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TIME-PERIODIC SOLUTIONS OF A QUASILINEAR BEAM EQUATION VIA ACCELERATED CONVERGENCE METHODS

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Summary. The author investigates time-periodic solutions of the quasilinear beam equation with the help of accelerated convergence methods. Using the Newton iteration scheme, the problem is approximated by a sequence of linear equations solved via the Galerkin method. The derivative loss inherent to this kind of problems is compensated by taking advantage of smoothing operators.

Provided that the right-hand side of the equation is small and smooth, the existence of at least one solution is established.

Keywords: quasilinear beam equation, periodic solutions, accelerated convergence method.

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In his Thesis [8], M. Štědrý succeeded in proving the existence of at least one periodic solution to beam equations involving both damping terms and a small nonlinear right-hand side. Being achieved with the help of the abstract Moser theorem, his results cover many important cases except the situation when all "space" derivatives of the unknown function occur in the nonlinearity mentioned above. Consequently, no information is gained concerning e.g. quasilinear equations in spite of their frequent appearance in the so called physically nonlinear elasticity.

The forced transversal vibrations of a damped beam with simply supported ends can be modelled by the equation

$$\text{(E)} \qquad \begin{gathered} u_{tt} + \alpha(u_t) + \sigma(u_{xx})_{xx} = g \\ u = u(x, t), \quad x \in (0, l), \quad t \in \mathbb{R} \end{gathered}$$

with the boundary conditions

$$\text{(B)} \qquad u(0, t) = u(l, t) = u_{xx}(0, t) = u_{xx}(l, t) = 0, \quad t \in \mathbb{R}$$

and with the periodicity condition

$$\text{(P)} \qquad \begin{gathered} u(x, t + \omega) = u(x, t) \\ x \in (0, l), \quad t \in \mathbb{R}. \end{gathered}$$

In this paper, existence of at least one solution to the problem just outlined will be established if, roughly speaking, the function g is small and smooth. For a more precise formulation we refer to Section 2, Theorem 2.1.

The generalized Newton iteration scheme our proof leans on requires to solve but the linearized equations related to (E). To this end, we have to restrict ourselves to functions α, σ which are smooth on an open neighbourhood of $0 \in \mathbb{R}$ and satisfy

$$\alpha(0) = \sigma(0) = 0\,, \quad \alpha'(0) = d > 0\,, \quad \sigma'(0) = a > 0\,. \tag{0.1}$$

If quasilinear equations are involved, there are other methods to solve them, see e.g. [4]. Our approach seems to have the benefit of allowing to add another nonlinear term to (E) containing e.g. all derivatives of u up to the order 2. Speaking about a beam equation, we should mention the work [6] of H. Petzeltová for completeness.

To agree upon notation, let the symbol $\mathbb{R}$ stand for the set of all reals, while $\mathbb{Z}$ is the set of all integers, and $\mathbb{Z}^+$ denotes its positive part including 0. Throughout the paper, the symbols $c(v)$ are used to denote strictly positive constants depending on the quantity v only.

1. Function spaces. Following [7], we determine the spaces in which the solution is to be looked for. Seeing that the concrete values of l, ω do not matter, we may put $\omega = 2\pi$, $l = \pi$, and $Q = (0, \pi) \times (0, 2\pi)$.

To begin with, the symbols $L_p = L_p(Q)$ are reserved for the spaces of integrable functions with the norm $\| \; \|_{L_p}$ defined in the standard way.

Next, we consider anisotropic Sobolev spaces of periodic functions $H^{k,j}$, $\{\dot{H}^{k,j}\}$ determined as the completion of all smooth (real-valued) functions satisfying (P), {(P), (B)} with respect to the norm

$$\|v\|_{H^{k,j}} = \max\left(\{\|\partial_x^l v\|_{L_2} \mid l = 0, 1, \ldots, k\}, \{\|\partial_t^i v\|_{L_2} \mid i = 0, 1, \ldots, j\}\right)$$

(see [9]).

