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VARIATIONAL-HEMIVARIATIONAL INEQUALITIES IN NONLINEAR ELASTICITY. THE COERCIVE CASE

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Summary. Existence of a solution of the problem of nonlinear elasticity with non-classical boundary conditions, when the relationship between the corresponding dual quantities is given in terms of a nonmonotone and generally multivalued relation. The mathematical formulation leads to a problem of non-smooth and nonconvex optimization, and in its weak form to hemivariational inequalities and to the determination of the so called substationary points of the given potential.

Keywords: Variational-hemivariational inequalities, nonlinear elasticity, substationary points of the potential.

AMS Classification: 49A29, 49A99, 35A15

1. INTRODUCTION

In mechanics and physics there is a variety of variational inequality formulations which arise when the material laws and/or the boundary conditions are derived by a convex, generally not everywhere differentiable and finite superpotential (cf. [1], [2], [3]). The variational inequalities have a precise physical meaning: they express the principle of virtual work (or power) in its inequality form, introduced by Fourier in 1823 and since then only very rarely used (cf. e.g. [3] p. 124, 374). Prototypes of BVP's leading to variational inequalities are the Signorini-Fichera problem [4], [5] and the friction problem in the theory of elasticity [2]. The convexity of the superpotentials implies the monotonicity of the corresponding stress-strain or reaction-displacement laws. However, there exists a variety of nonmonotone laws which manifests the need for the derivation of variational formulations for nonconvex and not everywhere differentiable and finite energy functions (nonconvex superpotentials). Such variational formulations have been called by the author hemivariational inequalities [6], [7] and describe large families of important problems in physics and engineering. Similarly to the variational inequalities, the hemivariational inequalities express the principle of virtual work (or power) in its inequality form and therefore we call all the corresponding BVP's, both in the case of convexity and nonconvexity, unilateral BVP's. It should also be noted that the hemivariational inequalities are closely connected to the notion of the generalized gradient of Clarke-Rockafellar (see e.g. [8], [9]), which in the case of lack of convexity plays the same role as the subdifferential in the case of convexity (at least for static mechanical problems).

In [10], [11], [12] we studied coercive and semicoercive hemivariational inequalities arising in the static theory of Kirchhoff and von Kármán plates, whereas in [3], [13]. [14] we dealt with static hemivariational inequalities in the theory of nonmonotone semipermeability problems. Several applications in engineering can be found in [7], [15] and [16].

Here we formulate hemivariational inequalities for twodimensional and three-dimensional coercive problems in the theory of nonlinear elasticity, holonomic elastoplasticity, and the theory of locking materials, and study the resulting mathematical problems. Compactness arguments are combined with monotonicity arguments to yield approximation and existence results for BVP's arising for materials which obey monotone stress-strain laws and are formulated for nonmonotone boundary conditions [3].

2. CLASSICAL FORMULATIONS OF THE PROBLEMS AND DERIVATION OF THE VARIATIONAL EXPRESSIONS

Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ occupied by a deformable body in its undeformed state. We denote by Γ the boundary of Ω which is assumed to be Lipschitzian.

Let $\sigma = \{\sigma_{ij}\}$ and $\varepsilon = \{\varepsilon_{ij}\}$, $i, j = 1-3$ be respectively the stress and strain tensors of the body and let $f = \{f_i\}$ $u = \{u_i\}$ be the volume force and displacement vectors, respectively denote by $n = \{n_i\}$ the outward unit normal vector to Γ; then $S_i = \sigma_{ij}n_j$ (summation convention) are the boundary forces. Let S_N ans S_T be their normal and tangential components, respectively. The corresponding boundary displacement components are u_N and u_T (see Fig. 1a). We assume further that the boundary is divided into three disjoint open subsets Γ_U, Γ_F, and Γ_S, i.e. $\Gamma = \bar{\Gamma}_U \cup \bar{\Gamma}_F \cup \bar{\Gamma}_S$. On Γ_U the displacements are given, i.e.,

$$u_i = U_i\,, \quad U_i = U_i(x) \quad \text{on} \quad \Gamma_U\,, \tag{2.1}$$

on Γ_F the forces are prescribed, i.e.,

$$S_i = F_i\,, \quad F_i = F_i(x) \quad \text{on} \quad \Gamma_F\,, \tag{2.2}$$

and on Γ_S nonmonotone boundary conditions hold causing, as we shall see further, the formulation of the problem as a hemivariational inequality ([3] Ch. 4). We consider the following model problems:

Problem 1 (P_1): We assume that the tangential forces are given on Γ_S, i.e.

(2.3) $$S_{Ti} = C_{Ti}\,, \quad C_{Ti} = C_{Ti}(x)\,,$$

and that if

(2.3) $$u_N < 0 \quad \text{then} \quad S_N = 0$$

and if

(2.3b) $$u_N \geqq 0 \quad \text{then} \quad -S_N = k(u_N)\,,$$

where $k = k(u_N)$ is generally a nonmonotone function of u_N. Relations (2.3a, b) describe the unilateral contact problem of a deformable body with a granular support or concrete, which causes the nonmonotone reaction-displacement diagram. As we shall see further the function $k = k(u_N)$ may be very general and may include jumps

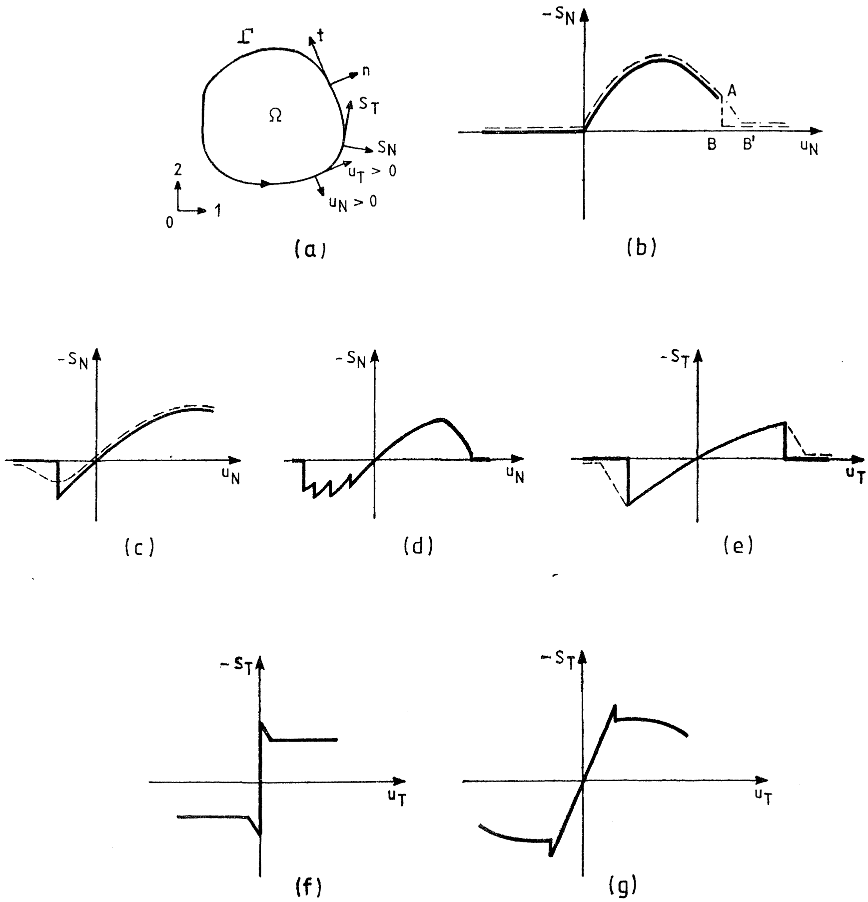

Fig. 1. Nonmonotone boundary conditions.

which describe local crushing effects. So, e.g., in Fig. 1b – dotted line – we have a crushing of the support at point A with ideally brittle (AB) or semibrittle behaviour (AB′).

