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ON NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES

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Summary. The author defines the numerical solution of a first order ordinary differential equation on a bounded interval in the way covering the general form of the so called one-step methods, proves convergence of the method (without the assumption of continuity of the right-hand side) and gives a sufficient condition for the order of convergence to be $O(h^\nu)$.

Keywords: numerical solution of differential equations, one-step method, order of convergence.

1. INTRODUCTION

Let an ordinary differential equation

(1) $$y'(t) = f(t, y(t)),\ t \in I = [a, b],$$

together with an initial condition

(2) $$y(a) = \eta$$

be given, where $f: I \times R^m \to R^m$. A function $\varphi: I \to R^m$ is a solution of $(1-2)$ if it is absolutely continuous on I and satisfies the condition (2) and the equation (1) almost everywhere on I, i.e., except on a set of Lebesgue measure zero. We assume that the function f satisfies the Perron condition

$$\|f(t, y_1) - f(t, y_2)\| \leqq \Omega(t, \|y_1 - y_2\|),$$

where f and Ω are of Cartheodory's type. It is known that $(1-2)$ has a solution φ (see for example [3], [9]).

We assume that the problem $(1-2)$ has a bounded solution φ. In numerical calculations this solution is approximated by a numerical solution only for points $t_i^h = a + ih$ with $h = h_N = (b - a)/N$. Here N is a natural number. Now let $\{v_i^h\} \subset R^m$ be an arbitrary sequence such that

$$v_0^h = \eta,$$
$$\|v_{i+1}^h - \Phi_i(t_{i+1}^h)\| \leqq h\,\varepsilon_1(h), \quad \varepsilon_1(h) \to 0,$$

where Φ_i denotes the solution of (1) passing through (t_i^h, v_i^h). Then $v^h = \{v_0^h, \ldots, v_N^h\}$ is a numerical solution of $(1-2)$. Using the above assumptions we can prove convergence of v^h to the solution φ of $(1-2)$. We also give a sufficient condition of its convergence provided $\Omega(t, u) = Lu$, $L \geqq 0$ and

$$\|v_i^h - \varphi(t_i^h)\| = 0(h^\nu)\,, \quad \text{where } \nu \text{ is a positive constant}\,.$$

A similar problem was considered in [6] but only when the function f was continuous with a linear comparison function $\Omega(t, u) = Lu$. The sequence $\{v_i^h\}$ may be generated by a one-step method so that the results of the paper are a slight generalization of the known ones. Numerical solutions of $(1-2)$ were also considered for example in [2, 4, 7].

2. CONVERGENCE

We are now able to prove

Theorem 1. *Suppose that*

1° *the function $f: I \times R^m \to R^m$ is bounded, measurable with respect to the first variable for any fixed value of the second, and continuous with respect to the second variable for any fixed value of the first;*

2° *there exists a function $\Omega: I \times R_+ \to R_+ = [0, \infty)$ such that for $t \in I$, $y_1, y_2 \in R^m$ we have*

$$\|f(t, y_1) - f(t, y_2)\| \leqq \Omega(t, \|y_1 - y_2\|)\,;$$

3° *Ω is bounded and nondecreasing with respect to the second variable and $\Omega(t, 0) \equiv 0$;*

4° *Ω is measurable with respect to the first variable for any fixed value of the second, and continuous with respect to the second variable uniformly with respect to the first;*

5° *the function $u(t) \equiv 0$ is the only absolutely continuous solution of the problem*

$$u'(t) = \Omega(t, u(t))\,, \quad t \in I\,,$$
$$u(a) = 0\,;$$

6° *the sequence $\{v_i^h\} \subset R^m$ is arbitrary and such that*

$$\|v_{i+1}^h - \Phi_i(t_{i+1}^h)\| \leqq h\,\varepsilon_1(h)\,, \quad i \in R_{N-1} = \{0, 1, \ldots, N-1\}\,, \quad v_0^h = \eta\,,$$

where $\varepsilon_1(h) \to 0$ and Φ_i denotes the solution of (1) passing through (t_i^h, v_i^h).

