Aplikace matematiky

Vladimír Drápalík; Vladimír Janovský

On a potential problem with incident wave as a field source

Aplikace matematiky, Vol. 33 (1988), No. 6, 443–455

Persistent URL: `http://dml.cz/dmlcz/104323`

33 (1988) APLIKACE MATEMATIKY No. 6, 443—455

ON A POTENTIAL PROBLEM WITH INCIDENT WAVE AS A FIELD SOURCE

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

(Received March 13, 1986)

Summary. A field source which is given by an incident wave in a neighborhood of an inhomogeneous body (in $\mathbf{R}^2$) yields an integral equation on the boundary of Ω. This integral equation may serve as a boundary condition for the field equation on Ω. If Ω is a circle then the existence and uniqueness of the new boundary value problem is proved and an algorithm for the approximate solution is proposed.

Keywords: diffraction, nonlocal boundary condition, finite elements.

AMS Subject classification: 31A30, 65N30, 35J15, 35J67, 78A20, 78A45.

1. INTRODUCTION

We investigate a classical problem of the wave scattering: Let $f = f(x)$ be the density of an electric charge in $\mathbb{R}^2$ (the support of f, $\operatorname{supp} f$, is assumed to be compact in $\mathbb{R}^2$). Let $w = w(x)$ on $\mathbb{R}^2$ be the potential of the electric field in vacuum. Provided an inhomogeneous body Ω (Ω is a bounded domain in $\mathbb{R}^2$, $\bar{\Omega} \cap \operatorname{supp} f = \emptyset$) is present, the field is changed. If $u = u(x)$ is the potential of the resulting field in $\mathbb{R}^2$, find u on Ω.

The classical mathematical formulation reads as follows: We say that u is a smooth solution if $u = u(x)$ is continuous and bounded in $\mathbb{R}^2$, all the first derivatives of u are piecewise continuous in $\mathbb{R}^2$, and

$$(1.1) \qquad \mathbf{A}u \equiv -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j}\right) = f(x)$$

on $\mathbb{R}^2$ in the sense of distributions. Here and in the sequel, the summation convention of repeated indices i and j is used.

We assume

(i) strong ellipticity of $\mathbf{A}$, i.e. $a_{ij} \in L_\infty(\mathbb{R}^2)$, there exists a positive constant c such that

$$a_{ij}\xi_i\xi_j \geqq c\xi_i\xi_i \quad \text{for each} \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2\,;$$

(ii) $a_{ij}(x) \equiv \delta_{ij}$ outside Ω, where Ω is a bouded domain in $\mathbb{R}^2$, $\partial\Omega$ is "smooth enough";

(iii) $f \in L_2(\mathbb{R}^2)$, $\operatorname{supp} f$ is compact in $\mathbb{R}^2$, $\operatorname{supp} f \cap \bar{\Omega} = \emptyset$.

The function $w = w(x)$ is called the *incident wave.* We assume w to be continuous and bounded in $\mathbb{R}^2$, and to satisfy

$$-\Delta w = f \tag{1.2}$$

in $\mathbb{R}^2$ (in the sense of distributions).

Thus, in order to find u on Ω, one has to solve (1.1) in $\mathbb{R}^2$. Traditional numerical procedures were based on an approximate reduction of $\mathbb{R}^2$ to a "sufficiently large" bounded domain containing Ω (of course, the larger domain, the better).

The aim of this paper is to formulate a properly posed boundary value problem for *u on Ω.* It means that u should satisfy $\boldsymbol{A}u = 0$ on Ω with a *boundary condition on $\partial\Omega$.* Moreover, the trace of w on $\partial\Omega$ are the only data of the new boundary value problem.

If Ω is a circle then the boundary condition on $\partial\Omega$ is considerably simplified. We give some proposals concerning numerical solution of the relevant boundary value problem.

We suggest a practical strategy in the case of a "general" Ω: Replace the given Ω by *any* circle Ω' which a) contains Ω and b) does not intersect $\operatorname{supp} f$; define $a_{ij} \equiv \delta_{ij}$ on $\Omega' - \Omega$, of course. Then $\boldsymbol{A}u = 0$ on Ω' with a comparatively simple boundary condition on $\partial\Omega'$. Naturally, the trace of w on $\partial\Omega'$ is equal to the data.

2. BOUNDARY CONDITION ON $\partial\Omega$

According to our assumptions, the functions u and w are harmonic and bounded in a neighborhood of ∞. It is well known that both u and w are continuous at ∞, i.e.

$$\lim_{|x| \to +\infty} u(x) = u_\infty\,, \tag{2.1}$$

$$\lim_{|x| \to +\infty} w(x) = w_\infty\,. \tag{2.2}$$

Moreover, the first derivatives vanish at ∞, namely

$$\frac{\partial u}{\partial x_i}(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to +\infty\,, \tag{2.3}$$

$$\frac{\partial w}{\partial x_i}(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to +\infty \tag{2.4}$$

for $i = 1, 2$.