Problem 2 (P_2): Again (2.3) holds and S_N is related to u_N by a law whose graph is depicted in Fig. 1c or Fig. 1d. The first graph describes the behaviour of adhesive joints (the joint can sustain a small traction) or of boundary cracks, the second graph describes the stress-strain diagram of springs simulating the behaviour of reinforced concrete (e.g. in the case of anchoring). In this respect Scanlon's effect for tensile stress in reinforced concrete is worth noting (see e.g. [3] p. 152, [17] and [18], and cf. Fig. 1d). Due to the multivalued character of the precious laws we may write them in the form

$$-S_N \in \hat{\beta}_N(u_N) \tag{2.4}$$

where $\hat{\beta}_N\colon \mathbb{R} \to \mathbb{R}$ are multivalued functions with graphs $(\xi, \hat{\beta}_N(\xi))$ given in Figs. 1c and 1d.

Problem 3 (P_3): We assume that in this problem $\Omega \subset \mathbb{R}^2$ and that S_N is given on Γ_S. i.e.

$$S_N = C_N,\, C_N = C_N(x)\,, \tag{2.5}$$

and that

$$-S_T \in \hat{\beta}_T(u_T) \tag{2.6}$$

where $\hat{\beta}_T\colon \mathbb{R} \to \mathbb{R}$ is a multivalued function.

We can have, for instance, the law of Fig. 1e which describes cracking and/or adhesive behaviour in the tangential direction, or the laws of Figs. 1f and 1g which describe more realistic frictional effects and nonmonotone shearing.

Due to the nonmonotone character of the multivalued functions (or multifunctions) $\hat{\beta}_N$ and $\hat{\beta}_T$ a convex analysis approach to this problem is not possible. Note that if $\hat{\beta}_N$ and/or $\hat{\beta}_T$ were monotone increasing, then we could determine convex, lower semicontinuous and proper functionals j_N and j_T such that $\hat{\beta}_N = \partial j_N$ and $\hat{\beta}_T = \partial j_T$ (here ∂ denotes the subdifferentiation operator, see e.g. [3], Ch. 3). As we shall see further, in the present nonmonotone cases, we can determine locally Lipschitz continuous functions $j_N\colon \mathbb{R} \to \mathbb{R}$ and $j_T\colon \mathbb{R} \to \mathbb{R}$ such that ([19])

$$\hat{\beta}_N = \bar{\partial}' j_N \quad \text{and} \quad \hat{\beta}_T = \bar{\partial}' j_T \tag{2.7}$$

where ∂' denotes the generalized gradient of Clarke (see e.g. [8], [9], [3]). j_N and j_T are the ""“potentials" of the reaction-displacement law or the nonconvex superpotentials in the terminology of [6], [7] and they result, roughly speaking, by "intergrating" $\hat{\beta}_N$ and $\hat{\beta}_T$ over $\mathbb{R}$.

In the framework of small strains and nonlinear monotone elastic behaviour of the body Ω we write the relations

$$\sigma_{ij,j} + f_i = 0\,, \tag{2.8}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \,, \tag{2.9}$$

$$\sigma \in \partial w(\varepsilon) \in \mathbb{R}^6 (\text{or } \mathbb{R}^4) \quad \text{if} \quad \Omega \subset \mathbb{R}^3 (\text{or } \mathbb{R}^2) \tag{2.10}$$

where the comma denotes differentiation, ∂ is the subdifferential of convex analysis and $w: \mathbb{R}^6 \to (-\infty, +\infty]$, $w \not\equiv \infty$, is a convex lower semicontinuous function. It is well-known, [20], [3] (Ch. 3 and Ch. 6) that with appropriate choice of w (2.10) describes in general Hooke's elastic materials, the elastic ideally locking materials, the elastic workhardening materials, the elastic-ideally "plastic" materials (Hencky's theory) and the materials obeying the law of the deformation theory of plasticity. The two last classes of materials belong to the so-called "holonomic" plasticity in order to distinguish them from the flow theory of plasticity. By definition (2.4) and (2.6) are equivalent (due to (2.7)) to the hemivariational inequalities (2.11)

$$j_N^0(u_N, v_N - u_N) \geqq -S_N(v_N - u_N) \quad \forall v_N \in \mathbb{R} \,, \tag{2.11}$$

$$j_T^0(u_T, v_T - u_T) \geqq -S_T(v_T - u_T) \quad \forall v_T \in \mathbb{R} \,, \tag{2.12}$$

respectively. Here $j_N^0(\cdot, \cdot)$ (and analogously $j_T^0(\cdot, \cdot)$) is the directional derivative of Clarke defined by

$$j_N^0(\zeta, z) = \limsup_{\substack{h \to 0 \\ \lambda \to 0+}} \frac{j_N(\zeta + h + \lambda z) - j_N(\zeta + h)}{\lambda} \,. \tag{2.13}$$

(2.10) is by definition equivalent to the variational inequality

$$w(\varepsilon^*) - w(\varepsilon) \geqq \sigma_{ij}(\varepsilon_{ij}^* - \varepsilon_{ij}) \quad \forall \varepsilon^* \in \mathbb{R}^4 \, (\text{or } \mathbb{R}^6) \,. \tag{2.14}$$

From (2.8) and (2.9) we obtain the variational equality (formal application of the Green-Gauss theorem)

$$\int_\Omega \sigma_{ij}(\varepsilon_{ij}(v) - \varepsilon_{ij}(u)) \, d\Omega = \int_\Omega f_i(v_i - u_i) \, d\Omega + \tag{2.15}$$

$$+ \int_{\Gamma_F} F_i(v_i - u_i) \, d\Gamma + \int_{\Gamma_S} [S_N(v_N - u_N) + S_T(v_T - u_T)] \, d\Gamma \quad \forall v \in U_{ad}$$

for $u \in U_{ad}$. We denote by U_{ad} the set of all kinematically admissible displacements, i.e. $U_{ad} = \{v \mid v \in U, v_i = U_i \text{ on } \Gamma_U\}$, where U is the displacement space which will be chosen later.

Using (2.15) we obtain from (2.14) with (2.11) and (2.12) the following variational-hemivariational inequalities:

Find $u \in U_{ad}$ with $w(\varepsilon(u)) < \infty$ such as to satisfy for (P_1) and (P_2)

$$\int_\Omega [w(\varepsilon(v)) - w(\varepsilon(u))] \, d\Omega + \int_{\Gamma_S} j_N^0(u_N, v_N - u_N) \, d\Gamma \geqq \tag{2.16}$$

$$\geqq \int_\Omega f_i(v_i - u_i)\,\mathrm{d}\Omega + \int_{\Gamma_F} F_i(v_i - u_i)\,\mathrm{d}\Gamma + \int_{\Gamma_S} C_{Ti}(v_{Ti} - u_{Ti})\,\mathrm{d}\Gamma \quad \forall v \in U_{\mathrm{ad}}\,,$$

and for (P_3)

$$(2.17) \qquad \int_\Omega [w(\varepsilon(v)) - w(\varepsilon(u))]\,\mathrm{d}\Omega + \int j_T^0(u_T, v_T - u_T)\,\mathrm{d}\Gamma \geqq$$

$$\geqq \int_\Omega f_i(v_i - u_i)\,\mathrm{d}\Omega + \int_{\Gamma_F} F_i(v_i - u_i)\,\mathrm{d}\Gamma + \int_{\Gamma_S} C_N(v_N - u_N)\,\mathrm{d}\Gamma \quad \forall v \in U_{\mathrm{ad}}\,.$$

It is worth noting that if the (σ, ε)-law is nonmonotone and is given by $\sigma \in \bar{\partial}' w(\varepsilon)$, where w is nonconvex (e.g. for composite materials, or for "complete" laws, see e.g. [3], [7] and [18](, then in (2.16) and (2.17) the term $w(\varepsilon(v)) - w(\varepsilon(u))$ is replaced by $w^\uparrow(\varepsilon(u), \varepsilon(v - u))$, where $w^\uparrow(\cdot, \cdot)$ is generally the upper-subderivative of Rockafellar [9]. (2.16) and (2.17) express the principle of virtual work for the respective problems. The hemivariational inequalities do not imply minimum problems but only substationarity problems for the total potential energy, see e.g. [3], [7]. It is worth noting that any local minimum of the potential energy is also a substationarity point but not conversely. Moreover, due to the lack of convexity there is generally nonuniqueness of the solution. As usual, for the corresponding dynamic problems f_i is replaced in (2.16) and (2.17) by the term $f_i - \varrho\, \partial^2 u_i / \partial t^2$. Initial conditions for the displacements and velocities must be considered on the additional assumption of small displacements.