Finally, the most important spaces are U_n, $n \in \mathbb{Z}^+$,

$$U_n = \{v \mid \partial_t^j v \in H^{4,2},\ j = 0, 1, \ldots, n\}\,,$$
$$\dot{U}_n = U_n \cap \dot{H}^{4,2}$$

with norms

$$\|v\|_{U_n} = \max\{\|\partial_t^j v\|_{H^{4,2}} \mid j = 0, 1, \ldots, n\}\,.$$

In a similar way, we set

$$I_n = \{v \mid \partial_t^j v \in H^{2,2},\ j = 0,1, \ldots, n\}\,,$$
$$\|v\|_{I_n} = \max\{\|\partial_t^j v\|_{H^{2,2}}|,\ j = 0, 1, \ldots, n\}\,,$$
$$G_n = \{v \mid \partial_t^j v \in L_2,\ j = 0, 1, \ldots, n\}\,,$$
$$\|v\|_{G_n} = \max\{\|\partial_t^j v\|_{L_2}|,\ j = 0, 1, \ldots, n\}\,.$$

Being linearized, the inverse operator related to (E) loses derivatives, which is an inherent difficulty with hyperbolic problems. To overcome it, smoothing operators are needed.

Every $v \in H^{0,0}$ can be expressed in the form

$$v = \sum_{i=\infty}^{\infty} v_i(x)\, z_i(t)\,, \quad v_i = \int_0^{2\pi} v(x, t)\, z_i(t)\, dt$$

where

$$z_i = 1/\sqrt{(\pi)} \sin(it)\,, \quad i > 0$$
$$1/\sqrt{(2\pi)}\,, \quad i = 0$$
$$1/\sqrt{(\pi)} \cos(it)\,, \quad i < 0\,, \quad i \in \mathbb{Z}\,.$$

For a fixed number $r > 1$, the sequence of linear operators $S_n = S_n(r)$, $n \in \mathbb{Z}^+$ is defined as

$$S_n v = \sum_{i \leqq r^n} v_i(x)\, z_i(t)\,. \tag{1.1}$$

It is a matter of routine (cf. [7]) to prove

$$\|S_n v - v\|_{X_k} \leqq r^{-nl} \|v\|_{X_{k+l}}\,, \tag{1.2}$$

$$\|S_n v\|_{X_{k+l}} \leqq r^{nl} \|v\|_{X_k}\,, \quad l, k \in \mathbb{Z}^+ \tag{1.3}$$

where U, I, or G can be inserted instead of X.

Pursuing Hörmander's work [1], we draw from (1.2), (1.3) the interpolation inequality

$$\|v\|_{X_n} \leqq c(k, l, n)\, \|v\|_{X_k}^{\lambda}\, \|v\|_{Xl}^{1-\lambda}\,, \tag{1.4}$$

$$n = \lambda k + (1 - \lambda)\, l\,, \quad n, k, l \in \mathbb{Z}^+\,, \quad \lambda \in [0, 1]\,.$$

Combining (1.4) with the well known relations $ab \leqq a^p/p + b^q/q$, $1/p + 1/q = 1$, we conclude (cf. [3])

$$\|v_1\|_{X_{n2}} \|v_2\|_{Y_{m2}} \leqq c(n_1 + m_1)\left(\|v_1\|_{X_{n1}} \|v_2\|_{Y_{m1}} + \|v_1\|_{X_{n3}} \|v_2\|_{Y_{m3}}\right), \tag{1.5}$$

$$n_i, m_i \in \mathbb{Z}^+\,, \quad n_1 + m_1 = n_2 + m_2 = n_3 + m_3\,, \quad i = 1, 2, 3\,,$$

$$n_1 \leqq n_2 \leqq n_3$$

where the symbols X, Y stand for U, I, or G.

2. MAIN RESULT

Theorem 2.1. *Let an integer $M \geqq 10$ be given.*

If (0.1) holds, we are able to find $\varepsilon = \varepsilon(M) > 0$ such that the problem (**E**), (**B**), (**P**) *possesses at least one solution* $u \in \dot{U}_{[M/2]}$ *(in the sense of generalized derivatives) whenever* $g \in G_M$ *and*

$$\|g\|_{G_M} < \varepsilon . \tag{2.1}$$

If, moreover, g is continuous, then the solution u is a classical one.

Remark. $[M/2]$ denotes the greatest integer less or equal to $M/2$.

The main tool to prove the above theorem is the generalized Newton iteration scheme which will be introduced in Section 3. In connection with differential equations, this method is often called Nash's or Moser's method, see e.g. [5].

To solve linear equations, some auxiliary results formulated in Section 4 and proved in Sections 6, 7 are of interest. The proof of Theorem 2.1 will be carried out in Section 5.

3. ITERATION SCHEME

We proceed analogously as in [2], [3].