Lemma 2.1. *The equality*

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} f(y)\log|x-y|\,\mathrm{d}y = w_\infty - w(x) \tag{2.5}$$

holds for each $x \in \mathbb{R}^2$.

Proof. The above formula is one of the classical Green's formulae. Thus, the proof is omitted.

Notation. Let Ω^c be the complement of $\bar{\Omega}$ in $\mathbb{R}^2$. We denote by $\mu = \mu(y)$ the outward normal vector at $y \in \partial\Omega^c$ with respect to Ω^c. If $y \in \partial\Omega^c$ then the symbol $\partial u/\partial\mu(y)$ means the derivative of u at the point y along the direction $\mu(y)$ "with respect to Ω^c", i.e.

$$\frac{\partial u}{\partial\mu(y)} \overset{\text{def}}{=} \mu_i(y) \lim_{\substack{z\to y\\ z\in\Omega^c}} \frac{\partial u}{\partial x_i}(z)$$

where $\mu = (\mu_1, \mu_2)$.

Lemma 2.2. *The equality*

$$\frac{1}{2\pi}\int_{\Omega^c} f(y)\log|x-y|\,\mathrm{d}y = u_\infty - \frac{1}{2}u(x) - \tag{2.6}$$
$$- \frac{1}{2\pi}\int_{\partial\Omega^c}\left\{\frac{\partial u}{\partial\mu(y)}\log|x-y| - u(y)\frac{\partial\log|x-y|}{\partial\mu(y)}\right\}\mathrm{d}\sigma(y)$$

holds for each $x \in \partial\Omega^c$.

Proof. Let $x \in \partial\Omega^c$ be given. We define the following sets depending on positive parameters R and δ, $R > \delta$:

$$\begin{aligned}
B_{R,\delta} &= \{y\in\mathbb{R}^2 : \delta < |x-y| < R\}, & U_{R,\delta} &= B_{R,\delta}\cap\Omega^c,\\
S_R &= \{y\in\mathbb{R}^2 : |x-y| = R\}, & S_\delta &= \{y\in\mathbb{R}^2 : |x-y| = \delta\},\\
K_\delta &= S_\delta \cap \partial U_{R,\delta}, & D_\delta &= \partial\Omega^c \cap \partial U_{R,\delta}.
\end{aligned}$$

We assume R's large and δ's small enough so that $\operatorname{supp} f \subset U_{R,\delta}$. Clearly, $\partial U_{R,\delta} = S_R \cup K_\delta \cup D_\delta$.

Let us extend the definition of $\mu(y)$ to S_R and K_δ: If $y \in S_R \cup K_\delta$ then $\mu = \mu(y)$ is the outward normal vector at y with respect to $U_{R,\delta}$, see Fig. 1. Moreover, we set

$$\frac{\partial u}{\partial\mu(u)} \overset{\text{def}}{=} \mu_i(y) \lim_{\substack{x\to y\\ x\in U_{R,\delta}}} \frac{\partial u}{\partial x_i}(x), \quad \mu = (\mu_1, \mu_2).$$

We define

$$I_{R,\delta} = \frac{1}{2\pi}\int_{U_{R,\delta}} f(y)\log|x-y|\,\mathrm{d}y. \tag{2.7}$$

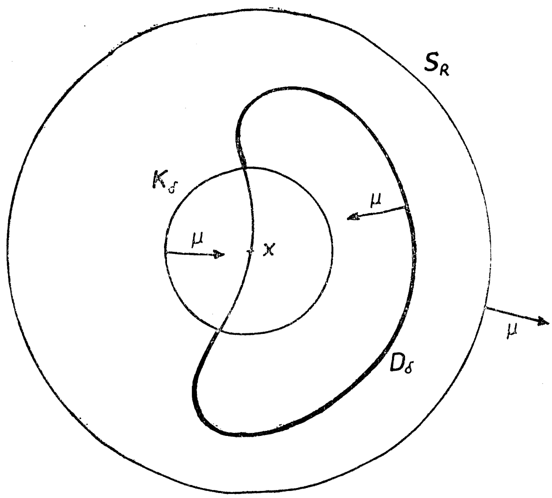

Fig. 1.