3. FUNCTIONAL FRAMEWORK AND IMPLEMENTATION OF THE VARIATIONAL EXPRESSIONS

We further assume that $u_i, v_i \in W^{1,p}(\Omega)$ with $p > 3$ for $\Omega \subset \mathbb{R}^3$ and $p > 2$ for $\Omega \subset \mathbb{R}^2$ (the well-known Sobolev space, see e.g., [21]) and that $F_i \in L^{q'}(\Gamma_F)$ and C_N and C_{Ti} are elements of $L^{q'}(\Gamma_S)$ $(1/q + 1/q' = 1$ and $q \geqq 1$ arbitrary). Moreover, we assume that $U_i \in \dot{V}(\Gamma_U)$ which is a space with the property that there exists $u_i^* \in \in W^{1,p}$ such that $u_i^*/\Gamma = U_i$ on $\Gamma_U(u_i^*/\Gamma$ is the trace of u_i^* on Γ which is an element on $W^{1-1/p,p}(\Gamma))$. We further assume that Γ_U is nonempty. For the sake of simplicity let $U_i = 0$ on Γ_U and thus $U_{\mathrm{ad}} = \{v / v_i \in W^{1,p}(\Omega),\ v_i = 0 \text{ on } \Gamma_U\}$. (If $U_i \neq 0$ on Γ_U we perform the translation $\tilde{v} = v - u^*$ and $\tilde{u} = u - u^*$). We also assume that $f_i \in L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$, and let $(\cdot, \cdot)$ denote the duality pairing on $L^p(\Omega) \times \times L^{p'}(\Omega)$.

If grad $w(\cdot)$ exists as is the case in the deformation theory of plasticity, the polygonal stress-strain laws etc., then it is easy to show that (2.16), for instance, is equivalent to the hemivariational inequality

$$(3.1) \qquad \int_\Omega \left[\frac{\partial w(\varepsilon(u))}{\partial \varepsilon}\right]_{ij} \varepsilon_{ij}(\bar{v} - u)\,\partial\Omega + \int_{\Gamma_S} j_N^0(u_N, \bar{v}_N - u_N)\,\mathrm{d}\Gamma \geqq$$

$$\geqq \int_\Omega f_i(\bar v_1 - u_i)\, d\Omega + \int_{\Gamma_F} F_i(\bar v_i - u_i)\, d\Gamma + \int_{\Gamma_S} C_{Ti}(\bar v_{Ti} - u_{Ti})\, d\Gamma \quad \forall \bar v \in U_{ad}\,.$$

This results easily by setting in (2.16) $v = u + \lambda(\bar v - u)$, $\lambda > 0$, for $\lambda \to 0_+$. Note that $j_N^0(u_N, \cdot)$ is positively homogeneous. Conversely, from (3.1) and the obvious inequality

$$w(\varepsilon(v)) - w(\varepsilon(u)) \geqq \left[\frac{\partial \varepsilon(u)}{\partial \varepsilon}\right]_{ij} \varepsilon_{ij}(v - u) \tag{3.2}$$

holding for every $\varepsilon(v) \in \mathbb{R}^6$ we get (2.16). It is also easy to verify that (2.16) (or (3.1)) is equivalent to the hemivariational inequality

$$\int_\Omega \left[\frac{\partial w(\varepsilon(\bar v))}{\partial \varepsilon}\right]_{ij} \varepsilon_{ij}(\bar v - u)\, d\Omega + \int_{\Gamma_S} j_N^0(u_N, \bar v_N - u_N)\, d\Gamma \geqq \tag{3.3}$$

$$\geqq \int_\Omega f_i(\bar v_i - u_i)\, d\Omega + \int_{\Gamma_F} F_i(\bar v_i - u_i)\, d\Gamma + \int_{\Gamma_S} C_{Ti}(\bar v_{Ti} - u_{Ti})\, d\Gamma \quad \forall \bar v \in U_{ad}\,.$$

Indeed, (3.1) together with the monotonicity inequality

$$\left[\frac{\partial w(\varepsilon(\bar v))}{\partial \varepsilon} - \frac{\partial w(\varepsilon(u))}{\partial \varepsilon}\right]_{ij} \varepsilon_{ij}(\bar v - u) \geqq 0 \quad \forall \varepsilon(\bar v), \varepsilon(u) \in \mathbb{R}^6 \text{ (or } \mathbb{R}^4) \tag{3.4}$$

implies (3.1). Conversely, in (3.3) we put $\bar v = u + \lambda(v - u)$, $0 < \lambda < 1$ and due to the monotonicity of $\lambda \to [(\text{grad } w(\varepsilon(u + \lambda(v - u)))_{ij}\ \varepsilon_{ij}(v - u)]$ we get the inequality (3.1) as the limit for $\lambda \to 0_+$. Analogously we may argue for (2.17) and for every variational hemivariational inequality of this form. Let us set $v_i - u_i = \pm \varphi_i \in C_c^\infty(\Omega)$ in (3.1). This implies

$$[\text{grad } w(\varepsilon(u))]_{ij,j} + f_i = 0 \tag{3.5}$$

in the sense of distributions on Ω. But due to $f_i \in L^{p'}(\Omega)$ we may apply the Green-Grauss theorem and write (2.15) in the given functional framework, for $\sigma_{ij} = [\text{grad } w(\varepsilon(u))]_{ij}$. The resulting expression together with (3.3) implies first the boundary conditions (2.2) and (2.3) as equalities in the space $[W^{1-1/p,p}(\Gamma)]'$, and secondly the boundary condition (2.4) in the weak form,

$$\int_{\Gamma_S} j_N^0(u_N, v_N - u_N)\, d\Gamma \geqq -\langle S_N, v_N - u_N\rangle \quad \forall v_N \in W^{1-1/p,p}(\Gamma)\,. \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $W^{1-1p,p}(\Gamma) \times [W^{1-1/p,p}(\Gamma)]'$.