Let us put

$$F(u) = u_{tt} + \alpha(u_t) + \sigma(u_{xx})_{xx} ,$$

$$F'(u)\, y = y_{tt} + \alpha'(u_t)\, y_t + (\sigma'(u_{xx})\, y_{xx})_{xx} .$$

The following sequence of linear equations (for y_n) is to be solved:

$$F'(S_n u_n)\, y_n = h_n , \quad n \in \mathbb{Z}^+ \tag{3.1$_n$}$$

where the function u_0 is determined by the equation

$$F'(0)\, u_0 = g . \tag{3.1$_{-1}$}$$

Our aim is to obtain the solution u as a limit of functions u_n, $u_n \to u$ where u_n are given successively as

$$u_n = u_0 + \sum_{k=0}^{n-1} y_k .$$

The only terms we are to pick out are the functions h_n. Following the Taylor expansion formula, we can express

$$F(u_{n+1}) = F(u_n) + F'(u_n)\, y_n + e^1_{n+1} . \tag{3.2}$$

Setting

$$e^2_{n+1} = (F'(u_n) - F'(S_n u_n))\, y_n \tag{3.3}$$

we get by induction

$$F(u_{n+1}) = F(u_0) + \sum_{k=0}^{n} h_k + \sum_{k=1}^{n+1} e_k , \tag{3.4}$$

$$e_k = e^1_k + e^2_k .$$

To accomplish $F(u_n) \to g$, we put

$$(3.5) \qquad S_n(g - F(u_0)) = \sum_{k=0}^{n} h_k + S_n \sum_{k=1}^{n} e_k .$$

Accepting the convention $e_0 = F(u_0) - g$, we easily derive

$$(3.6) \qquad h_n = -S_n e_n - (S_n - S_{n-1}) \sum_{k=0}^{n-1} e_k .$$

Combining (3.4) with (3.5), we conclude that

$$(3.7) \qquad F(u_{n+1}) = g + e_{n+1} + (id - S_n) \sum_{k=0}^{n} e_k .$$

4. AUXILIARY RESULTS

To succeed in solving $(3.1)_n$, the following assertion is of interest.

Proposition 4.1. *Given a fixed number $N \in \mathbb{Z}^+$, then $\delta_1(N) > 0$ can be found in such a way that there is a unique solution $y \in \dot{U}_N$ to the equation*

$$F'(u)\, y = h$$

provided that

$$(4.1) \qquad \|u\|_{U_4} < \delta_1 , \quad u \in \bigcap_{n=0}^{\infty} U_n .$$

Moreover, we have the estimates

$$(4.2) \qquad \|y\|_{U_l} \leqq c(l) \left(\|h\|_{G_{l+1}} + \|u\|_{U_{l+4}} \|h\|_{G_1} \right) l = 0, 1, \ldots, N .$$

Remark. According to embedding theorems (see [9]) $U_4 \subsetneq C^2(\bar{Q})$. Consequently, δ_1 being chosen close to 0, $F'(u)\, y$ is defined and the equation is satisfied in the sense of generalized derivatives.

We postpone the proof to Section 6.

The next proposition enables us to estimate the norm of the nonlinear terms appearing in Section 3.

Proposition 4.2. *Let us consider a smooth function ϱ, $\varrho: (-\mu, \mu) \to \mathbb{R}$. The embedding $H^{2,2} \subsetneq C(\bar{Q})$ being taken into account, the composition $\varrho(v(x, t))$ makes sense provided that*

$$(4.3) \qquad \|v\|_{I_0} < \delta_2 ,$$

$\delta_2 > 0$ being sufficiently small.

Under these circumstances, we have the estimate

$$(4.4) \qquad \|\varrho(v)\|_{I_l} \leqq c(l, \varrho) \left(\|\varrho(v)\|_{L_2} + \|v\|_{I_l} \right)$$

for any $v \in I_l$, $l \in \mathbb{Z}^+$.

Remark. If I_l were ordinary Sobolev spaces, the theorem above would coincide with that presented by Moser in [5].

For the proof we refer to Section 7.

Taking advantage of Proposition 4.2, we will estimate the quantities e_n. To begin with, we assume

$$\|u_n\|_{U_2} + \|y_n\|_{U_2} < \delta_3 , \tag{4.5}$$

$\delta_3 > 0$ small enough, for we want all terms involved to be well defined. Observe that the space $H^{2,2}$ is a Banach algebra, which seems to be crucial in what follows.