Since $f = \boldsymbol{A}u = -\Delta u$ on $U_{R,\delta}$, the classical Green's formula yields $I_{R,\delta} = \gamma_1 + \gamma_2$, where

$$\gamma_1 = -\frac{1}{2\pi}\int_{\partial U_{R,\delta}} \frac{\partial u}{\partial \mu(y)} \log|x - y| \, \mathrm{d}\sigma(y)\,,$$

$$\gamma_2 = \frac{1}{2\pi}\int_{\partial U_{R,\delta}} u(y)\,\frac{\partial \log|x - y|}{\partial \mu(y)}\, \mathrm{d}\sigma(y)\,.$$

Clearly, $\gamma_1 = \gamma_{1,1} + \gamma_{1,2} + \gamma_{1,3}$ where $\gamma_{1,1}$ and $\gamma_{1,2}$ and $\gamma_{1,3}$ are equal to

$$-\frac{1}{2\pi}\int_\Gamma \frac{\partial u}{\partial \mu(y)} \log|x - y| \, \mathrm{d}\sigma(y)$$

where Γ is to be replaced by K_δ and D_δ and S_R, respectively. Similarly, $\gamma_2 = \gamma_{2,1} + \gamma_{2,2} + \gamma_{2,3}$ where $\gamma_{2,1}$ and $\gamma_{2,2}$ and $\gamma_{2,3}$ are equal to

$$\frac{1}{2\pi}\int_\Gamma u(y)\,\frac{\partial \log|x - y|}{\partial \mu(y)}\, \mathrm{d}\sigma(y)$$

where $\Gamma := K_\delta$ and D_δ and S_R, respectively. All γ's are functions of R and δ.

Passing $R \to +\infty$ and $\delta \to 0_+$. we obtain $\gamma_{1,1} \to 0$, $\gamma_{2,1} \to -\frac{1}{2}\,u(x)$ (using the smoothness of $\partial\Omega$),

$$\gamma_{1,2} \to -\frac{1}{2\pi}\int_{\partial\Omega^c} \frac{\partial u}{\partial \mu(y)} \log|x - y| \, \mathrm{d}\,\sigma(y)\,,$$

$$\gamma_{2,2} \to \frac{1}{2\pi}\int_{\partial\Omega^c} u(y)\,\frac{\partial \log|x - y|}{\partial \mu(y)}\, \mathrm{d}\,\sigma(y)\,.$$

Moreover, by virtue of (2.3) and (2.1) we can prove $\gamma_{1,2} \to 0$ and $\gamma_{2,3} \to u_\infty$. Finally, (2.7) implies that

$$I_{R,\delta} \to \frac{1}{2\pi} \int_{\Omega^c} f(y) \log |x - y| \, \mathrm{d}y \, .$$

The formula (2.6) immediately follows from the above facts. Q.E.D.

Corollary. *We have*

$$\frac{1}{2} u(x) - \frac{1}{2\pi} \int_{\partial\Omega^c} \left\{ u(y) \frac{\partial \log |x - y|}{\partial \mu(y)} - \frac{\partial u}{\partial \mu(y)} \log |x - y| \right\} \mathrm{d}\sigma(y) = = w(x) + u_\infty - w_\infty \tag{2.8}$$

for each $x \in \partial\Omega^c$.

Proof. Since $\operatorname{supp} f \cap \bar{\Omega} = \emptyset$, we have

$$\int_{\Omega^c} f(y) \log |x - y| \, \mathrm{d}y = \int_{\mathbb{R}^2} f(y) \log |x - y| \, \mathrm{d}y \, .$$

Then (2.8) follows directly from (2.5) and (2.6). Q.E.D.

Notation. Let $y \in \partial\Omega$. We denote by $\partial u / \partial \nu(y)$ the derivarive of u along the co-normal at y with respect to Ω, i.e.

$$\frac{\partial u}{\partial \nu(y)} \stackrel{\text{def}}{=} a_{ij}(y) \, . \, (-\mu_i(y)) \, . \lim_{\substack{z \to y \\ z \in \Omega}} \frac{\partial u(z)}{\partial x_j} \, .$$

Due to (1.1) and the Gauss theorem,

$$\frac{\partial u}{\partial \nu(y)} = - \frac{\partial u}{\partial \mu(y)} \tag{2.9}$$

at any $y \in \partial\Omega$. The equation (2.9) expresses the continuity of "fluxes" through $\partial\Omega$.

Theorem 2.1. *If u is a smooth solution then*

$$\frac{1}{2} u(x) - \frac{1}{2\pi} \int_{\partial\Omega} \left\{ u(y) \frac{\partial \log |x - y|}{\partial \mu(y)} + \right. \left. + \frac{\partial u}{\partial \nu(y)} \log |x - y| \right\} \mathrm{d}\sigma(y) = w(x) + c \tag{2.10}$$

for each $x \in \partial\Omega$, where $c = u_\infty - w_\infty$.