Similarly we may argue for (2.17). The aforementioned arguments do not hold in the general case. i.e. when $w(\cdot)$ is nondifferentiable, for a locking material. As usual in the variational approach (cf. [2] p. 286), in this case (2.16) and (2.17) may be considered as definitions, from the standpoint of mechanics, of the problem. For a material obeying the law of classical deformation theory of plasticity we obtain, using the notation of [23] Ch. 8, that

$$(3.7) \qquad w(\varepsilon(u)) = \tfrac{1}{2}K(\varepsilon_{ii}(u))^2 + \tfrac{1}{2}\int_0^{\Gamma(u,u)} \mu(\zeta)\,\mathrm{d}\zeta$$

where K is the bulk modulus of the material. Note that $w(\cdot)$ is a strictly convex and continuously differentiable function of ε. Then (2.16) and (2.17) become (cf. also (3.1)): Find $u \in U_{\mathrm{ad}}$ such as to satisfy for (P_1) and (P_2)

$$(3.8) \quad \int_\Omega [(K - \tfrac{2}{3}\,\mu(\Gamma^2(u)))\,\varepsilon_{ii}(u)\,\varepsilon_{ii}(v-u) + 2\,\mu(\Gamma^2(u))\,\varepsilon_{ij}(u)\,\varepsilon_{ij}(v-u)]\,\mathrm{d}\Omega +$$
$$+ \int_{\Gamma_S} j_N^0(u_N, v_N - u_N)\,\mathrm{d}\Gamma \geqq \int_\Omega f_i(v_i - u_i)\,\mathrm{d}\Omega + \int_{\Gamma_F} F_i(v_i - u_i)\,\mathrm{d}\Gamma +$$
$$+ \int_{\Gamma_S} C_{Ti}(v_{Ti} - u_{Ti})\,\mathrm{d}\Gamma \quad \forall v \in U_{\mathrm{ad}}\,.$$

For (P_3) an analogous formulation is obtained. For such a material and under the assumptions of [23] Ch. 8 a study in H^1 – space would be more adequate. However, in the functional framework presented herein we may study a generalization of such a material resulting by replacing the linear relation between σ_{ii} and ε_{ii} by a superlinear one e.g. by assuming that in (3.7) K is not a constant but an appropriate differentiable and convex function of $\varepsilon_{ii}(u)$, and by modifying appropriately the assumptions on μ.

For a linear elastic ideally locking material we have

$$(3.9) \qquad w(\varepsilon) = \tfrac{1}{2}C_{ijhk}\varepsilon_{ij}\varepsilon_{hk} + I_{\tilde{K}}(\varepsilon)$$

where $C = \{C_{ijhk}\}$, with $C_{ijhk} \in L^\infty(\Omega)$, is Hooke's strain tensor with the well-known symmetry and ellipticity properties, and $\tilde{K}$ is a convex closed subset of the strain space defined by the locking criterion; $I_{\tilde{K}}(\varepsilon) = \{0 \text{ if } \varepsilon \in K,\ \infty \text{ otherwise}\}$. For a three-dimensional generalization of a polygonal stress-strain law we refer to [3] p. 97. Note that in order to treat cases where $w(\cdot)$ can take the value $+\infty$ we define the functional

$$(3.10) \qquad W(\varepsilon) = \begin{cases} \displaystyle\int_\Omega w(\varepsilon)\,\mathrm{d}\Omega & \text{if} \quad w(\varepsilon) \in L^1(\Omega)\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

$W(\cdot)$ is a convex, proper and lower semicontinuous functional on $[L^p(\Omega)]^6$. For $\sigma_{ij} \in L^{p'}(\Omega)$ the relation $\sigma \in \partial W(\varepsilon)$ is the extension for $\varepsilon_{ij}, \sigma_{ij} \in L^p(\Omega)$ (cf. [20]) of the relation (2.10) holding a.e. on Ω.

The method which we will follow for the study of the arising variational-hemivariational inequalities assumes much more general "functions" $\hat{\beta}_N$ rnd $\hat{\beta}_T$ than the ones leading to (P_1), (P_2) and (P_3): Let $b\colon \mathbb{R} \to \mathbb{R}$ be a locally bounded measurable function, i.e. $b \in L^\infty(I)$ on every compact subset I of $\mathbb{R}$ (see Fig. 2a). Function $\hat{b}$ (Fig. 2b) results, roughly speaking, from b by "filling in the discontinuities" of the graph of b, and is a multivalued function. Mathematically the same can be achieved in the following way:

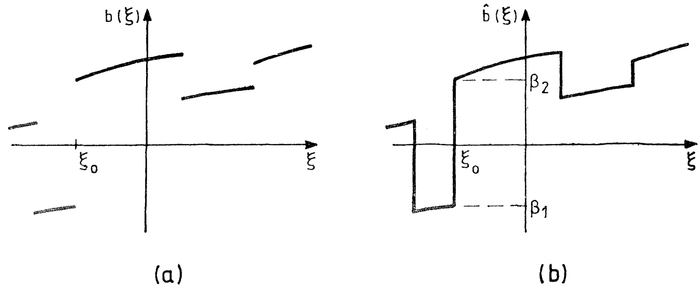

Fig. 2. Illustration of the general form of the boundary conditions.

For $\delta > 0$ and $\xi \in \mathbb{R}$ we define

$$\bar{b}_\delta(\xi) = \operatorname*{esssup}_{|\xi-\xi_1|<\delta} b(\xi_1) \quad \text{and} \tag{3.11}$$

$${}_-b_\delta(\xi) = \operatorname*{essinf}_{|\xi-\xi_1|<\delta} b(\xi_1) \tag{3.12}$$

which are increasing and decreasing functions of δ, respectively. Therefore the limit as $\delta \to 0$ exists. We denote by $\bar{b}(\xi)$ and ${}_-b(\xi)$ the limits $\lim_{\delta-0} \bar{b}_\delta(\xi)$ and $\lim_{\delta\to\infty} {}_-b_\delta(\xi)$, respectively, and define the multivalued function

$$\hat{b}(\xi) = [{}_-b(\xi), \bar{b}(\xi)] \tag{3.13}$$

where $[\cdot, \cdot]$ denotes a closed interval in $\mathbb{R}$. So, e.g., in Fig. 2a we have ${}_-b(\xi_0) = \beta_1$ and $\bar{b}(\xi_0) = \beta_2$ and therefore $\hat{b}(\xi_0) = [\beta_1, \beta_2]$.

Chang [19] has shown that if $b(\xi_{\pm 0})$ exists at every $\xi \in \mathbb{R}$, then a locally Lipschitz continuous function $\varphi: \mathbb{R} \to \mathbb{R}$ can be determined such that

$$\hat{b}(\xi) = \bar{\partial}'\varphi(\xi)\,. \tag{3.14}$$

Here we can write, up to a constant, the equality $\varphi(\xi) = \int_0^\xi b(\xi_1)\, d\xi_1$. However, in what follows, only the expression $\varphi^0(\xi, z)$ defined by

$$\varphi^0(\xi, z) = \limsup_{\substack{h\to 0\\ \lambda\to 0_+}} \int_{\xi+h}^{\xi+h+\lambda z} b(\xi_1)\, d\xi_1 \tag{3.15}$$

is needed; for the mechanical problem this constitutes a remark of major importance. According to [7] the $\varphi(\cdot)$ is called superpotential of the boundary constraint and $\varphi^0(\xi, z)$ is simply the virtual wotk of the constraint at a displacement ξ for a virtual displacement z.

Let us further introduce a linear continuous functional on the displacement space, $l = \{l_i\}$, $l_i \in [W^{1,p}(\Omega)]'$ defined by the right hand sides of (2.16) and (2.17) and let us denote by V the kinematically admissible subspace

$$V = \{v \mid v = \{v_i\},\ v_i \in W^{1,p}(\Omega),\ v|_\Gamma = o \text{ on } \Gamma_U\}\,. \tag{3.16}$$

In the next sections we deal with the following problem:

(P): Find $u \in V$ such that

$$W(\varepsilon(v)) - W(\varepsilon(u)) + \int_{\Gamma_S} \varphi^0(u_N, v_N - u_N)\,\mathrm{d}\Gamma \geqq (l, v - u) \quad \forall v \in V\,. \tag{3.17}$$

The study is completely analogous when φ represents j_T and not j_N. We shall distinguish two cases: in the first, called "differentiable case" we assume that grad $w(\cdot)$ exists everywhere, in the second w may take the value $+\infty$, $w \not\equiv \infty$ and is generally lower semicontinuous and not everywhere differentiable ("nondifferentiable case").