Let us, for instance, treat the most difficult term f appearing in e^1_{n+1},

$$f = \partial^2_x \int_0^1 (1 - s)\, \sigma''(\partial^2_x(u_n + s y_n))\, (\partial^2_x y_n)^2 \,\mathrm{d}s .$$

For $0 \leqq j \leqq l$ we get

$$\|\partial^j_t f\|_{L_2} \leqq c(l) \sum_{j1+j2+j3=j} \|\partial^2_x y_n\|_{I_{j1}} \|\partial^2_x y_n\|_{I_{j2}} \int_0^1 \|\sigma''(\partial^2_x(u_n + s y_n))\|_{I_{j3}} \,\mathrm{d}s .$$

Using (4.4), we conclude

$$\|f\|_{G_l} \leqq c(l) \sum_{l1+l2+l3=l} (1 + \|u_n\|_{U_{l1+2}} + \|y_n\|_{U_{l1+2}})\, \|y_n\|_{U_{l2+2}} \|y_n\|_{U_{l3+2}} .$$

Repeating this procedure, we obtain the estimates

$$\|e_0\|_{G_l} \leqq c(l) \sum_{l1+l2+l3=l} (1 + \|u_0\|_{U_{l1+2}})\, \|u_0\|_{U_{l2+2}} \|u_0\|_{U_{l3+2}} , \tag{4.6}$$

$$\|e^l_{n+l}\|_{G_l} \leqq c(l) \sum_{l1+l2+l3=l} (1 + \|u_n\|_{U_{l1+2}} + \|y_n\|_{U_{l1+2}})\, \|y_n\|_{U_{l2+2}} \|y_n\|_{U_{l3+2}} , \tag{4.7}$$

$$\|e^2_{n+1}\|_{G_l} \leqq c(l) \sum_{l1+l2+l3=l} (1 + \|u_n\|_{U_{l1+2}})\, \|y_n\|_{U_{l2+2}} \|u_n - S_n u_n\|_{U_{l3+2}} \tag{4.8}$$

where $l \in \mathbb{Z}^+$, $u_n, y_n \in U_{l+2}$.

5. PROOF OF THEOREM 2.1.

First of all, we put $N = M - 1$ where M appears in Theorem 2.1. Further, a number β is chosen such that $2\beta \in (N + 1, N + 2)$. Consequently, $\beta > [M/2] \geqq$ $\geqq 5$. Finally, let the relation (2.1) hold for some $\varepsilon > 0$.

In view of Proposition 4.1, there exists a (unique) solution u_0 of $(3.1)_{-1}$ satisfying

$$\|u_0\|_{U_l} \leqq c(l)\, \|g\|_{G_{l+1}},\; l \leqq N,\; l \in \mathbb{Z}^+ . \tag{5.1}$$

In order to solve $(3.1)_n$, $n \in \mathbb{Z}^+$, we require both the relation (4.1) for $S_n u_n$ and (4.5) for $m = 0, 1, \ldots, n - 1$ where we have set $u_{-1} = u_0$, $y_{-1} = 0$. To fulfil that, we have to keep u_m, y_{m-1} small, more precisely,

$$(5.2)_n \qquad \|u_m\|_{U_4} + \|y_{m-1}\|_{U_4} \leqq \delta_4 \quad \text{for all} \quad m = 0, 1, \ldots, n$$

is required where $\delta_4 > 0$ is sufficiently small.

Observe that $(5.2)_0$ holds provided the number $\varepsilon > 0$ has been chosen small enough. According to Proposition 4.1, $(3.1)_0$ is solvable and the solution y_0 satisfies

$$\|y_0\|_{U_l} \leqq c(l)\left(\|h_0\|_{G_{l+1}} + \|S_0 u_0\|_{U_{l+4}} \|h_0\|_{G_l}\right), \quad l \leqq N .$$

For an arbitrarily chosen $\delta > 0$, we are able to find $\varepsilon > 0$, $\varepsilon \leqq \delta$ such that

$$(5.3)_0 \qquad \|y_0\|_{U_l} \leqq \delta \quad \text{for all} \quad l = 0, 1, \ldots, N$$

and ε appearing in (2.1). To see that, we combine (4.6), (5.1) with the above estimate.

Following [3], our goal is to choose $\delta > 0$ so small that all equations $(3.1)_n$ may be solvable and

$$(5.3)_n \qquad \|y_n\|_{U_l} \leqq \delta r^{(l-\beta)n}$$

may hold for all $l = 0, 1, \ldots, N$.

At this stage, we proceed by induction. We intend to prove $(5.3)_{n+1}$ having already shown $(5.3)_m$ for all $m = 0, 1, \ldots, n$.

