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A monotonicity method for solving hyperbolic problems with hysteresis

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A MONOTONICITY METHOD FOR SOLVING HYPERBOLIC PROBLEMS WITH HYSTERESIS

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Summary. A version of the Minty-Browder method is used for proving the existence and uniqueness of a weak ω-periodic solution to the equation $u_{tt} - \operatorname{div} F(\operatorname{grad} u) = g$ in a bounded domain $\Omega \subset \mathbb{R}^N$ with the boundary condition $u = 0$ on $\partial\Omega$, where g is a given (generalized) ω-periodic function and F is the Ishlinskiĭ hysteresis operator.

Keywords: Quasilinear hyperbolic equation, Ishlinskiĭ hysteresis operator, periodic solution.

AMS Classification: 35B10, 35L70

INTRODUCTION

Hyperbolic equation with a hysteresis operator in the "elliptic" part describe in a natural way the behavior of systems of evolution with hysteresis, e.g. vibrations of non-perfectly elastic bodies in the sense of Ishlinskiĭ [7], where Hooke's law is of a hysteresis type, or the electromagnetic field in ferromagnetic media.

For the sake of simplicity we demonstrate the method of solving such problems by choosing the scalar equation

$$(*) \qquad u_{tt} - \operatorname{div}\left(F(\operatorname{grad} u)\right) = g(x, t)\,, \quad x \in \Omega\,, \quad t \geqq 0\,,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, g is a given (generalized) function which is ω-periodic with respect to t, and F is the Ishlinskiĭ hysteresis operator ([8], [2]). Using the Minty-Browder technique we prove that there exists a unique weak ω-periodic solution to $(*)$ with the boundary condition

$$(**) \qquad u(x, t) = 0 \quad \text{for} \quad x \in \partial\Omega\,.$$

1. NOTATION, FUNCTION SPACES

In the sequel, Ω denotes a bounded open domain in $\mathbb{R}^N$ with a Lipschitzian boundary. Partial derivatives with respect to x_i, $(x_1, \ldots, x_N) \in \Omega$ and $t \in \mathbb{R}^1$ are denoted by

∂_i, ∂_t, respectively. We introduce the following spaces: L^p_ω, $1 \leqq p \leqq \infty$: the Lebesgue space of all measurable ω-periodic function $v: \mathbb{R}^1 \to \mathbb{R}^1$ such that

$$|v|_p = (\int_0^\omega |v(t)|^p \, dt)^{1/p} < \infty \quad \text{for} \quad p < \infty$$

and

$$|v|_\infty = \sup \operatorname{ess} \{|v(t)|, t \in \mathbb{R}^1\} \quad \text{for} \quad p = \infty\,, \quad \text{with the norm} \quad |\cdot|_p\,;$$

C_ω: the Banach space of all continuous real ω-periodic functions with the norm $|\cdot|_\infty$; $L^p(\Omega; L^q_\omega)$, $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$: the space of all measurable functions: $u: \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ such that $u(x, \cdot) \in L^q_\omega$ for a.e. $x \in \Omega$ and

$$|u|_{p,q} = (\int_\Omega |u(x, \cdot)|_q^p \, dx)^{1/p} < \infty\,, \quad \text{with the norm} \quad |\cdot|_{p,q}\,;$$

for $p = q$ we write simply $L^p_\omega(\Omega)$; $L^p(\Omega; C_\omega)$, $1 \leqq p < \infty$: the subspace of all functions $u \in L^p(\Omega; L^\infty_\omega)$ such that $u(x, \cdot) \in C_\omega$ for almost all $x \in \Omega$.

The spaces $L^p(\Omega; L^q_\omega)$ are Banach spaces (cf. [6]), and the same is true for $L^p(\Omega; C_\omega)$, which is a closed subspace of $L^p(\Omega; L^\infty_\omega)$.