Proof. The assertion follows from (2.8) and (2.9). Q.E.D.

3. BOUNDARY CONDITION ON A CIRCLE

Let us assume Ω to be a circle; without loss of generality, $\Omega \equiv \{x \in \mathbb{R}^2 : |x| < R\}$. One easily calculates that

$$(3.1) \qquad \frac{\partial}{\partial \mu(y)} \log |x - y| = - \frac{1}{2R}$$

for each $x \in \partial\Omega$, $y \in \partial\Omega$, $x \neq y$.

Introducing polar coordinates (r, α): $x_1 = r \cos \alpha$, $x_2 = r \sin \alpha$, we can write (2.10) as

$$(3.2) \quad \frac{1}{2} u(R, \alpha) + \frac{1}{4\pi} \int_0^{2\pi} u(R, \beta) \, d\beta + \int_0^{2\pi} \mathscr{K}(\alpha - \beta) \frac{\partial u}{\partial \nu} (R, \beta) \, d\beta = w(R, \alpha) + c \,,$$

where $c = u_\infty - w_\infty$ and

$$(3.3) \qquad \mathscr{K}(\alpha) \overset{\text{def}}{=} - \frac{R}{2\pi} \left[\log R \sqrt{(2)} + \frac{1}{2} \log (1 - \cos \alpha) \right];$$

the symbol $(\partial u / \partial \nu)(R, \beta)$ means the value of $\partial u / \partial \nu(y)$ at the point y with polar coordinates (R, β).

Lemma 3.1. *We have*

$$(3.4) \qquad \int_0^{2\pi} \mathscr{K}(\alpha - \beta) \cos k\beta \, d\beta = \frac{R}{2k} \cos k\alpha \,,$$

$$(3.5) \qquad \int_0^{2\pi} \mathscr{K}(\alpha - \beta) \sin k\beta \, d\beta = \frac{R}{2k} \sin k\alpha$$

for $k = 1, 2, \ldots$ *and any* $0 \leqq \alpha < 2\pi$. *Moreover,*

$$(3.6) \qquad \int_0^{2\pi} \mathscr{K}(\alpha - \beta) \, d\beta = -R \log R$$

for any $0 \leqq \alpha < 2\pi$.

Proof. It can be done by a straightforward verification.

The above lemma describes the spectrum of an operator $\boldsymbol{K}$ which is (formally) defined as follows:

$$(\boldsymbol{K} v)(\alpha) \overset{\text{def}}{=} \int_0^{2\pi} \mathscr{K}(\alpha - \beta) \, v(\beta) \, d\beta \,.$$

Later, we will make clear on which spaces $\boldsymbol{K}$ acts.

Notation. We introduce

$$s_k = s_k(\alpha) = \sin k\alpha \,, \quad c_k = c_k(\alpha) = \cos k\alpha$$

(functions of α) and parameters

$$\lambda_k = \frac{R}{2k} \quad \text{for} \quad k = 1, 2, \ldots .$$

Let $H^r(\partial\Omega)$, r real, be the usual Sobolev space with the norm $\|\cdot\|_{r,\partial\Omega}$. The 1-$D$ manifold $\partial\Omega$ is isomorphic to the interval $[0, 2\pi)$, and it is well known that the norm $\|\cdot\|_{r,\partial\Omega}$ can be equivalently defined by means of the Fourier series:

Identifying $L_2(\partial\Omega)$ with its dual, let $\langle\cdot, \cdot\rangle$ be the pairing of $H^r(\partial\Omega)$ and $H^{-r}(\partial\Omega)$. If $v \in H^r(\partial\Omega)$ then

$$v = v_0 + \sum_{k=1}^{\infty} (v_k s_k + v_{-k} c_k), \quad \text{where} \quad v_0 = \frac{\langle v, 1\rangle}{2\pi R},$$

$$v_k = \frac{\langle v, s_k\rangle}{\pi R}, \quad v_{-k} = \frac{\langle v, c_k\rangle}{\pi R}.$$

We set

$$\|v\|^2_{r,\partial\Omega} \stackrel{\text{def}}{=} \pi R\Big[v_0^2 + \sum_{k=1}^{\infty} (1 + k)^{2r} (v_k^2 + v_{-k}^2)\Big].$$

We want the operator $\boldsymbol{K}$ to positive. Thus according to (3.6) the operator $\boldsymbol{K}$ should act on functions satisfying the condition $\int_0^{2\pi} v(\alpha)\, d\alpha = 0$. This motivates the following