4. STUDY OF THE DIFFERENTIABLE CASE

Let us consider a mollifier p, i.e., $p \in C_c^\infty(-1, +1)$ with $p \geqq 0$ and $\int_{-\infty}^{+\infty} p(\zeta)\,\mathrm{d}\zeta = 1$. Let

$$p_s(\xi) = \frac{1}{\varepsilon}\, p\left(\frac{\xi}{\varepsilon}\right) \tag{4.1}$$

and let us form the convolution

$$b_\varepsilon = p_\varepsilon * b\,, \quad \varepsilon > 0\,. \tag{4.2}$$

It is well-known that $b_\varepsilon \in C^\infty(\mathbb{R})$. We call b_ε the "regularized form of b". We may now pose the following variational equality (the regularized form of (P)).

(P_ε): Find $u_\varepsilon \in V$ such that

$$(\text{grad } w(\varepsilon(u_\varepsilon)), \varepsilon(v)) + \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon})\, v_N\,\mathrm{d}\Gamma = (l, v) \quad \forall v \in V\,. \tag{4.3}$$

In order to discretize (P_ε) we consider a basis $\{w_i\}$ of V. Let V_m be the corresponding m-dimensional subspace of V. Then we formulate the finite-dimensional problem $(P_{\varepsilon m})$.

$(P_{\varepsilon m})$: Find $u_{\varepsilon m} \in V_m$ such that

$$(\text{grd } w(\varepsilon(u_{\varepsilon m})), \varepsilon(v)) + \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, v_N\,\mathrm{d}\Gamma = (l, v) \quad \forall v \in V_m\,. \tag{4.4}$$

Further, we assume that for some ξ,

$$\operatorname*{ess\,sup}_{(-\infty, -\xi)} b(\xi) \leqq \operatorname*{ess\,inf}_{(\xi, +\infty)} b(\xi)\,. \tag{4.5}$$

Then, without loss of generality, using an appropriate translation of the coordinate axes we can assume that for some ξ

$$\operatorname*{essup}_{(-\infty,-\xi)} b(\xi) \leqq 0 \leqq \operatorname*{essinf}_{(\xi,+\infty)} b(\xi) \tag{4.6}$$

Moreover, we assume that the energy function w has the following property: For every $u = \{u_i\}$, $u_i \in W^{1,p}(\Omega)$ there exists $c > 0$ such that

$$(\operatorname{grad} w(\varepsilon(v)), \varepsilon(v)) \geqq c \int_\Omega [\varepsilon_{ij}(v)\,\varepsilon_{ij}(v)]^{p/2}\, d\Omega\,. \tag{4.7}$$

It can be easily verified that (4.7) is fulfilled for the three- or twodimensional generalizations of the polynomial laws (e.g. for $\sigma \sim \varepsilon^\varrho$, $\varrho = n - 1 + r$, $n = 2$ or 3 and $r > 0$) [28], and the superlinear generalizations of the deformation theory of plasticity. At this point we can also mention the regularized elastici deally locking laws, a wide class of materials of the hyperelastic type, as well as the three- or twodimensional generalizations of rubber-like materials [29].

Lemma 4.1. *Suppose that* (4.6) *and* (4.7) *hold. Then* ($P_{\varepsilon m}$) *has a solution.*

Proof. The imbedding $W^{1,p}(\Omega) \subset C^0(\bar{\Omega}) \subset C^0(\Gamma) \subset L^\infty(\Gamma)$ for $p > n$ implies that we can write (4.4) in the form

$$(T(u_{\varepsilon m}), v) = 0 \quad \forall v \in V_m\,, \tag{4.8}$$

where $T\colon V_m \to V_m'$. We shall apply Brouwer's fixed point theorem in the following well-known version ([24], p. 53): if $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous and such that for $r > 0$, $\sum_i (f(a))_i\, a_i \geqq 0\ \forall a = \{a_i\}$ with $|a| = r$, then a_0 exists with $|a_0| \leqq r$ such that $f(a_0) = 0$. Indeed, (4.6) implies that we can determine $\varrho_1 > 0$ and $\varrho_2 > 0$ such that

$$b_\varepsilon(\xi) \geqq 0 \quad \text{if} \quad \xi > \varrho_1,\ b_\varepsilon(\xi) \leqq 0 \quad \text{if} \quad \xi < \varrho_1 \quad \text{and} \quad |b_\varepsilon(\xi) \leqq \varrho_2 \quad \text{if} \quad |\xi| < \varrho_1\,.$$

Moreover, due to (4.7) and Korn's first inequality in $W^{1,p}(\Omega)$-spaces $p > 1$ we have the relation(1)

$$(\operatorname{grad} w(\varepsilon(u)), \varepsilon(u)) \geqq c_1 \int_\Omega [\varepsilon_{ij}(u)\,\varepsilon_{ij}(u)]^{p/2}\, d\Omega \geqq c_2 \|u\|^p \quad \forall u \in V, \tag{4.9}$$

$$c_1, c_2 \text{ const} > 0$$

where $\|\cdot\|$ denotes the $[W^{1,p}(\Omega)]$-norm, with $n = 3$ or 2. Thus we find that

(1) We refer the reader to [20] [27] [30] [31]. For a counterexample in the case $p = 1$ cf. [32]. Korn's inequality implies that $\varepsilon\cdot\colon u \to \varepsilon(u)$ is a continuous linear function from $[W^{1,p}(\Omega)]^n$ to $[L^p(\Omega)^{2n}$.

$$(4.10)\qquad (T(u_{\varepsilon m}), u_{\varepsilon m}) \geqq c_1\|u_{\varepsilon m}\|^p + \int_{\Gamma_S} u_{N\varepsilon m}\, b_\varepsilon(u_{N\varepsilon m})\, \mathrm{d}\Gamma - \|l\|_* \|u_{\varepsilon m}\| \geqq$$

$$\geqq c_1\|u_{\varepsilon m}\|^p - c_2\|u_{\varepsilon m}\| - c_3 \quad p > 3 \quad \text{for} \quad \Omega \subset \mathbb{R}^3, p > 2 \quad \text{for} \quad \Omega \subset \mathbb{R}^2$$

since

$$(4.11)\qquad \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}\, \mathrm{d}\Gamma = \int_{|u_{N\varepsilon m}(x)| > \varrho_1} \ldots\ldots + \int_{|u_{N\varepsilon m}(x)| \leqq \varrho_1} \ldots\ldots \geqq 0 - \varrho_1\varrho_2 \operatorname{mes} \Gamma_S\,.$$

Here $\|\cdot\|_*$ denotes the $[W^{1,p}(\Omega)']^n$-norm and c_1, c_2, c_3 are positive constants. (4.10) implies, by setting in the foregoing version of Brouwer's fixed point theorem $u_{\varepsilon m} = \sum_{i=1}^{m} a_i w_i$ and $f(a)_i = (T(u_{\varepsilon m}), w_i)$, that (4.8) has a solution such that $\|u_{\varepsilon m}\| < c$, q.e.d.

Further, we shall investigate the behaviour of the solution $u_{\varepsilon m}$ of the finite dimensional problem $(P_{\varepsilon m})$ as $\varepsilon \to 0$ and $m \to \infty$. Due to the fact that $\{u_{\varepsilon m}\}$ is bounded in V we may extract a subsequence again denoted by $\{u_{\varepsilon m}\}$ such that

$$(4.12)\qquad u_{\varepsilon m} \to u \quad \text{weakly in } V.$$

However, ([33], p. 344) $W^{1,p}(\Omega)$ is compactly imbedded into $L^q(\Gamma)$, q > 1; thus denoting here for the sake of simplicity the trace on Γ as the function itself, we obtain that$(p > n)$

$$(4.13)\qquad u_{\varepsilon m} \to u \quad \text{strongly in } [L^2(\Gamma)]^n$$

and accordingly

$$(4.14)\qquad u_{\varepsilon m} \to u \quad \text{a.e. on } \Gamma\,.$$

Further, we shall investigate the behaviour of $b_\varepsilon(u_{N\varepsilon m})$ as $\varepsilon \to 0$ and $m \to \infty$.