Let $\beta_1, \ldots, \beta_N$ be positive numbers, $\beta_0 = \min\{\beta_i, i = 1, \ldots, N\}$. We denote by Z the space of all $u \in L^{1+\beta_0}_\omega(\Omega)$ such that $\partial_t u \in L^2_\omega(\Omega)$, $\partial_i u$, $\partial_i \partial_t u \in L^{1+\beta_i}(\Omega; L^3_\omega)$, with the norm

$$|u|_Z = |u|_{1+\beta_0, 1+\beta_0} + |\partial_t u|_{2,2} + \sum_{i=1}^N (|\partial_i u|_{1+\beta_i, 3} + |\partial_i \partial_t u|_{1+\beta_i, 3})\,.$$

Let $\{e_k(x), k = 1, 2, \ldots\}$ be a complete system of eigenfunctions of the Laplacian in Ω with zero Dirichlet boundary condition on $\partial\Omega$, i.e.

$$\Delta e_k = -\lambda_k e_k\,, \quad e_k(x) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \quad 0 < \lambda_1 < \lambda_2 \leqq \ldots.$$

We define

$$(1.1) \qquad w_{jk}(x, t) = \begin{cases} \sin \dfrac{2\pi j}{\omega} t \, e_k(x)\,, & k \geqq 1\,, \quad j \geqq 1\,, \\ \cos \dfrac{2\pi j}{\omega} t \, e_k(x)\,, & k \geqq 1\,, \quad j \leqq 0\,. \end{cases}$$

Let us denote the closure of the linear hull of $\{w_{jk}, j \text{ integer}, k \geqq 1\}$ in Z by Z^0, the closure of $\operatorname{Lin}\{w_{jk}, j \neq 0, k \geqq 1\}$ in Z by $W^\perp$, and the closure of $\operatorname{Lin}\{w_{0k}, k \geqq 1\}$ in Z by W. We can identify W with the anisotropic Sobolev space $W_0^{1,1+\beta}(\Omega) = \{u \in L^{1+\beta_0}(\Omega);\ \partial_i u \in L^{1+\beta_i}(\Omega),\ u = 0 \text{ on } \partial\Omega\}$. We have $W^\perp = \{u \in Z^0, \int_0^\omega u(x, t)\, dt = 0\}$ and $Z^0 = W \oplus W^\perp$.

Notice that for $u \in Z$ we have $\partial_i u \in L^{1+\beta_i}(\Omega; C_\omega)$.

Let us recall a useful lemma for periodic functions which follows immediately from the Fubini theorem.

(1.2) Lemma. *Let $\varrho \in C_0^\infty(\mathbb{R}^1)$ be an odd function with support in $(-\omega/2, \omega/2)$. Then for each $f \in L^1_\omega$ we have* $\int_0^\omega \int_{-\infty}^\infty \varrho(s - t) f(s) f(t)\, dt\, ds = 0$.

Throughout the paper, c, c_k denote any independent positive constants.

Let $F_1, \ldots, F_N$ be Ishlinskiĭ operators (cf. [8], [2], [3]) with the following properties

(2.1) F_i is an odd continuous operator $C_\omega \to C_\omega$,

(2.2) $\varphi_i: (0, \infty) \to (0, \infty)$

are given twice continuously differentiable functions such that

(i) φ_i is increasing, $\varphi_i(0+) = 0$, $0 < \varphi_i'(0+) < +\infty$,

(ii) $\varphi_i(h) \leqq c_1 h^{\beta_i}$ for every $h > 0$, where $\beta_i \in (0,1)$

(iii) $\gamma_i(r) \geqq c_2 r^{\beta_i - 2}$ for $r \geqq r_0$, where

$$\gamma_i(r) = \inf\{-\varphi_i''(h), 0 < h \leqq r\},$$

(2.3) $|F_i(u) - F_i(v)|_\infty \leqq 2\varphi_i(|u - v|_\infty)$ for every $u, v \in C_\omega$,

(2.4) $\int_\omega^{2\omega} F_i(v)\, v''' \, dt \leqq -\frac{1}{4}\gamma_i(|v|_\infty) \int_0^\omega |v'|^3 \, dt$

for every $v \in C_\omega$ such that v'' is absolutely continuous;