To this end, we estimate u_{m+1} as follows:

$$\|u_{m+1}\|_{U_l} \leqq \|u_0\|_{U_l} + \sum_{k=0}^{m} \|y_k\|_{U_l}$$

(according to (5.1), (5.3))

$$\leqq \delta\, c(l)\left(1 + \sum_{k=0}^{m} r^{(l-\beta)k}\right).$$

Summing up the series on the right-hand side, we deduce

$$(5.4) \qquad \|u_{m+1}\|_{U_l} \leqq \delta\, c(l)\left(1 + r^{(l-\beta)(m+1)}\right) \quad \text{for all} \quad l \leqq N, \quad m \leqq n .$$

Consequently, $(5.2)_{n+1}$ is satisfied and we are able to solve $(3.1)_{n+1}$.

The unique solution y_{n+1} fulfils

$$\|y_{n+1}\|_{U_l} \leqq c(l)\left(\|h_{n+1}\|_{G_{l+1}} + \|S_{n+1} u_{n+1}\|_{U_{l+4}} \|h_{n+1}\|_{G_1}\right).$$

To accomplish $(5.3)_{n+1}$, we are to treat but the term $\|h_{n+1}\|_{G_l}$, namely, we are going to prove

$$(5.5) \qquad \|h_{n+1}\|_{G_l} \leqq \delta^2\, c(l)\, r^{(l-2\beta+4)(n+1)}, \quad l \in \mathbb{Z}^+ .$$

Indeed, we can pick out $\delta > 0$ such that $\delta\, c(l) \leqq 1$ for all $l \leqq N$. Since

$$\|S_{n+1} u_{n+1}\|_{U_{l+4}} \leqq r^{l(n+1)} \delta_4 ,$$

we get $\|y_{n+1}\|_{U_l} \leqq \delta r^{(l+5-2\beta)(n+1)}$.

Since $\beta > 5$, $(5.3)_{n+1}$ follows.

We focus our attention upon (5.5). To estimate e_{m+1}, we derive

$$\|(\mathrm{id} - S_{m+1})\, u_{m+1}\|_{U_l} \leqq r^{(l-N)(m+1)} \|u_{m+1}\|_{U_N} .$$

In accordance with (5.4), we have

$$\|(id - S_{m+1})\, u_{m+1}\|_{U_l} \leqq \delta\, c(l)\, r^{(l-\beta)(m+1)} \quad \text{for all} \quad l \leqq N\,, \quad m \leqq n\,. \tag{5.6}$$

Combining (4.6)–(4.8) with (5.4)–(5.6) we obtain

$$\|e_{m+1}\|_{G_l} \leqq \delta^2\, c(l)\, r^{(l-2\beta+4)(m+1)} \quad \text{for all} \quad l \leqq N-2\,, \quad m \leqq n\,. \tag{5.7}$$

Reasoning as in (5.4) we get

$$\Big\|\sum_{k=0}^{m} e_k\Big\|_{G_l} \leqq \delta^2\, c(l)\, \big(1 + r^{(l-2\beta+4)(m+1)}\big)\,, \quad l \leqq N-2\,. \tag{5.8}$$

Due to $N + 2 > 2\beta$, we obtain similarly as in (5.6)

$$\Big\|(\mathrm{id} - S_j) \sum_{k=0}^{n} e_k\Big\|_{G_l} \leqq \delta^2\, c(l)\, r^{(l-2\beta+4)(n+1)} \tag{5.9}$$

$$\text{for all} \quad l \leqq N-2\,, \quad j = n\,, \quad n+1\,.$$

The hardest term to estimate appearing in h_{n+1} is

$$\Big\|(S_{n+1} - S_n) \sum_{k=0}^{n} e_k\Big\|_{G_l} \leqq \delta^2\, c(l)\, r^{(l-2\beta+4)(n+1)} \tag{5.10}$$

where l is supposed to satisfy $l \leqq N - 2$.

On the other hand, if $l > N - 2$, we have

$$\Big\|(S_{n+1} - S_n) \sum_{k=0}^{n} e_k\Big\|_{G_l} \leqq \Big\|S_{n+1} \sum_{k=0}^{n} e_k\Big\|_{G_l} + \Big\|S_n \sum_{k=0}^{n} e_k\Big\|_{G_l} .$$

According to (1.3), (5.8), one has (5.10) even for $l > N - 2$. Consequently, (5.5) has been proved.

Solving the equations $(3.1)_n$, $n \in \mathbb{Z}^+$, we obtain the sequence $\{u_n\}_{n=0}^{\infty}$. Moreover, according to (5.3), this sequence admits a limit $u \in \dot{U}_{[M/2]}$ and, a fortiori, $u_n \to u$ in $\dot{U}_5$. Using (3.7), we get the estimate

$$\|F(u_{n+1}) - g\|_{G_0} \leqq \|e_{n+1}\|_{G_0} + \Big\|(\mathrm{id} - S_n) \sum_{k=0}^{n} e_k\Big\|_{G_0} \leqq$$

(according to (5.8), (5.9))

$$\leqq \delta^2 c r^{(-2\beta+4)(n+1)} .$$

Consequently, $F(u_{n+1}) \to g$ in G_0.