Remark 3.1. Any solution u to (1.1) is determined up to a constant shift. Thus, we may assume the smooth solution u to satisfy

$$\int_0^{2\pi} u(R, \beta)\, d\beta = 0. \tag{3.7}$$

The same can be assumed about the incident wave w, i.e.

$$\int_0^{2\pi} w(R, \beta)\, d\beta = 0. \tag{3.8}$$

We define $\tilde{H}^r(\partial\Omega) = \{v \in H^r(\partial\Omega): \langle v, 1\rangle = 0\}$ for each real r, with the natural norm $|\cdot|_r$. If $v \in \tilde{H}^r(\partial\Omega)$ and $v = \sum_{k=1}^{\infty} (v_k s_k + v_{-k} c_k)$ is the relevant Fourier series then

$$|v|_r^2 \stackrel{\text{def}}{=} \pi R \sum_{k=1}^{\infty} k^{2r} (v_k^2 + v_{-k}^2).$$

By virtue of Lemma 3.1, we can specify our definition of the operator $\boldsymbol{K}$ on $\tilde{H}^r(\partial\Omega)$: Let $v \in \tilde{H}^r$ and let $v = \sum_{k=1}^{\infty} (v_k s_k + v_{-k} c_k)$ be its Fourier series. Then

$$\boldsymbol{K} v \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \lambda_k (v_k s_k + v_{-k} c_k) \tag{3.9}$$

and

$$\boldsymbol{K}^{-1} v \stackrel{\text{def}}{=} \sum_{k-1}^{\infty} \lambda_k^{-1} (v_k s_k + v_{-k} c_k). \tag{3.10}$$

Lemma 3.2. *The operators*

$$\boldsymbol{K}\colon \quad \tilde{H}^r(\partial\Omega) \to \tilde{H}^{r+1}(\partial\Omega),$$

$$\boldsymbol{K}^{-1}\colon \tilde{H}^r(\partial\Omega) \to \tilde{H}^{r-1}(\partial\Omega)$$

are bounded for each real r. Moreover, $\mathbf{K}\mathbf{K}^{-1} = \mathbf{K}^{-1}\mathbf{K} =$ *identity* (*on* $\tilde{H}^r(\partial\Omega)$).

Proof. The proof follows easily from the asymptotic properties of the eigenvalues λ_k.

Suppose u is a smooth solution. We easily observe that $\mathbf{A}u = 0$ on Ω. Thus, integrating by parts (Green's theorem),

$$0 = \int_\Omega \mathbf{A}u \, \mathrm{d}x = -R \int_0^{2\pi} \frac{\partial u}{\partial \nu}(R, \beta) \, \mathrm{d}\beta .$$

The same is true in the weak sense, see Lemma 3.3.

Notation. $$\mathfrak{a}(w, v) \overset{\text{def}}{=} \int_\Omega a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} \, \mathrm{d}x$$

for each w and v from $H^1(\Omega)$.

Remark 3.2. If $u \in H^1(\Omega)$ then the assumption $\mathbf{A}u = 0$ on Ω in the sense of distributions is equivalent to the condition

$$\mathfrak{a}(u, v) = 0 \quad \text{for each} \quad v \in H_0^1(\Omega). \tag{3.11}$$

It is well known (see e.g. [5]) that the operator $\partial/\partial\nu$ maps $H^1(\Omega)$ onto $H^{-1/2}(\partial\Omega)$ continuously. Assuming $u \in H^1(\Omega)$ and (3.11), we can equivalently define $\partial u/\partial\nu$ as follows:

$$\left\langle \frac{\partial u}{\partial \nu}, v \right\rangle = \mathfrak{a}(u, v) \quad \text{for each} \quad v \in H^1(\Omega) . \tag{3.12}$$

Lemma 3.3. *Let* $u \in H^1(\Omega)$ *satisfy* $\mathbf{A}u = 0$ *on* Ω *in the sense of distributions. Then* $\partial u/\partial\nu \in \tilde{H}^{-1/2}(\partial\Omega)$.

Proof. It suffices to take $v \equiv 1$ in (3.12). Q.E.D.

We shall proceed with the formulation of a weak solution to the problem (1.1).

Notation. $\mathscr{H} = \{u \in H^1(\Omega)$: the trace of u on $\partial\Omega$ belongs to $\tilde{H}^{1/2}(\partial\Omega)\}$.