(2.5) given $z \in \mathbb{R}^1$ and $v \in C_\omega$, the difference

$F_i(v + z)(t) - F_i(v)(t)$ is independent of t for $t \geqq \omega$. We have

$$\psi_i(v, z) \equiv F_i(v + z)(t) - F_i(v)(t) =$$
$$= \operatorname{sign}(\mu + z)\left[\varphi_i(\nu + |\mu + z|) - \varphi_i(\nu)\right] - \operatorname{sign}(\mu)\left[\varphi_i(\nu + |\mu|) - \varphi_i(\nu)\right],$$

where

$$\mu = \tfrac{1}{2}(\max v + \min v), \quad \nu = \tfrac{1}{2}(\max v - \min v).$$

The functions $\psi_i(v, \cdot)$ are continuously differentiable and for every $v \in C_\omega$, $z, z_1, z_2 \in \mathbb{R}^1$ we have

(i) $|\psi_i(v, z_1) - \psi_i(v, z_2)| \leqq 2\varphi_i(\frac{1}{2}|z_1 - z_2|)$,

(ii) $\partial/\partial z\, \varphi_i(v, z) \geqq \varphi_i'(|v|_\infty + |z|)$,

(iii) $\psi_i(v, 0) = 0$;

(2.6) let $u, v \in C_\omega$ be absolutely continuous. Then

$$\int_\omega^{2\omega} (F_i(u) - F_i(v))(u' - v')\, dt \geqq 0.$$

If moreover

$$\int_\omega^{2\omega} (F_i(u) - F_i(v))(u' - v')\, dt = 0, \quad \text{then} \quad u' = v' \quad \text{a.e.};$$

(2.7) for $u \in L^p(\Omega; C_\omega)$ we define $F_i(u)(x, t) = F_i(u(x, \cdot))(t)$

for a.e. $x \in \Omega$ and every $t \in \mathbb{R}^1$. We have

$$F_i(u) \in L^{p/\beta_i}(\Omega; C_\omega) \quad \text{and} \quad |F_i(u) - F_i(v)|_{p/\beta_i, \infty} \leqq c|u - v|_{p,\infty}^{\beta_i}$$

for each $u, v \in L^p(\Omega; C_\omega)$.

3. EXISTENCE AND UNIQUENESS THEOREM

(3.1) **Theorem.** *Let* $F = (F_1, \ldots, F_N)$ *satisfy* (2.1)–(2.7),

$$F(\text{grad } u) = (F_1(\partial_1 u), \ldots, F_N(\partial_N u)),$$

and let $G = (G_1, \ldots, G_N)$ *be such that* $G_i, \partial_t^2 G_i \in L^{1+1/\beta_i}(\Omega; L_\omega^{3/2})$. *Then there exists a unique* $u \in Z^0$ *such that for every* $z \in Z^0$ *we have*

$$\text{(3.2)} \qquad \int_\Omega \int_\omega^{2\omega} (-\partial_t u \, . \, \partial_t z + \langle F(\text{grad } u), \text{grad } z\rangle + \langle G, \text{grad } z\rangle) \, dt \, dx = 0,$$

where $\langle \cdot, \cdot \rangle$ *denotes the scalar product in* $\mathbb{R}^N$.

The method of the proof is classical (cf. e.g. [5]). We decompose u into $v + w$, where $v \in W^\perp$, $w \in W$ are solutions of auxiliary problems I, II.

Auxiliary problem I. Find $v \in W^\perp$ such that

$$\text{(3.3)} \qquad \int_\Omega \int_\omega^{2\omega} (-\partial_t v \, \partial_t z + \langle F(\text{grad } v) + G, \text{grad } z\rangle) \, dt \, dx = 0$$

for every $z \in W^\perp$.

(3.4) Lemma. *Let the assumptions of* (3.1) *be fulfilled. Then there exists a unique solution* $v \in W^\perp$ *to* (3.3).