Lemma 4.2. *On the assumption* (4.6), $\{b_\varepsilon(u_{N\varepsilon m})\}$ *is weakly precompact in* $L^1(\Gamma_S)$.

Proof. By the Dunford-Pettis theorem (see [25], p. 139) we will show that for each $\alpha > 0$ a constant can be determined such that for $g \subset \Gamma_S$ with mes $g < \gamma$

$$(4.15)\qquad \int_g |b_\varepsilon(u_{N\varepsilon m})\, \mathrm{d}\Gamma| < \alpha$$

holds. Let us consider the inequality (cf. [26])

$$(4.16)\qquad |b_\varepsilon(\xi)\, \xi| + \xi_0 \sup_{|\xi| \leqq \xi_0} |b_\varepsilon(\xi)| \geqq \xi_0 |b_\varepsilon(\xi)|$$

which implies that

$$(4.17)\qquad \int_g |b_\varepsilon(u_{N\varepsilon m})|\, \mathrm{d}\Gamma \leqq \frac{1}{\xi_0}\int_{\Gamma_S} |b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}|\, \mathrm{d}\Gamma + \int_g \sup_{|u_{N\varepsilon m}(x)| \leqq \xi_0} |b_\varepsilon(u_{N\varepsilon m})|\, \mathrm{d}\Gamma\,.$$

Due to (4.9) and $\|u_{\varepsilon m}\| < c$ we have

$$(4.18) \qquad \int_{\Gamma_S} |b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}|\, d\Gamma = \int_{|u_{N\varepsilon m}(x)| > \varrho_1} b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}\, d\Gamma +$$

$$+ \int_{|u_{N\varepsilon m}(x)| \leqq \varrho_1} |b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}|\, d\Gamma \leqq \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}\, d\Gamma +$$

$$+ 2 \int_{|u_{N\varepsilon m}(x)| \leqq \varrho_1} |b_\varepsilon(u_{N_{\varepsilon m}})\, u_{N\varepsilon m}|\, d\Gamma = - (\text{grad}\, w(\varepsilon(u_{\varepsilon m})), \varepsilon(u_{\varepsilon m})) +$$

$$+ (l, u_{\varepsilon m}) + 2 \int_{|u_{N\varepsilon m}(x)| \leqq \varrho_1} |b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}|\, d\Gamma \leqq c + 2\varrho_1\varrho_2 \text{ mes } \Gamma_S .$$

We choose ξ_0 such that

$$(4.19) \qquad \frac{1}{\xi_0} \int_{\Gamma_S} |b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}|\, d\Gamma < \frac{\alpha}{2}$$

for all m and ε. Let γ be such that for mes$g < \gamma$ (cf. also (4.2))

$$(4.20) \qquad \int_g \sup_{|u_{N\varepsilon m}(x)| \leqq \xi_0} b_\varepsilon(u_{N\varepsilon m})|\, d\Gamma \leqq \gamma \operatorname*{essup}_{|\xi| \leqq \xi_0 + 1} |b(\xi)| < \frac{\alpha}{2} .$$

Then (4.17), (4.18), (4.19) and (4.20) imply (4.15), qfe.d.

From Lemma 4.2 we find that as $\varepsilon \to 0$ and $m \to \infty$,

$$(4.21) \qquad b_\varepsilon(u_{N\varepsilon m}) \to \chi \quad \text{weakly in} \quad L^1(\Gamma_S) .$$

Proposition 4.1. *u is a solution of the problem (P).*

Proof. In order to pass to the limit $\varepsilon \to 0$, $m \to \infty$ in (4.4) we have to pay special attention to the term grad $\varepsilon(u_{\varepsilon m})$, $\varepsilon(v))$. The following method is a combination of the well-known monotonicity argument (Minty's method) which is often encountered in the theory of variational inequalities, with a compactness argument.

a) We shall first show that (4.4) and (4.21) imply, as $\varepsilon \to 0$ and $m \to \infty$, the equality

$$(4.22) \qquad (\text{grad}\, w(\varepsilon(u)), \varepsilon(v)) + \int_{\Gamma_S} \chi v_N\, d\Gamma = (l, v) \quad \forall v \in V_m .$$

Indeed, $v \in V_m \subset [W^{1,p}(\Omega)]^n$, $v|_\Gamma \in [L^\infty(\Gamma)]^n$ and thus $v_N \in L^\infty(\Gamma)$ $(n = \{n_i\} \in$ $\in [L^\infty(\Gamma)^2$, cf. [23] p. 19). Accordingly, $\int b_\varepsilon v_N \to \int \chi v_N$. Let us formulate the non-negative expression

$$(4.23) \quad X_m = (\text{grad}\, w(\varepsilon(u_{\varepsilon m})) - \text{grad}\, w(\varepsilon(\varphi)), \varepsilon(u_{\varepsilon m}) - \varepsilon(\varphi)) \geqq 0 \quad \forall \varphi \in [W^{1,p}(\Omega)]^n$$

which by means of (4.4) becomes

$$(4.24) \qquad X_m = (l, u_{\varepsilon m}) - \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}\, \mathrm{d}\Gamma - (\operatorname{grad} w(\varepsilon(u_{\varepsilon m})), \varepsilon(\varphi)) -$$
$$- (\operatorname{grad} w(\varepsilon(\varphi)), \varepsilon(u_{\varepsilon m} - \varphi)) \geqq 0 \quad \forall \varphi \in [W^{1,p}(\Omega)]^n$$

From (4.4) we easily obtain that the sequence $|(\operatorname{grad} w(\varepsilon(u_{\varepsilon m})), \varepsilon(v_m))|$, where $u_{\varepsilon m}, v_m \in \in V_m$, is bounded and thus $\|\operatorname{grad} w(\varepsilon(u_{\varepsilon m}))\|_{[L^{p'}(\Omega)]^{\tilde{n}}} \leqq c$. Therefore, as $\varepsilon \to 0$, $m \to \infty$,

$$(4.25) \qquad \operatorname{grad} w(\varepsilon(u_{\varepsilon m})) \to \Psi \quad \text{weakly in} \quad [L^{p'}(\Omega)]^{\tilde{n}}, \quad \tilde{n} = \frac{n(n+1)}{2}.$$

Thus (4.4) (4.21) and (4.25) yield the equality

$$(4.26) \qquad (\Psi, \varepsilon(v)) + \int_{\Gamma_S} \chi v_N \, \mathrm{d}\Gamma = (l, v) \quad \forall v \in V_m$$

and its extension by density to V (i.e. $\forall\, v \in V$). We denite this extension by (4.26a). We assume at the moment that

$$(4.27) \qquad \lim \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m}\, \mathrm{d}\Gamma = \int_{\Gamma_S} \chi u_N \, \mathrm{d}\Gamma .$$

This last relation will be shown later in lemma 4.3. From (4.27) and (4.24), (4.25) (4.26a) we find that

$$(4.28) \quad \lim X_m = (\Psi, \varepsilon(u - \varphi)) - (\operatorname{grad} w(\varepsilon(\varphi)), \varepsilon(u - \varphi)) \geqq 0 \quad \forall \varphi \in [W^{1,p}(\Omega)]^n .$$

In (4.28) let us set $u - \varphi = \lambda\theta$, $\lambda > 0$. We get the relation

$$(4.29) \qquad (\Psi, \varepsilon(\theta)) - (\operatorname{grad} w(\varepsilon(u - \lambda\theta)), \varepsilon(\theta)) \geqq 0 \quad \forall \theta \in [W^{1,p}(\Omega)]^n .$$