Problem. *Let* $w \in \tilde{H}^{1/2}(\partial\Omega)$ *be given. Find* $u \in \mathscr{H}$ *such that*

$$\mathbf{A}u = 0 \quad on \quad \Omega \quad \textit{in the sense of distribution} , \tag{3.13}$$

and

$$\frac{1}{2} u + K \frac{\partial u}{\partial \nu} = w \quad on \quad \partial\Omega . \tag{3.14}$$

Theorem 3.1. *Let u be the smooth solution to* (1.1) *satisfying* (3.7). *Let w be the incident wave satisfying* (3.8). *Then u is a weak solution, i.e., u satisfies* (3.13), (3.14).

Proof. It is obvious that $u \in H^1(\Omega)$. By virtue of the assumption (3.7), the trace of u belongs to $\tilde{H}^{1/2}(\partial\Omega)$.

The restriction of (1.1) to Ω implies (3.11). As the boundary condition on $\partial\Omega$ is concerned, we have already shown that a smooth solution u satisfies (3.2) on $\partial\Omega$. Thus, taking into account (3.7), we have

$$(3.15) \qquad \frac{1}{2}u + \boldsymbol{K}\frac{\partial u}{\partial v} = w + c$$

on $\partial\Omega$, $c = u_\infty - w_\infty$. It suffices to show that $c = 0$ since then (3.13) implies (3.12).

Taking into account Lemma 3.3, we observe that $(\partial u/\partial v)\ (R, \cdot) \in \tilde{H}^{-1/2}(\partial\Omega)$. Then Lemma 3.2 implies that $\boldsymbol{K}\, \partial u/\partial v \in \tilde{H}^{1/2}(\partial\Omega)$. According to the assumption (3.8), the trace of w belongs to $\tilde{H}^{1/2}(\partial\Omega)$. Since $u \in \tilde{H}^{1/2}(\partial\Omega)$ ass well, (3.15) implies $c \in \tilde{H}^{1/2}(\partial\Omega)$, i.e. $c = 0$. Q.E.D.

Assuming u to be a weak solution, then, by virtue of Lemmas 3.3 and 3.2, we can write the boundary condition (3.14) as follows:

$$(3.16) \qquad -\frac{\partial u}{\partial v} = \frac{1}{2}\boldsymbol{K}^{-1}u - K^{-1}w\,.$$

Since (3.13) implies (3.12), we observe that u satisfies

$$(3.17) \qquad a(u, v) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1}u, v\rangle = \langle \boldsymbol{K}^{-1}w, v\rangle$$

for each $v \in \mathscr{H}$.

On the other hand, if $u \in \mathscr{H}$ and (3.17) holds for each $v \in \mathscr{H}$ then (3.11) is clearly satisfied, i.e. (3.13) is true. Then (3.17) and (3.12) imply that

$$\left\langle \frac{\partial u}{\partial v}, v \right\rangle = -\frac{1}{2}\langle \boldsymbol{K}^{-1}u, v\rangle + \langle \boldsymbol{K}^{-1}w, v\rangle$$

for each $v \in \mathscr{H}$. This statement means that (3.16) holds, i.e. (by virtue of Lemma 3.2 again), u satisfies (3.14).

We have proved the following

Remark 3.3. (Variational formulation.) u is a weak solution if and only if $u \in \mathscr{H}$ and (3.17) holds for each $v \in \mathscr{H}$.

Notation. $|v|_\Omega \overset{\text{def}}{=} \sqrt{(a(v, v))}$ for each $v \in \mathscr{H}$. Clearly, $|\cdot|_\Omega$ is an equivalent norm on $\mathscr{H}$.

Theorem 3.2. *The weak solution u uniquely exists for every choice of data.*

Proof. We verify the assumption of Lax-Milgram's theorem which is to be applied to (3.17).

a) The right hand side $\langle \boldsymbol{K}^{-1}w, \cdot\rangle$ is a linear bounded functional on $\mathscr{H}$, since $\boldsymbol{K}^{-1}w \in \tilde{H}^{-1/2}(\partial\Omega)$ (see Lemma 3.2) and $\mathscr{H}$ is continuously embedded into $\tilde{H}^{1/2}(\partial\Omega)$.

b) The bilinear form on the left hand side is elliptic in the sense that

$$\alpha(v, v) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1}v, v\rangle \geqq |v|_\Omega^2 \quad \text{for each} \quad v \in \mathscr{H}\,.$$

In fact, $\alpha(v, v) = |v|_\Omega^2$ and (see (3.10))

$$\langle \boldsymbol{K}^{-1}v, v\rangle = \sum_{k,m=1}^{\infty} \lambda_k^{-1}\langle v_k s_k + v_{-k}c_k, v_m s_m + v_{-m}c_m\rangle =$$

$$= \pi R \sum_{k=1}^{\infty} \lambda_k^{-1}(v_k^2 + v_{-k}^2) = \frac{2}{R}\,|v|_{1/2,\partial\Omega}^2 \geqq 0\,.$$

This completes the verification. Lax-Milgram's theorem implies the existence and uniqueness of u. Q.E.D.