Proof of (3.4). Put $v_m(x, t) = \sum_{k=1}^{m} \sum_{\substack{j=-m \\ j \neq 0}}^{m} v_{jk} \, w_{jk}(x, t)$, where v_{jk} satisfy

$$\text{(3.5)} \qquad \int_\Omega \int_\omega^{2\omega} (-\partial_t v_m \, \partial_t w_{jk} + \langle F(\text{grad } v_m) + G, \text{grad } w_{jk}\rangle) \, dt \, dx = 0,$$

$$k = 1, \ldots, m, \quad j = -m, \ldots, -1, 1, \ldots, m.$$

We first derive apriori estimates which ensure the existence of v_{jk} satisfying (3.5). We have

$$\int_\Omega \int_\omega^{2\omega} \langle F(\text{grad } v_m), \text{grad } \partial_t^3 v_m\rangle \, dt \, dx = \int_\Omega \int_0^\omega \langle G, \text{grad } \partial_t^3 v_m\rangle \, dt \, dx.$$

Using (2.4) and the relation $|\partial_i v_m(x, \cdot)|_\infty \leqq c_1 |\partial_i \, \partial_t v_m(x, \cdot)|_3$ we obtain

$$\text{(3.6)} \qquad \sum_{i=1}^{N} \int_\Omega \gamma_i(c_1 |\partial_i \, \partial_t v_m(x, \cdot)|_3) \, |\partial_i \, \partial_t v_m(x, \cdot)|_3^3 \, dx \leqq$$

$$\leqq \sum_{i=1}^{N} \int_\Omega |\partial_t^2 \, G_i(x, \cdot)|_{3/2} \, |\partial_i \, \partial_t \, v_m(x, \cdot)|_3 \, dx.$$

Putting $M_+^i = \{x \in \Omega; |\partial_i \, \partial_t v_m(x, \cdot)|_3 \geqq r_0/c_1\}$, where r_0 is defined in (2.2) (iii), $M_-^i = \Omega \setminus M_+^i$, we have $\int_{M^i_-} |\partial_i \, \partial_t v_m(x, \cdot)|_3^{1+\beta_i} \, dx \leqq (r_0/c_1)^{1+\beta_i} \text{ meas } \Omega$, and (3.6) yields

(3.7) (i) $|\partial_i \, \partial_t v_m|_{1+\beta_i,3} \leqq c$, hence

(ii) $|\partial_i v_m|_{1+\beta_i,\infty} \leqq c$,

(iii) $|F_i(\partial_i v_m)|_{1+1/\beta_i,\infty} \leqq c$.

Moreover, from (3.5) we derive

$$\int_\Omega \int_\omega^{2\omega} (-(\partial_t v_m)^2 + \langle F(\text{grad } v_m) + G, \text{grad } v_m\rangle) \, dt \, dx = 0 ,$$

hence

$$|\partial_t v_m|_{2,2} \leqq c , \quad |v_m|_{2,\infty} \leqq c . \tag{3.8}$$

The estimates (3.7), (3.8) imply the solvability of (3.5) (cf. e.g. [2]). Moreover, we find a subsequence $\{v_n\} \subset \{v_m\}$ and $v \in W^\perp$ such that $v_n \to v$, $\partial_t v_n \to \partial_t v$ in $L^2_\omega(\Omega)$ weak, $\partial_i \partial_t v_n \to \partial_i \partial_t v$ in $L^{1+\beta_i}(\Omega; L^3_\omega)$ weak. We can assume that $F_i(\partial_i v_n)$ is weakly convergent e.g. in $L^{1+1/\beta_i}_\omega(\Omega)$; we denote its weak limit by χ_i, $\chi = (\chi_1, \ldots, \chi_N)$. Passing to the limit in (3.5) we obtain

$$\int_\Omega \int_\omega^{2\omega} (-\partial_t v \, \partial_t z + \langle \chi + G, \text{grad } z\rangle) \, dt \, dx = 0 \tag{3.9}$$

for every $z \in W^\perp$. It remains to prove that v satisfies (3.3). We apply the Minty-Browder method (cf. [5]).