On the other hand, $U_5 \subsetneq H^{4,7}$. In view of embedding theorems (see [9]), all derivatives of u_n up to the order 3 are continuous and converge to the corresponding derivatives of u in $C(\bar{Q})$. Thus $F(u_{n+1}) \to F(u)$ in G_0, and we get

$$F(u) = g\,. \tag{5.11}$$

If $g \in C(\bar{Q})$, then $u_{xxxx} \in C(\bar{Q})$ due to (5.11). Theorem 2.1 has been proved.

6. Linear equations. In this section, our aim is to prove Proposition 4.1. The problem consists in solving the linear equation

$$\text{(L)} \qquad Ly + \big(b(x,t)\, y_{xx}\big)_{xx} + w(x,t)\, y_t = h$$

where a solution y is to satisfy the conditions (B), (P). We have denoted $Ly = y_{tt} + dy_t + ay_{xxxx}$, $b = \sigma'(u_{xx}) - a$, $w = \alpha'(u_t) - d$. According to (4.1), we have

$$b, w \in \bigcap_{n=0}^{\infty} I_n .$$

To begin with, we claim that there is an equivalent norm on $\dot{H}^{4,2}$ given by

$$\text{(6.1)} \qquad \|v\|_{H^{4,2}} = \max\left\{\|\partial_x^4 v\|_{L_2}, \|\partial_t^2 v\|_{L_2}\right\}$$

(see [9]).

Seeing that the standard Galerkin method is applicable to our situation, we are going to derive *a priori estimates* only. For this purpose, we are allowed to assume y to be smooth.

We put (cf. [6])

$$\Theta(y) = dy + 2y_t - dy_{tt} - 2y_{ttt} + \frac{da}{2}\, y_{xxxx} .$$

For $l \in \mathbb{Z}^+$ we set $\Xi(y, l) = \Theta((-1)^l\, \partial_t^{2l} y)$.

Let us multiply the equation (L) successively by $\Xi(y, l)$ and integrate over Q. Using integration by parts, the following relations are obtained:

$$\text{(6.2)} \qquad \int_Q Ly\, \Xi(y,l) = d\|\partial_t^{l+1} y\|_{L_2}^2 + d\|\partial_t^{l+2} y\|_{L_2}^2 + da\|\partial_x^2\, \partial_t^l y\|_{L_2}^2 + \frac{da}{2}\|\partial_x^2\, \partial_t^{l+1} y\|_{L_2}^2 + \frac{da^2}{2}\|\partial_x^4\, \partial_t^l y\|_{L_2}^2 ,$$

$$\text{(6.3)} \qquad \left|\int_Q h\, \Xi(y,l)\right| \leqq c(d,a)\, \|h\|_{G_{l+1}}\, \|y\|_{U_l} .$$

Next, we are going to show that

$$\text{(6.4)} \qquad \left|\int_Q w y_t\, \Xi(y,l)\right| \leqq c(l) \sum_{k=0}^{l} \|w\|_{I_{l+1-k}}\, \|y\|_{U_k}\, \|y\|_{U_l} .$$

The most difficult term is A,

$$A = \int_Q w y_t\, \partial_t^{2l+3} y = (-1)^{l+1} \sum_{k=0}^{l+1} \int_Q \binom{l+1}{k}\, \partial_t^{l+1-k} w\, \partial_t^{k+1} y\, \partial_t^{l+2} y .$$

With the help of the Hölder inequality, we get

$$|A| \leqq c(l) \sum_{k=0}^{l} \|\partial_t^{l+1-k} w\|_{C(\bar{Q})}\, \|y\|_{U_k}\, \|y\|_{U_l} + \|w\|_{C(\bar{Q})}\, \|y\|_{U_l}^2 .$$

Owing to $H^{2,2} \subsetneq C(\bar{Q})$, (6.4) follows.