Due to the monotonicity of $\lambda \to (\operatorname{grad} w(\varepsilon(u - \lambda\theta)), \varepsilon(\theta))$, we may take $\lambda \to 0_+$ in (4.29), thus finding for $\pm\theta$ that

$$(4.30) \qquad \Psi = \operatorname{grad} w(\varepsilon(u)) .$$

(4.30) and (4.26) imply (4.22).

b) It remains to show that $\chi \in \bar{\partial}'\varphi(u_N)$ a.e. on Γ_S, or equivalently, that

$$(4.31) \qquad \chi \in \tilde{b}(u_N) \quad \text{i.e.} \quad {}_{-}b(u_N) \leqq \chi \leqq \bar{b}(u_N) \quad \text{a.e. on} \quad \Gamma_S .$$

Given $\gamma > 0$, due to (4.14) we may choose $g \subset \Gamma_S$ with mes$g < \gamma$ such that, as $\varepsilon \to 0$ $m \to \infty$ (Egoroff's theorem),

$$(4.32) \qquad u_{N\varepsilon m} \to u_N \quad \text{uniformly on} \quad \Gamma_S - g$$

where $u_N \in L^\infty(\Gamma_S - g)$. (4.32) implies that for any $\delta > 0$ we may find $m_0 > 2/\delta$ and $\varepsilon_0 < \delta/2$ such that for $m > m_0$ and $\varepsilon < \varepsilon_0$,

$$(4.33) \qquad |u_{N\varepsilon m}(x) - u_N(x) < \delta/2 \quad \text{for all} \quad x \in \Gamma_S - g .$$

From (4.2) we obtain

$$(4.34)\qquad b_\varepsilon(\xi) = (p_\varepsilon * b)(\xi) = \int_{-\varepsilon}^{+\varepsilon} b(\xi - t)\, p_\varepsilon(t)\, \mathrm{d}t \leqq$$

$$\leqq \operatorname*{esssup}_{|t| \leqq \varepsilon} b(\xi - t) \int_{-\varepsilon}^{+\varepsilon} p_\varepsilon(t)\, \mathrm{d}t = \operatorname*{esssup}_{|t| \leqq \varepsilon} b(\xi - t)$$

i.e.

$$(4.35)\qquad b_\varepsilon(u_{N\varepsilon m}(x)) \leqq \operatorname*{esssup}_{|u_{N\varepsilon m}(x) - \xi| \leqq \varepsilon} b(\xi)\,.$$

But due to (4.33) and (3.12) (4.35) implies that

$$(4.36)\qquad \operatorname*{esssup}_{|u_{N\varepsilon m}(x) - \xi| \leqq \varepsilon} b(\xi) \leqq \operatorname*{esssup}_{|u_{N\varepsilon m}(x) - \xi| < \delta/2} b(\xi) \leqq \operatorname*{essup}_{|u_N(x) - \xi| \leqq \delta} b(\xi) = \bar{b}_\delta(u_N(x))$$

and analogously

$$(4.37)\qquad {}_-b_\delta(u_N(x)) \leqq b_\varepsilon(u_{N\varepsilon m}(x))\,.$$

From (4.36, 37) we obtain for $e \geqq 0$ a.e. on $\Gamma_S - g$ and $e \in L^\infty(\Gamma_S - g)$ that

$$(4.38)\qquad \int_{\Gamma_S - g} {}_-b_\delta(u_N(x))\, e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} b_\varepsilon(u_{N\varepsilon m}(x))\, e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} \bar{b}_\delta(u_N(x))\, e\, \mathrm{d}\Gamma\,,$$

which implies as $\varepsilon \to 0$, $m \to \infty$,

$$(4.39)\qquad \int_{\Gamma_S - g} {}_-b_\delta(u_N(x))\, e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} \chi e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} \bar{b}_\delta(u_N(x))\, e\, \mathrm{d}\Gamma\,.$$

This last relation implies by passing to the limit $\delta \to 0_+$ the relation

$$(4.40)\qquad \int_{\Gamma_S - g} {}_-b(u_N(x))\, e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} \chi e\, \mathrm{d}\Gamma \leqq \int_{\Gamma_S - g} \bar{b}(u_N(x))\, e\, \mathrm{d}\Gamma\,.$$

From (4.40) we obtain, since e is arbitrary, that

$$(4.41)\qquad \chi \in [{}_-b(u_N), \bar{b}(u_N)] \quad \text{a.e. on} \quad \Gamma_S - g\,,$$

and by taking γ as small as possible we get (4.31), q.e.d.
In order to complete the proof we have to show (4.27).

Lemma 4.3. (4.27) *holds.*

Proof. We recall (4.12), (4.13) and (4.21) and form the difference

$$(4.42)\qquad \int_{\Gamma_S} [b_\varepsilon(u_{N\varepsilon m})\, u_{N\varepsilon m} - \chi u_N]\, \mathrm{d}\Gamma = \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\, (u_{N\varepsilon m} - u_N)\, \mathrm{d}\Gamma +$$

$$+ \int_{\Gamma_S} u_N(b_\varepsilon(u_{N\varepsilon m}) - \chi)\, \mathrm{d}\Gamma = A + B\,.$$

Due to imbeddings $W^{1,p}(\Omega) \subset C^0(\bar{\Omega}) \subset C^0(\Gamma) \subset L^\infty(\Gamma)$, $u_N \in L^\infty(\Gamma)$ and due to (4.21), $\lim B = 0$. We estimate the first summand in (4.42). From $W^{1,p}(\Omega) \subset L^\infty(\Gamma)$ and (4.12) we get that

$$(4.43) \qquad u_{N\varepsilon m} \to u_N \text{ strongly in } L^\infty(\Gamma)\,.$$

Similarly (4.21) implies

$$(4.44) \qquad \|b_\varepsilon(u_{N\varepsilon m})\|_{L^1(\Gamma_S)} < c_2\,.$$

Therefore

$$(4.45) \quad |A| = \left|\int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon m})\,(u_{N\varepsilon m} - u_N)\,\mathrm{d}\Gamma\right| \leqq \|b_\varepsilon(u_{N\varepsilon m})\|_{L^1(\Gamma_S)}\,\|u_{N_{\varepsilon m}} - u_N\|_{L^\infty(\Gamma_S)}\,.$$

From (4.44) and (4.45) we obtain that $\lim (A + B) = 0$, q.e.d.

5. STUDY OF THE NONDIFFERENTIABLE CASE

If in problem (P) the internal energy function $w(\cdot)$ is nondifferentiable, as is the case e.g. for ideally locking materials, then we introduce a sequence of convex functions w_ϱ depending on a parameter ϱ such that

(i) as $\varrho \to 0$,

$$(5.1) \qquad \int_\Omega w_p(\varepsilon(v))\,\mathrm{d}\Omega \to W(\varepsilon(v)) \quad \forall v \in V\,;$$

(ii) if $v_\varrho \to v$ weakly in V for $\varrho \to 0$ and $\int_\Omega w_\varrho(\varepsilon(v_\varrho))\,\mathrm{d}\Omega < c$, then

$$(5.2) \qquad \liminf_{\varrho \to 0} \int_\Omega w_\varrho(\varepsilon(v_\varrho))\,\mathrm{d}\Omega \geqq W(\varepsilon(v))\,;$$

(iii) relation (4.7) holds for every w_p with c independent of ϱ.

Now we define the following regularized problem $(P_{\varepsilon\varrho})$.