Let $\varrho \in C_0^\infty(\mathbb{R}^1)$ be a nonnegative even function,

$$\int_{-\infty}^{\infty} \varrho(s) \, ds = 1 , \quad \text{supp } \varrho \subset \left(-\frac{\omega}{2}, \frac{\omega}{2}\right).$$

For $\varepsilon \in (0, 1)$ we set

$$v_\varepsilon(x, t) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varrho\left(\frac{1}{\varepsilon}(t - s)\right) v(x, s) \, ds = \int_{-\infty}^{\infty} \varrho(s) \, v(x, t - \varepsilon s) \, ds . \tag{3.10}$$

Setting $z = \partial_t v_\varepsilon$ and using (1.2), we obtain from (3.9)

$$\int_\Omega \int_\omega^{2\omega} \langle \chi + G, \text{grad } \partial_t v_\varepsilon\rangle \, dt \, dx = 0 .$$

Since $\partial_i \partial_t v_\varepsilon \to \partial_i \partial_t v$ in $L^{1+\beta_i}_\omega(\Omega)$ strong as $\varepsilon \to 0+$, this identity implies

$$\int_\Omega \int_\omega^{2\omega} \langle \chi + G, \text{grad } \partial_t v\rangle \, dt \, dx = 0 .$$

On the other hand, (3.5) yields

$$\lim_{n\to\infty} \int_\Omega \int_\omega^{2\omega} \langle F(\text{grad } v_n), \text{grad } \partial_t v_n\rangle \, dt \, dx =$$

$$= - \int_\Omega \int_\omega^{2\omega} \langle G, \text{grad } \partial_t v\rangle \, dt \, dx = \int_\Omega \int_\omega^{2\omega} \langle \chi, \text{grad } \partial_t v\rangle \, dt \, dx .$$

In particular, for $z \in W^\perp$ we have (cf. (2.6))

$$\int_\Omega \int_\omega^{2\omega} \langle F(\text{grad } v_n) - F(\text{grad } z), \text{grad } \partial_t v_n - \text{grad } \partial_t z\rangle \, dt \, dx \geqq 0 .$$

Passing to the limit we obtain for $z = v - \varkappa w$, $\varkappa > 0$, $w \in W^\perp$

$$\int_\Omega \int_\omega^{2\omega} \langle \chi - F(\text{grad } v - \varkappa \text{ grad } w), \text{grad } \partial_t w\rangle \, dt \, dx \geqq 0 .$$

Consequently, for $\varkappa \to 0+$ we conclude (notice that $w \in W^\perp$ is arbitrary and F_i is continuous from $L^{1+\beta_i}(\Omega; C_\omega)$ into $L^{1+1/\beta_i}(\Omega; C_\omega)$)

$$\int_\Omega \int_\omega^{2\omega} \langle \chi - F(\text{grad } v), \text{grad } z\rangle \, dt \, dx = 0$$

for every $z \in W^{\perp}$, hence v satisfies (3.3). The uniqueness in Lemma (3.4) follows easily from (2.6). Indeed, let v^1, v^2 be two solutions of (3.3). We put $v = v^1 - v^2$ and $z = \partial_t v_\varepsilon$, where v_ε is given by (3.10). From (1.2) we obtain for $\varepsilon \to 0+$

$$\int_\Omega \int_\omega^{2\omega} \langle F(\text{grad } v^1) - F(\text{grad } v^2), \text{grad } \partial_t v^1 - \text{grad } \partial_t v^2 \rangle \, dt \, dx = 0$$

and (2.6) yields $v^1 = v^2$. The proof of (3.4) is complete.