Remark 3.4. Taking $v = u$ in (3.17), we easily obtain the following a priori bound of the weak solution u:

$$|u|_\Omega^2 + \frac{1}{R}\,|u|_{1/2,\partial\Omega}^2 \leqq \frac{2}{R}\,|w|_{1/2,\partial\Omega}\,|u|_{1/2,\partial\Omega}\,,$$

which implies

$$|u|_{1/2,\partial\Omega} \leqq 2|w|_{1/2,\partial\Omega}$$

and

$$|u|_\Omega \leqq 2R^{-1/2}|w|_{1/2,\partial\Omega}\,.$$

4. HINTS AT NUMERICAL SOLUTION

In this section we assume $\Omega \equiv \{x \in \mathbb{R}^2 : |x| < R\}$ again. Remark 3.3 offers the Ritz-Galerkin approximation to the solution u. Let S^h be a finite dimensional subspace of $H^1(\Omega)$; let S^h be spanned by a basis $\{\varphi_1, \ldots, \varphi_N\}$. Define $\tilde{S}^h = \{v \in S^h : \langle v, 1\rangle = 0\}$. The space $\tilde{S}^h$ is a natural approximation to $\mathscr{H}$.

We define $u^h \in \tilde{S}^h$ to be the Ritz approximation to u at $\tilde{S}^h$, i.e.

$$\alpha(u^h, \tilde{v}^h) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1}u^h, \tilde{v}^h\rangle = \langle \boldsymbol{K}^{-1}w, \tilde{v}^h\rangle \quad \text{for each} \quad \tilde{v}^h \in \tilde{S}^h\,.$$

One easily observes that $\alpha(u^h, 1) = \langle \boldsymbol{K}^{-1}u^h, 1\rangle = \langle \boldsymbol{K}^{-1}w, 1\rangle = 0$ and $S^h = \mathbb{R}^1 \oplus \tilde{S}^h$. It means that u^h can be defined equivalently as follows:

Find $u^h \in \tilde{S}^h$ such that

$$\text{(4.1)} \qquad \begin{cases} \langle u^h, 1\rangle = 0 \quad \text{and} \\ \alpha(u^h, v^h) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1}u^h, v^h\rangle = \langle \boldsymbol{K}^{-1}w, v^h\rangle \\ \text{for each} \quad v^h \in S^h. \end{cases}$$

We extend $\boldsymbol{K}^{-1}$ to S^h: If $v \in S^h$, let

$$v = v_0 + \sum_{k=1}^{\infty}(v_k s_k + v_{-k}c_k) \quad \text{on} \quad \partial\Omega\,.$$

We define

$$\boldsymbol{K}^{-1}v \overset{\text{def}}{=} \boldsymbol{K}^{-1}(v - v_0) = \frac{2}{R}\sum_{k=1}^{\infty} k(v_k s_k + v_{-k}c_k).$$

Then u^h solves (4.1) if and only if

$$u^h = \sum_{i=1}^{N} \alpha_i \varphi_i ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)^{\mathsf{T}} \in \mathbb{R}^N$ satisfies

$$\sum_{i=1}^{N} \alpha_i \langle \varphi_i, 1 \rangle = 0 \tag{4.2}$$

and

$$\boldsymbol{B}\alpha + \boldsymbol{M}\alpha = \boldsymbol{f} ; \tag{4.3}$$

$$\begin{aligned}
\boldsymbol{B} &\overset{\text{def}}{=} \{b_{ij}\}_{i,j=1,\ldots,N} , & b_{ij} &= a(\varphi_j, \varphi_i) ,\\
\boldsymbol{f} &\overset{\text{def}}{=} (f_1, \ldots, f_N)^{\mathsf{T}} , & f_i &= \langle K^{-1}w, \varphi_i \rangle ,\\
\boldsymbol{M} &\overset{\text{def}}{=} \{m_{ij}\}_{i,j=1,\ldots,N} , & m_{ij} &= \tfrac{1}{2}\langle K^{-1}\varphi_j, \varphi_i \rangle .
\end{aligned}$$

The matrix $\boldsymbol{M}$ is symmetric. Its entries are calculated from the definition of $\boldsymbol{K}^{-1}$.

Let
$$\varphi_j|_{\partial\Omega} = v_0^j + \sum_{k=1}^{\infty} (v_k^j s_k + v_{-k}^j c_k)$$

be the Fourier expansion of φ_j on $\partial\Omega$. Then

$$\langle \boldsymbol{K}^{-1}\varphi_j, \varphi_i \rangle = 2\pi \sum_{k=1}^{\infty} k(v_k^j v_k^i + v_{-k}^j v_{-k}^i) .$$

In the actual computation, the infinite sum should be replaced by a finite approximation. The same problem arises in the computation of the entries of $\boldsymbol{f}$.