$(P_{\varepsilon\varrho})$: Find $u_{\varepsilon p} \in V$ such that

$$(5.3) \qquad (\operatorname{grad} w_\varrho(\varepsilon(u_{\varepsilon\varrho})), \varepsilon(v)) + \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon\varrho})\,v_N\,\mathrm{d}\Gamma = (l, v) \quad \forall v \in V\,,$$

and by means of a basis the problem $(P_{\varepsilon\varrho m})$.

$(P_{\varepsilon\varrho m})$: Find $u_{\varepsilon\varrho m} \in V_m$ such that

$$(5.4) \qquad (\operatorname{grad} w_\varrho(\varepsilon(u_{\varepsilon\varrho m})), \varepsilon(v)) + \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon\varrho m})\cdot v_N\,\mathrm{d}\Gamma = (l, v) \quad \forall v \in V_m\,.$$

Proposition 5.1. *Suppose that* (4.6) *holds and that* w_ϱ *satisfies* (i), (ii), (iii). *Then* (P) *has a solution.*

Proof. For problem $(P_{\varepsilon\varrho m})$ we can prove as in Lemma 4.1 that a solution exists and that

$$\|u_{\varepsilon\varrho m}\| < c \tag{5.5}$$

where c is independent of ε, ϱ and m. Thus as $\varepsilon \to 0$, $m \to \infty$,

$$u_{\varepsilon\varrho m} \to u_\varrho \quad \text{weakly in} \quad V, \tag{5.6}$$

and thus strongly in $[L^2(\Gamma)]^2$ and a.e. on Γ. Moreover, Lemma 4.2 obviously holds for $\{b_\varepsilon(u_{N\varepsilon\varrho m})\}$ with α in (4.15) independent of ε, m and ϱ. Thus

$$b_\varepsilon(u_{N\varepsilon\varrho m}) \to \chi_\varrho \quad \text{weakly in} \quad L^1(\Gamma_S). \tag{5.7}$$

Note that (5.5) implies that, as $\varrho \to 0$,

$$u_\varrho \to u \quad \text{weakly in} \quad V. \tag{5.8}$$

On the other hand, from (4.15) we easily find that

$$\chi_\varrho \to \chi \quad \text{weakly in} \quad L^1(\Gamma_S). \tag{5.9}$$

From (5.4) we obtain that

$$\|\text{grad } w_\varrho(\varepsilon(u_{\varepsilon\varrho m}))\|_{[L^{p'}(\Omega)]^{\tilde{n}}} < c \tag{5.10}$$

where c is independent of ε, m, ϱ and thus, as $\varepsilon \to 0$, $n \to \infty$,

$$\text{grad } w_\varrho(\varepsilon(u_{\varepsilon\varrho m})) \to \Psi_\varrho \quad \text{weakly in} \quad [L^{p'}(\Omega)]^{\tilde{n}}. \tag{5.11}$$

From (5.4) we obtain, by passing to the limit $\varepsilon \to 0$ $m \to \infty$ and by density, the variational equality

$$(\Psi_\varrho, \varepsilon(v)) + \int_{\Gamma_S} \chi_\varrho v_N \, d\Gamma = (l, v) \quad \forall v \in V. \tag{5.12}$$

As in Proposition 4.1, by means of the monotonicity argument we obtain that

$$\Psi_\varrho = \text{grad } w_\varrho(\varepsilon(u_\varrho)) \tag{5.13}$$

where we make use of the fact that

$$\lim_{\substack{\varepsilon \to 0 \\ m \to \infty}} \int_{\Gamma_S} b_\varepsilon(u_{N\varepsilon\varrho m})\, u_{N\varepsilon\varrho m} \, d\Gamma = \int_{\Gamma_S} \chi_\varrho u_{N\varrho} \, d\Gamma. \tag{5.14}$$

The proof of this last assertion is the same as that of Lemma 4.3.

Further, we pass to the limit with respect to ϱ. From (5.12) and (5.13) we find, due to the convexity of w_ϱ (use (3.3)), the relation

$$\int_\Omega [w_\varrho(\varepsilon(v)) - w_\varrho(\varepsilon(u_\varrho))] \, d\Omega + \int_{\Gamma_S} \chi_\varrho(v_N - u_{N\varrho}) \, d\Gamma \geqq (l, v - u_\varrho) \quad \forall v \in V. \tag{5.15}$$

In (5.15) let us take v such that $W(\varepsilon(v)) < \infty$. Then, due to (5.1), $\int w_\varrho(\varepsilon(v))\, d\Omega < c$ and from (5.15) we obtain that

$$\int_\Omega w_\varrho(\varepsilon(u_\varrho))\, d\Omega < c \tag{5.16}$$

and therefore due to (5.8) the relation (5.2) holds. From (5.15) we find, as $\varrho \to 0$,

$$\liminf \left[\int_\Omega w_\varrho(\varepsilon(v))\, d\Omega + \int_\Omega \chi_\varrho v_N\, d\Gamma\right] \geqq \tag{5.17}$$

$$\geqq \liminf \left[\int_\Omega w_\varrho(\varepsilon(u_\varrho))\, d\Omega + \int_{\Gamma_S} \chi_\varrho u_{N\varrho}\, d\Gamma + (l, v - u_\varrho)\right]$$

for every $v \in V$ with $W(\varepsilon(v)) < \infty$. But from (5.1), (5.2), (5.9) and the fact that

$$\lim_{\varrho \to 0} \int_{\Gamma_S} \chi_\varrho u_{N\varrho}\, d\Gamma = \int_{\Gamma_S} \chi u_N\, d\Gamma \tag{5.18}$$

(the proof is the same as that of Lemma 5.3) we conclude that the inequality

$$W(\varepsilon(v)) - W(\varepsilon(u)) + \int_{\Gamma_S} \chi(v_N - u_N)\, d\Gamma \geqq (l, v - u)\ \forall v \in V \tag{5.19}$$

is satisfied by $u \in V$ with $W(\varepsilon(u)) < \infty$. Finally, as in part b) of the proof of Proposition 4.1 we show (4.31) and thus u is a solution of problem (P), q.e.d.

It should be noted that the regularization of w defined by (i), (ii), (iii) seems to be very reasonable and useful for practical applications (cf. also [2], [3]). Regularizations of more technical character, as the one based on the Yosida approximation of w, can also be applied affecting only slightly the proof of Proposition 5.1. It is wotrh noting that the imbedding $W^{1,p}(\Omega) \subset L^\infty(\Gamma)$ is necessary for the whole proof and especially for the proof of Lemma 4.3. Therefore we have assumed that $p > n$ where where $n = 2$ or 3 depending on the dimensions of the body consideted. Of course, the proofs are general and hold for any n.

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Souhrn

VARIAČNÍ-HEMIVARIAČNÍ NEROVNOSTI V NELINEÁRNÍ ELASTICITĚ. KOERCIVNÍ PŘÍPAD

P. D. PANAGIOTOPOULOS

Je dokázána existence řešení problému nelineární elasticity s neklasickými okrajovými podmínkami, kdy vztah mezi odpovídajícími duálními veličinami je dán pomocí nemonotonní a obecně nehladké relace. Matematická formulace vede na problematiku nehladké *h* nekonvexní optimalizace a ve slabé formě k nalezení tzv. substacionárních bodů daného potennciálu.

Резюме

ВАРИАЦИОННЫЕ - ПОЛУВАРИАЦИОННЫЕ НЕРАВЕНСТВА В НЕЛИНЕЙНОЙ ЭЛАСТИЧНОСТИ. КОЭРЦИТИВНЫЙ СЛУЧАЙ

P. D. PANAGIOTOPOULOS

Доказано существование решения проблемы нелинейной эластичности с неклассическими краевыми условиями в случае, когда соответствующие двойственные величины связаны немонотонным и, вообще говоря, не гладким отношением. Математическая формулировка ведет к проблеме негладкой и невыпуклой оптимизации и в слабой форме к отысканию т. н. субстационарных точек данного потенциала.

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