Auxiliary problem II. Find $w \in W$ such that

(3.11) $\int_\Omega \langle \Psi(\text{grad } v(x, \cdot), \text{grad } w(x)), \text{grad } z(x) \rangle \, dx = \int_\Omega \langle \bar{G}(x), \text{grad } z(x) \rangle \, dx$

for every $z \in W$, where $\Psi(\text{grad } v(x, \cdot), \text{ grad } w(x)) = (\psi_1(\partial_1 v(x, \cdot), \partial_1 w(x)), \ldots$
$\ldots, \psi_N(\partial_N v(x, \cdot), \partial_N w(x)))$ (cf. (2.5)),

$$\bar{G}(x) = -\frac{1}{\omega} \int_\omega^{2\omega} (G(x, t) + F(\text{grad } v)(x, t)) \, dt ,$$

and v is the solution of (3.3).

(3.12) **Lemma.** *There exists a unique solution of* (3.11).

Proof of (3.12). The space W is reflexive. Let $((\cdot, \cdot))$ denote the duality between W and W^*. For $w, z \in W$ we denote by $((Tw, z))$ the left-hand side of (3.11). We verify that the mapping $T: W \to W^*$ thus defined is demicontinuous, bounded, strictly monotone and coercive. The demicontinuity, boundedness and monotonicity follow immediately from (2.5) (i)–(iii). To prove the coercivity of T we denote

$$M_1^i = \{x \in \Omega; |\partial_i w(x)| > \max\{r_0, |\partial_i v(x, \cdot)|_\infty\}\} ,$$
$$M_2^i = \{x \in \Omega, |\partial_i w(x)| \leqq |\partial_i v(x, \cdot)|_\infty\} ,$$
$$M_3^i = \Omega \setminus (M_1^i \cup M_2^i) ,$$

where r_0 is defined in (2.2) (iii). By (2.5) (ii) we have

$$\int_\Omega \psi_i(\partial_i v(x, \cdot), \partial_i w(x)) \, \partial_i w(x) \, dx \geqq$$
$$\geqq \int_\Omega |\partial_i w(x)|^2 \, \varphi_i'(|\partial_i v(x, \cdot)|_\infty + |\partial_i w(x)|) \, dx \geqq c_1 \int_{M_1^i} |\partial_i w(x)|^{1+\beta_i} \, dx .$$

On the other hand, $\int_{M^i_2 \cup M^i_3} |\partial_i w(x)|^{1+\beta_i} \, dx \leqq c$, hence for arbitrary $w \in W$ we obtain the inequality

$$((Tw, w)) \geqq c_2 \sum_{i=1}^{N} \int_\Omega |\partial_i w(x)|^{1+\beta_i} \, dx - c ,$$

which implies the coercivity of T. Lemma (3.12) follows now from the Minty-Browder theorem (cf. e.g. [1]).

Proof of Theorem (3.1). We put $u = v + w$, where $v \in W^{\perp}$, $w \in W$ are the solutions of (3.3) and (3.11), respectively. An easy verification of (3.2) completes the proof.

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Souhrn

METODA MONOTONIE PRO ŘEŠENÍ HYPERBOLICKÝCH ÚLOH S HYSTEREZÍ

Pomocí jisté verze Mintyho-Browderovy metody je dokázána existence a jednoznačnost slabého ω-periodického řešení rovnice $u_{tt} - \operatorname{div} F(\operatorname{grad} u) = g$ v omezené oblasti $\Omega \subset \mathbb{R}^N$ s okrajovou podmínkou $u = 0$ na $\partial\Omega$, kde g je zadaná (zobecněná) ω-periodická funkce a F je hysterezní operátor Išlinského.

Резюме

МЕТОД МОНОТОННОСТИ ДЛЯ РЕШЕНИЯ ГИПЕРБОЛИЧЕСКИХ ЗАДАЧ С ГИСТЕРЕЗИСОМ

PAVEL KREJČÍ

Некоторый вариант метода Минти-Браудера применяется к доказательству существования и единственности слабого ω-периодическогo решения уравнения $u_{tt} - \operatorname{div} F(\operatorname{grad} u) = g$ в ограниченной области $\Omega \subset R^N$ с краевым условием $u = 0$ на $\partial\Omega$, где g — заданная (обобщенная) ω-периодическая функция и F — гистерезисный оператор Ишлинского.

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