5. EXTENSIONS

The operator $\boldsymbol{A}$ *inside* Ω can be nonlinear, e.g.

$$\boldsymbol{A}u \equiv -\frac{\partial}{\partial x_i} a_{ij}(x, u, \nabla u) \frac{\partial u}{\partial x_j} \quad \text{for} \quad x \in \Omega$$

($\boldsymbol{A}u = -\Delta u$ outside Ω, of course). Then the boundary condition (2.10) will not be affected. Ine case of a circular Ω, Theorem 3.1 remains true. Moreover, assuming a suitable concept of monotonicity (e.g. the strong monotonicity of $\boldsymbol{A}$: $H^1(\Omega) \to \to (H^1(\Omega))'$), one can prove Theorem 3.2. as well.

The analogous problem in three dimensions can be treated similarly. For example, if Ω is a ball $\{x \in \mathbb{R}^3 : |x| < R\}$ the boundary condition on $\partial\Omega$ reads as follows:

$$w = \frac{1}{2} u + \boldsymbol{K} \frac{\partial u}{\partial \nu} + \frac{1}{2R} \boldsymbol{K} u \,,$$

where w is the trace of the incident wave on $\partial\Omega$ and $\boldsymbol{K}$ is an integral operator. In the spherical coordinates (r, α, ϑ):

$$(\boldsymbol{K}v)(R, \alpha, \vartheta) \overset{\text{def}}{=} \frac{R}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{v(R, \alpha', \vartheta') \sin \alpha' \, d\alpha' \, d\vartheta'}{\sqrt{(2(1 - \sin \alpha \sin \alpha' \cos (\vartheta - \vartheta') - \cos \alpha \cos \alpha'))}} \,.$$

Both the analysis and the numerical treatment rely on the spectral properties of the operator $\boldsymbol{K}$. Namely

$$\boldsymbol{K}v = \frac{R}{2k + 1} v$$

where $v = Y_k(\alpha, \vartheta)$ is any spherical function of order k, $k = 0, 1, \ldots$. Details are to appear in a forthcoming paper.

References

[1] *V. Janovský, I. Marek, J. Neuberg:* Maxwell's equations with incident wave as a field source. Technical Report KNM-0105057/81, Charles University, Prague 1981.

[2] *V. Janovský, I. Marek, J. Neuberg:* Maxwell's equations with incident wave as a field source. Proceedings of Equadiff 5, Teubner-Texte zur Mathematik, Leipzig 1982, 237—240.

[3] *A. N. Tichonov, A. A. Samarskij:* Equations of Mathematical Physics (in Russian), Nauka, Moscow 1977.

[4] *M. Brelot:* Lectures on Potential Theory. Tata Institute of Fundamental Research, Bombay 1960.

[5] *J. Nečas:* Les Méthodes Directes en Théorie des Equations Elliptiques, Academia, Prague 1967.

[6] *C. Johnson, J. C. Nedelec:* On the coupling of boundary integral and finite elements methods. Math. Comp. 35 (1980), 1063—1079.

Souhrn

O JEDNÉ ÚLOZE TEORIE POTENCIÁLU SE ZADANOU DOPADAJÍCÍ VLNOU

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

Vyšetřuje se dvoudimenzionální model difrakce elektrostatického pole na omezeném nehomogenním tělese. Je formulována nelokální okrajová podmínka, umožňující řešení problému na omezené fiktivní oblasti (např. kruhu), která obsahuje zadanou nehomogenitu. Takto redukovaná úloha je aproximována metodou konečných prvků v kombinaci s Fourierovou metodou na hranici fiktivní oblasti.

Резюме

ОБ ОДНОЙ ПРОБЛЕМЕ ТЕОРИИ ПОТЕНЦИАЛА С ЗАДАННОЙ ПАДАЮЩЕЙ ВОЛНОЙ

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

Рассматривается двумерная задача дифракции электростатического поля в заданной (ограниченной) неоднородной среде. Формулируется интегральное граничное условие, при помощи которого задача корректно определяется внутри ограниченной фиктивной области (круга). Задача аппроксимируется методом конечных элеменков внутри и методом разложения Фурье на границе фиктивной области.

Authors' address: RNDr. *Vladimír Drápalík*, doc. RNDr. *Vladimír Janovský*, CSc., MFF UK, Malostranské nám. 25, 118 00 Praha 1.