = 0$ and the latter relation can be rewritten as

$$\Delta_h^2 a(\lambda) = \Delta_h^2 a(0)\,.$$

Solving the difference equation $\Delta_h^3 a(\lambda) = 0$ we obtain that

$$a(\lambda) = K_0 + K_1\lambda + K_2\lambda^2\,.$$

As we need $a(0) = 0$, we have $K_0 = 0$. In this way we have proved that every continuous at zero solution of (5) has the form

$$\phi(\lambda) = e^{K\lambda^2} \,.\, e^{Q\lambda}$$

where $K \in R_1$, $Q \in C$.

Corollary 1. *Let $\{x(t), t \in R_1\}$ be a harmonizable locally stationary process*

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, d\xi(\lambda)\,.$$

Then the process $\{y(t), t \in \mathbb{R}_1\}$, $y(t) = \int_{-\infty}^{+\infty} e^{it\lambda} e^{K\lambda^2} e^{Q\lambda} d\xi(\lambda)$ *with* $K \leqq 0$, $Q \in \mathbb{C}$ *is locally stationary, too.*

Proof. It is evident that

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} e^{K(\lambda^2+\mu^2)} e^{Q\lambda} e^{\bar{Q}\lambda} dd\gamma(\lambda, \mu) .$$

By means of the transformation T: $(\lambda + \mu)/2 = u$, $\lambda - \mu = v$ and the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$ we get

$$R_y(s, t) =$$
$$= \iint_{-\infty}^{+\infty} e^{iu(s-t)} e^{iv(s+t)/2} e^{2Ku^2} e^{(Q+\bar{Q})u} e^{K(v^2/2)} e^{(Q-\bar{Q})v/2} dF_1(u) dF_2(v) .$$

As $Q + \bar{Q} = 2 \operatorname{Re} Q$ and $e^{2Ku} . e^{2\operatorname{Re}Qu} > 0$,

$$\int_{-\infty}^{+\infty} e^{iu(s-t)} e^{2Ku^2} e^{2\operatorname{Re}Qu} dF_1(u)$$

is a stationary covariance function. Similarly $(Q - \bar{Q})/2 = i \operatorname{Im} Q$ and hence

$$\int_{-\infty}^{+\infty} e^{iv(s+t)/2} e^{i\operatorname{Im}Qv} e^{K(v/2)} dF_2(v) =$$
$$= \int_{-\infty}^{+\infty} e^{iv(s+t)/2} \left(\int_{-\infty}^{+\infty} e^{ivx} \frac{1}{\sqrt{(-2\pi K)}} e^{(x-\operatorname{Im}Q)^2/2K} dx\right) dF_2(v) =$$
$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{iv[(s+t)/2+x]} dF_2(v)\right) \frac{1}{(-2\pi K)} e^{(x-\operatorname{Im}Q)^2/2K} dx \geqq 0$$

because under the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$ we have

$$\int_{-\infty}^{+\infty} e^{ivy} dF_2(v) \geqq 0$$

for every $y \in \mathbb{R}_1$. Q.E.D.

Corollary 2. *Every continuous locally stationary covariance function* $R(\cdot,\cdot)$ *of the type*

$$R(s, t) = \phi(s)\, \bar{\phi}(t) , \quad R(0, 0) = 1$$

has the form

$$R(s, t) = e^{-a(s^2+t^2)} . e^{bs+\bar{b}t} ,$$

where $a \geqq 0$, $b \in C$.

Proof. In order to be locally stationary the covariance function $\phi(\cdot)\, \bar{\phi}(\cdot)$ must satisfy

$$\phi(s)\, \bar{\phi}(t) = R_1\left(\frac{s+t}{2}\right) R_2(s-t)$$

where $R_1(\cdot) \geqq 0$ and $R_2(\cdot)$ is a stationary covariance. One immediately sees that

$$R_1(x) = |\phi(x)|^2 , \quad R_2(y) = \phi\left(\frac{y}{2}\right) \bar{\phi}\left(-\frac{y}{2}\right)$$

and thus the function $\phi(\cdot)$ must be a solution of the equation

$$\phi(s)\,\bar{\phi}(t) = \left|\phi\left(\frac{s+t}{2}\right)\right|^2 \phi\left(\frac{s-t}{2}\right)\bar{\phi}\left(\frac{t-s}{2}\right), \quad \phi(0) = 1\,.$$

As was proved above the continuous solution of this functional equation is

$$\phi(\lambda) = e^{a\lambda^2 + b\lambda}$$

where $a \in R_1$, $b \in C$.

Thus $R_1(x) = |e^{ax^2+bx}|^2 = e^{2ax^2} \, . \, e^{(b+\bar{b})x}$ and

$$R_2(y) = e^{a(y/2)^2 + by/2} \, . \, e^{a(-y/2)^2 + \bar{b}(-y/2)} = e^{ay^2/2} \, . \, e^{(b-\bar{b})y/2}\,.$$

Indeed, we obtain that $R_1(\cdot) \geqq 0$; $R_2(\cdot)$ must be a stationary covariance. As $R_2(\cdot)$ is continuous it will be a stationary covariance if and only if $R_2(\cdot)$ is a characteritic function. It means that thc coefficient a must be less or equal to zero because the inequality

$$|R_2(y)| = e^{ay^2/2} \leqq 1$$

must hold for every $y \in R_1$. Then

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyv} \frac{1}{2\pi(-a)} e^{(v-(b-\bar{b})/2)^2/2a}\, dv$$

in the case $a < 0$ and

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyv}\, dF_0\left(v - \frac{b-\bar{b}}{2}\right)$$

for $a = 0$ where $F_0(v) = 0$ for $v \leqq 0$, $F_0(v) = 1$ otherwise. Q.E.D.

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Souhrn

LINEÁRNÍ TRANSFORMACE LOKÁLNĚ STACIONÁRNÍCH PROCESŮ

JIŘÍ MICHÁLEK

V článku je řešena otázka, za jakých podmínek je lineární transformace harmonizovatelného slabě lokálně stacionárního procesu opět lokálně stacionární proces. Jsou nalezeny nutné a postačující podmínky pro funkci, kterou je tato lineární transformace určena.

Резюме

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ЛИНЕЙНЫЕ ПРЕОБРАЗОВАНИЯ ЛОКАЛЬНО СТАЦИОНАРНЫХ ПРОЦЕССОВ

В статье решен вопрос, при каких условиях линейное преобразование гармонизуемого в широком смысле локально стационарного процесса опять является локально стационарным. Найдены необходимые и достаточные условия для функции, определяющей такое линейное преобразование.

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