Finally, we estimate

$$(6.5) \qquad \left|\int_Q (by_{xx})_{xx}\, \Xi(y, l)\right| \leqq c(l) \sum_{k=0}^{l} \|b\|_{I_{l+2-k}} \|y\|_{U_k} \|y\|_{U_l}\,.$$

The hardest terms seem to be

$$B = \int_Q (by_{xx})_{xx}\, \partial_t^{2l+3} y\,, \quad D = \int_Q (by_{xx})_{xx}\, \partial_t^{2l} y_{xxxx}\,.$$

To cope with B, we use the embedding relation (see [9])

$$(6.6) \qquad \|v_{xxt}\|_{L_2} \leqq c\|v\|_{H^{4,2}}\,.$$

We have

$$B = (-1)^{l+2} \sum_{k=0}^{l+2} \int_Q \binom{l+2}{k} \partial_t^{l+2-k} b\, \partial_t^k y_{xx}\, \partial_t^l y_{xxt}$$

and, consequently,

$$|B| \leqq c(l) \sum_{k=0}^{l} \|\partial_t^{l+2-k} b\|_{C(\bar{Q})} \|y\|_{U_k} \|y\|_{U_l} + \|\partial_t b\|_{C(\bar{Q})} \|y\|_{U_l}^2\,.$$

As to the term D, we use the estimates (see [9])

$$(6.7) \qquad \|v_{xxx}\|_{L_4} \leqq c\|v\|_{H^{4,2}}\,,$$

$$(6.8) \qquad \|v_{xx}\|_{C(\bar{Q})} \leqq c\|v\|_{H^{4,2}}\,.$$

We can decompose

$$D = \int_Q b_{xx} y_{xx}\, \partial_t^{2l} y_{xxxx} + 2b_x y_{xxx}\, \partial_t^{2l} y_{xxxx} + b y_{xxxx}\, \partial_t^{2l} y_{xxxx}\,.$$

In view of (6.8), we deduce

$$D_1 = \int_Q b_{xx} y_{xx}\, \partial_t^{2l} y_{xxxx} = (-1)^l \sum_{k=0}^{l} \int_Q \binom{l}{k} \partial_t^{l-k} b_{xx}\, \partial_t^k y_{xx}\, \partial_t^l y_{xxxx}\,,$$

$$|D_1| \leqq c(l) \sum_{k=0}^{l} \|\partial_t^{l-k} b_{xx}\|_{L_2} \|\partial_t^k y_{xx}\|_{C(\bar{Q})} \|\partial_t^l y_{xxxx}\|_{L_2} \leqq c(l) \sum_{k=0}^{l} \|b\|_{I_{l-k}} \|y\|_{U_k} \|y\|_{U_l}\,.$$

The second term is treated in the following way:

$$D_2 = \int_Q b_x y_{xxx}\, \partial_t^{2l} y_{xxxx} = (-1)^l \sum_{k=0}^{l} \int_Q \binom{l}{k} \partial_t^{l-k} b_x\, \partial_t^k y_{xxx}\, \partial_t^l y_{xxxx}\,,$$

$$|D_2| \leqq c(l) \sum_{k=0}^{l} \|\partial_t^{l-k} b_x\|_{L_4} \|\partial_t^k y_{xxx}\|_{L_4} \|\partial_t^l y_{xxxx}\|_{L_2} \leqq c(l) \sum_{k=0}^{l} \|b\|_{I_{l-k}} \|y\|_{U_k} \|y\|_{U_l}\,.$$

Summarizing the results just achieved, we get by (6.1)

$$(6.9) \qquad \|y\|_{U_l}^2 \leqq c(l) \Big(\sum_{k=0}^{l} (\|w\|_{I_{l+1-k}} + \|b\|_{I_{l+2-k}}) \|y\|_{U_k} \|y\|_{U_l} + \|h\|_{G_{l+1}} \|y\|_{U_l} \Big)\,,$$
$$l = 0, 1, \ldots\,.$$

According to (1.5), we obtain

$$\|y\|_{U_l}^2 \leqq c(l)\left(\left(\|w\|_{I_{l+1}} + \|b\|_{I_{l+2}}\right)\|y\|_{U_0}\|y\|_{U_l} + \|h\|_{G_{l+1}}\|y\|_{U_l}\right) + \\ + c(l)\left(\|w\|_{I_1} + \|b\|_{I_2}\right)\|y\|_{U_l}^2, \quad l = 0, 1, \ldots .$$

For $N \geqq 0$ there exists $\delta_5 > 0$ such that

$$\|y\|_{U_l} \leqq c(l)\left(\left(\|w\|_{I_{l+1}} + \|b\|_{I_{l+2}}\right)\|h\|_{G_1} + \|h\|_{G_{l+1}}\right) \tag{6.10}$$

holds for all $l = 0, 1, \ldots, N$ whenever

$$\|w\|_{I_1} + \|b\|_{I_2} < \delta_5 . \tag{6.11}$$

To conclude this section, we claim that the a priori estimates (6.10) imply (4.2), while (6.11) corresponds to (4.1). Indeed, we can use Proposition 4.2 together with the relations

$$\|v_{xx}\|_{I_l} \leqq c(l)\|v\|_{U_{l+2}}, \quad \|v_t\|_{I_l} \leqq c(l)\|v\|_{U_{l+2}} .$$

7. PROOF OF PROPOSITION 4.2

We are to estimate $\|\partial_t^j \varrho(v)\|_{H^{2,2}}$, $j \leqq l$.

