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SHAPE OPTIMIZATION OF AN ELASTIC-PERFECTLY PLASTIC BODY

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Abstract. Within the range of Prandtl-Reuss model of elasto-plasticity the following optimal design problem is solved. Given body forces and surface tractions, a part of the boundary, where the (two-dimensional) body is fixed, is to be found, so as to minimize an integral of the squared yield function. The state problem is formulated in terms of stresses by means of a time-dependent variational inequality. For approximate solutions piecewise linear approximations of the unknown boundary, piecewise constant triangular finite elements for stress and backward differences in time are used. Convergence of the approximations to a solution of the optimal design problem is proven. As a consequence, the existence of an optimal boundary is verified.

Keywords: domain optimization, time-dependent variational inequality, elasto-plasticity, finite elements

AMS Subject class.: 65 K 10, 65 N 30, 73 E 99.

INTRODUCTION

The present paper is a continuation of the research in optimization of two-dimensional elastic [1] and elasto-plastic [2] bodies. Whereas in [2] the model of Hencky (cf. [3]. [4]) has been considered, here we apply the constituent law of Prandtl-Reuss, which leads to a weak formulation in terms of a variational inequality of evolution [3], [4], [5].

Given body forces, surfaces loads and material characteristics of an elasto-plastic two-dimensional body, we have to find the shape of a part of its boundary such that a cost functional is minimized. The latter functional is an integral of the square of the yield function over the time-space domain. Zero displacements are prescribed on the unknown part of the boundary.

We use piecewise linear approximation of the boundary, backward differences in time and piecewise constant (external) approximations of the stress field. Employing also some ideas of C. Johnson ([5], [6]), we prove the convergence of the approximations to a solution of the original optimal design problem.

1. FORMULATION OF THE OPTIMAL DESIGN PROBLEM

Let us recall the basic relations of the elasto-plastic bodies obeying the Prandtl-Reuss law.

Let $\Omega \subset \mathbb{R}^2$ be a given (bounded) domain with Lipschitz boundary $\partial\Omega$. Assume that

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_g \,, \quad \Gamma_u \cap \Gamma_g = \emptyset \,,$$

where each of the parts Γ_u, Γ_g is open in $\partial\Omega$.

Let $\mathbb{R}_\sigma$ be the space of symmetric 2×2 matrices (stress or strain tensor). A repeated index implies the summation over the range 1, 2. We introduce the following inner product in the space $\mathbb{R}_\sigma$

$$\langle \sigma, \tau \rangle = \sigma_{ij}\tau_{ij} \,, \quad \langle \sigma, \sigma \rangle^{1/2} = \|\sigma\| \,.$$

Let a yield function $f\colon \mathbb{R}_\sigma \to \mathbb{R}$ be given, which is convex, Lipschitz and satisfies the condition

$$(1) \qquad f(\lambda\sigma) = |\lambda| f(\sigma) \quad \forall\sigma \in \mathbb{R}_\sigma \,, \quad \forall\lambda \in \mathbb{R} \,.$$

These assumptions are fulfilled e.g. by the well-known von Mises function

$$f(\sigma) = K \,.\, [\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2]^{1/2} \,, \quad (K = \text{cost.}).$$

We introduce the following spaces and notations:

$$S(\Omega) = \{\tau\colon \Omega \to \mathbb{R}_\sigma \mid \tau_{ij} \in L^2(\Omega) \ \forall_{i,j}\} \,,$$

$$\langle \sigma, \mathbf{e} \rangle_\Omega = \int_\Omega \langle \sigma, \mathbf{e} \rangle \,\mathrm{d}x \,, \quad \|\sigma\|_{0,\Omega} = \langle \sigma, \sigma \rangle_\Omega^{1/2}$$

for $\sigma, \mathbf{e} \in S(\Omega)$.