Then the numerical solution v^h converges to a solution φ of $(1-2)$, i.e.

$$\lim_{N\to\infty} \max_{i \in R_N} \|v_i^h - \varphi(t_i^h)\| = 0\,,$$

and

$$\lim_{N\to\infty} \max_{i\in R_N} \sup_{[a,t_i{}^h]} \|\Phi_i(t) - \varphi(t)\| = 0\,,$$

Proof. It is known that our assumptions guarantee that there exists a unique solution φ of $(1-2)$ (see [3]). Let

$$a_i^h = \|v_i^h - \varphi(t_i^h)\|\,, \quad i \in R_N\,,$$

$$z_0^h = 0\,, \quad z_{i+1}^h = \sup_{[t^h{}_i, t^h{}_{i+1}]} \|\Phi_i(t) - \varphi(t)\|\,, \quad i \in R_{N-1}\,.$$

Then

$$a_i^h \leqq z_i^h + h\,\varepsilon_1(h)\,, \quad i \in R_N\,.$$

Further, we have

$$z_{i+1}^h = \sup_{[t^h{}_i, t^h{}_{i+1}]} \|v_i^h - \varphi(t_i^h) + \int_{t^h{}_i}^{t} [f(\tau, \Phi_i(\tau)) - f(\tau^\mathsf{T}, \varphi(\tau))]\,\mathrm{d}\tau \leqq$$

$$\leqq a_i^h + \int_{t^h{}_i}^{t^h{}_{i+1}} \Omega(\tau, \sup_{[t^h{}_i, t^h{}_{i+1}]} \|\Phi_i(\tau) - \varphi(\tau)\|)\,\mathrm{d}\tau\,,$$

and hence

$$z_{i+1}^h \leqq z_i^h + \int_{t^h{}_i}^{t^h{}_{i+1}} \Omega(\tau, z_{i+1}^h)\,\mathrm{d}\tau + h\,\varepsilon_1(h)\,, \quad i \in R_{N-1}\,.$$

Put

$$u_i^h = \sup_{[a,t^h{}_i]} \|\Phi_i(t) - \varphi(t)\|\,, \quad i \in R_N\,.$$

Evidently

$$u_{i+1}^h \geqq u_i^h\,, \quad i \in R_{N-1}\,.$$

Now we have

$$u_{i+1}^h = \max\big(\sup_{[a,t^h{}_i]} \|\Phi_i(t) - \varphi(t)\|\,, \sup_{[t^h{}_i, t^h{}_{i+1}]} \|\Phi_i(t) - \varphi(t)\|\big) \leqq$$

$$\leqq \max(u_i^h, z_{i+1}^h)\,, \quad i \in R_{N-1}\,.$$

Because $z_i^h \leqq u_i^h$ we have

$$u_{i+1}^h \leqq u_i^h + \int_{t^h{}_i}^{t^h{}_{i+1}} \Omega(\tau, u_{i+1}^h)\,\mathrm{d}\tau + h\,\varepsilon_1(h)\,, \quad i \in R_{N-1}\,.$$

In view of the boundedness of Ω there exists a constant $D > 0$ such that

$$0 \leqq u_{i+1}^h - u_i^h \leqq hD\,, \quad i \in R_{N-1}\,.$$

Moreover, by the continuity of Ω we get

$$u_{i+1}^h \leqq u_i^h + \int_{t^h{}_i}^{t^h{}_{i+1}} \Omega(\tau, u_i^h)\,\mathrm{d}\tau + h[\varepsilon_1(h) + \varepsilon_2(h)]\,,$$

where

$$\varepsilon_2(h) = \sup\{|\Omega(t, p) - \Omega(t, r)| : t \in I,\ r, p \in R_+,\ |r - p| \leqq hD\} \to 0\,.$$

Now we consider the initial-value problem

$$(3)\qquad \begin{cases}\lambda'(t) = \Omega(t, \lambda(t)) + \varepsilon(h), & \varepsilon(h) = \varepsilon_1(h) + \varepsilon_2(h), \\ \lambda(a) = 0 . \end{cases}$$

This problem has a solution λ^h which is a nondecreasing and absolutely continuous function (see [9]).