First of all, suppose $1 \leqq j \leqq l$. The term $\partial_t^j \varrho(v)$ can be expressed as the sum of quantities

$$(\partial^{\alpha_0}\varrho)(v)(\partial_t v)^{\alpha_1} \ldots (\partial_t^j v)^{\alpha_j}$$

where $\sum_{i=1}^{j} \alpha_i i \leqq j$ and at least one $\alpha_i \neq 0$.

Since the space $H^{2,2}$ is a Banach algebra, we get

$$\|\partial_t^j \varrho(v)\|_{H^{2,2}} \leqq c(l) \sum_{\text{fin}} \|\partial^{\alpha_0}\varrho(v)\|_{H^{2,2}} \|\partial_t v\|_{H^{2,2}}^{\alpha_1} \ldots \|\partial_t^j v\|_{H^{2,2}}^{\alpha_j} \leqq$$

$$\leqq c(l) \sum_{\text{fin}} \|\partial^{\alpha_0}\varrho(v)\|_{H^{2,2}} \|v\|_{I_1}^{\alpha_1} \ldots \|v\|_{I_j}^{\alpha_j} \leqq$$

(according to (1.5))

$$\leqq c(l, \delta_2) \sum_{\text{fin}} \|\partial^{\alpha_0}\varrho(v)\|_{H^{2,2}} \|v\|_{I_l} .$$

If $j = 0$, we estimate

$$\|\varrho(v)\|_{H^{2,2}} \leqq \|\varrho(v)\|_{L_2} + \|\partial_x \varrho(v)\|_{L_2} + \|\partial_x^2 \varrho(v)\|_{L_2} + \\ + \|\partial_t \varrho(v)\|_{L_2} + \|\partial_t^2 \varrho(v)\|_{L_2} .$$

Let us treat $\|\partial_x^2 \varrho(v)\|_{L_2}$, for example:

$$\|\partial_x^2 \varrho(v)\|_{L_2} \leqq \|\varrho''(v)\, v_x^2\|_{L_2} + \|\varrho'(v)\, v_{xx}\|_{L_2} \leqq$$

$$\leqq \|\varrho''(v)\|_{C(\bar{Q})} \|v_x\|_{L_4} \|v_x\|_{L_4} + \|\varrho'(v)\|_{C(\bar{Q})} \|v\|_{H^{2,2}} .$$

To complete the proof, we have but to realize that $H^{2,2} \hookrightarrow C(\bar{Q})$.

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Souhrn

ČASOVĚ PERIODICKÁ ŘEŠENÍ KVASILINEÁRNÍ ROVNICE TYČE — POUŽITÍ METOD URYCHLENÉ KONVERGENCE

EDUARD FEIREISL

Autor vyšetřuje časově periodická řešení kvasilineární rovnice tyče metodami urychlené konvergence. Na základě Newtonova iteračního schematu je úloha aproximována posloupností lineárních rovnic řešených standardní Galerkinovou metodou. Ztráta derivací, typická pro rovnice hyperbolického typu, je kompenzována užitím zhlazujících operátorů.

Je dokázána existence alespoň jednoho (klasického) řešení úlohy za předpokladu, že pravá strana rovnice je dostatečně malá a hladká.

Резюме

ИЗУЧЕНИЕ ПЕРИОДИЧЕСКИХ ВО ВРЕМЕНИ РЕШЕНИЙ КВАЗИЛИНЕЙНОГО УРАВНЕНИЯ СТЕРЖНЯ ПРИ ПОМОЩИ МЕТОДОВ УСКОРЕННОЙ СХОДИМОСТИ

EDUARD FEIREISL

Автор изучает периодические во времени решения квазилинейного уравнения стержня, пользуясь методом ускоренной сходимости. На основе итерационного метода Нютона задача аппроксимируется последовательностью линейных уравнений, которые решаются методом Галеркина. Типичная при решении гиперболических задач трудность — потеря производных — решается при помощи регуляризующих операторов. Показано, что задача обладает по крайней мере одним решением, если правая часть уравнения является малой гладкой функцией.

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