We can prove that

$$(4)\qquad \lambda^h(t_i^h) \geqq u_i^h, \quad i \in R_N .$$

This inequality is true for $i = 0$. We assume that (4) is true for a fixed i. Integrating (3) from t_i^h to t_{i+1}^h we get

$$\lambda^h(t_{i+1}^h) = \lambda^h(t_i^h) + \int_{t^h_i}^{t^h_{i+1}} \Omega(\tau, \lambda^h(\tau))\, d\tau + h\, \varepsilon(h) \geqq$$
$$\geqq \lambda^h(t_i^h) + \int_{t^h_i}^{t^h_{i+1}} \Omega(\tau, \lambda^h(t_i^h))\, d\tau + h\, \varepsilon(h) \geqq$$
$$\geqq u_i^h + \int_{t^h_i}^{t^h_{i+1}} \Omega(\tau, u_i^h)\, d\tau + h\, \varepsilon(h) \geqq u_{i+1}^h .$$

Now the inequality (4) follows by induction.

By the theorem on continuous dependence of the solution of the problem (3) on parameters and initial conditions we have

$$\lim_{h\to 0} \max_{t\in I} \lambda^h(t) = 0 ,$$

and

$$\max_{i\in R_N} z_i^h \leqq \max_{i\in R_N} u_i^h \leqq \max_{i\in R_N} \lambda^h(t_i^h) \leqq \max_{t\in I} \lambda^h(t) ,$$
$$\max_{i\in R_N} a_i^h \leqq h\, \varepsilon_1(h) + \max_{i\in R_N} z_i^h ,$$

which yields the assertion of our theorem.

3. REMARKS

(i) It is clear that Theorem 1 will remain true if we assume in 6° that the function Φ_i is a solution of the problem

$$\Phi_i'(t) = F_i(t, \Phi_i(t)) ,$$
$$\Phi_i(t_i^h) = v_i^h ,$$

where

$$\|f(t, w) - F_i(t, w)\| \leqq \varepsilon_3(h) \to 0 .$$

(ii) Theorem 1 is also valid if $\Omega(t, u) = Lu$, where L is a nonnegative constant. In this case Ω is not bounded but the sequence $\{u_i^h\}$ satisfies the condition

$$0 \leqq u_{i+1}^h \leqq u_i^h + Lhu_{i+1}^h + h\, \varepsilon_1(h) , \quad i \in R_{N-1} ,$$
$$u_0^h = 0 .$$

Taking N so large that $1 - Lh > 0$ and using Lemma 1.2 [5] we get

$$0 \leqq u_i^h \leqq \frac{1}{L} \left[\exp\left((b-a) L/(1-Lh)\right) - 1\right] \varepsilon_1(h), \quad i \in R_N,$$

$$0 \leqq z_i^h \leqq u_i^h.$$

From the above inequality we obtain the assertion of Theorem 1.

(iii) If $\Omega(t, u) = Lu$, $L \geqq 0$ and if there exists a constant $\nu > 0$ such that $\varepsilon_1(h) = O(h^\nu)$ then the order of convergence of the numerical solution v^h is ν, i.e.

$$\|v_i^h - \varphi(t_i^h)\| = O(h^\nu),$$

$$\sup_{[a, t_i^h]} \|\Phi_i(t) - \varphi(t)\| = O(h^\nu).$$

(iv) If

$$\begin{cases} v_0^h = \eta, \\ v_{i+1}^h = G_i(t_i^h, v_i^h, h), \quad i \in R_{N-1}, \end{cases}$$

then we have the general form of one-step methods considered by many authors (see for example [1, 5, 8]).

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Souhrn

O NUMERICKÉM ŘEŠENÍ OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC S NESPOJITOSTMI

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Autor definuje numerické řešení obyčejné diferenciální rovnice prvního řádu na omezeném intervalu způsobem, který zahrnuje obecný tvar tzv. jednokrokových metod, dokazuje konvergenci metody (bez předpokladu spojitosti pravé strany) a udává postačující podmínku pro rychlost konvergence řádu $O(h^\nu)$.

Резюме

О ЧИСЛЕННОМ РЕШЕНИИ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С РАЗРЫВАМИ

TADEUSZ JANKOWSKI

Автор определяет численное решение обыкновенного дифференциального уравнения первого порадка способом, который включает общий вид так называемых одношаговых методов, доказывает сходимост метода (без предположения непрерывности правой части) и приводит достаточное условие для скорости сходимости порядка $O(h^\nu)$.,

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