In $S(\Omega)$ we introduce also the energy scalar product

$$(\sigma, \tau)_\Omega = \langle b\sigma, \tau \rangle_\Omega \,, \quad \|\sigma\|_\Omega = (\sigma, \sigma)_\Omega^{1/2} \,,$$

where $b\colon S(\Omega) \to S(\Omega)$ is an isomorphism defined by the generalized Hooke's law

$$\mathbf{e} = b\sigma \Leftrightarrow e_{ij} = b_{ijkl}\sigma_{kl} \,, \quad \forall \mathrm{i,j} \,.$$

We assume that $b_{ijkl} \in L^\infty(\Omega)$, a positive constant b_0 exists such that

$$(2) \qquad b_0\sigma_{ij}\sigma_{ij} \leqq b_{ijkl}(x)\, \sigma_{ij}\sigma_{kl}$$

holds for almost all $\boldsymbol{x} \in \Omega$ and all $\sigma \in \mathbb{R}_\sigma$, and

$$b_{ijkl} = b_{klij} \,.$$

Then we have

$$(\sigma, \mathbf{e})_\Omega = \langle b\sigma, \mathbf{e} \rangle_\Omega = \langle \sigma, b\mathbf{e} \rangle_\Omega = \langle b\mathbf{e}, \sigma \rangle_\Omega = (\mathbf{e}, \sigma)_\Omega \,,$$

$$(3) \qquad b_0\|\sigma\|_{0,\Omega}^2 \leqq \|\sigma\|_\Omega^2 \leqq b_1\|\sigma\|_{0,\Omega}^2 \quad \forall\sigma \in S(\Omega) \,.$$

We consider a time interval $I = [0, T]$, $T < +\infty$.

Assume that the body forces $\mathbf{F}$ and the surface tractions $\mathbf{g}$ are of the following particular form:

$$\mathbf{F}(\mathbf{x}, t) = \gamma(t)\, \mathbf{F}^0(\mathbf{x})\,, \quad \mathbf{g}(\mathbf{x}, t) = \gamma(t)\, \mathbf{g}^0(\mathbf{x})\,,$$

where $\gamma \in C^2(I)$, $\gamma \geqq 0$, $\gamma(t) = 0$ in a "small" interval $[0, \gamma_0]$, $0 < \gamma_0 < T$.

We define the space of test functions

$$V(\Omega) = \{\mathbf{w} \in [H^1(\Omega)]^2 \mid \mathbf{w} = 0 \text{ on } \Gamma_u\}$$

and the set of statistically admissible stress field at the moment $t \in I$:

$$\mathscr{E}(\Omega; t) = \{\tau \in S(\Omega) \mid \langle \tau, \mathbf{e}(\mathbf{w})\rangle_\Omega = L_\Omega(\mathbf{w}, t)\ \forall\ \mathbf{w} \in V(\Omega)\}\,,$$

where

$$e(\mathbf{w})_{ij} = 2^{-1}(\partial w_i/\partial x_j + \partial w_j/\partial x_i)\,,$$

$$L_\Omega(\mathbf{w}, t) = \int_\Omega F_i(t)\, w_i \,\mathrm{d}x + \int_{\Gamma_g} g_i(t)\, w_i \,\mathrm{d}s\,.$$

The set of plastically admissible stress field is

$$P(\Omega) = \{\tau \in S(\Omega) \mid f(\tau) \leqq 1 \text{ a.e. in } \Omega\}\,.$$

Let us introduce the set

$$K(\Omega; t) = \mathscr{E}(\Omega; t) \cap P(\Omega)\,.$$

Let $C_0^1(I, S(\Omega))$ be the space of continuously differentiable functions on the interval I with values in $S(\Omega)$, which vanish at $t = 0$. We define $H_0^1(I, S(\Omega))$ as the closure of $C_0^1(I, S(\Omega))$ in the norm

$$\left(\int_0^T \|\dot{\sigma}\|_{0,\Omega}^2 \,\mathrm{d}t\right)^{1/2},$$

where $\dot{\sigma} \equiv \mathrm{d}\sigma/\mathrm{d}t$.

We observe that

$$\|\sigma(t) - \sigma(t_1)\|_{0,\Omega} \leqq |t - t_1|^{1/2}\, \|\sigma\|_{H_0^1(I,S(\Omega))} \tag{4}$$

is true for every $\sigma \in C_0^1(I, S(\Omega))$. Then continuity of functions from $H_0^1(I, S(\Omega))$ follows easily and (4) holds for all $\sigma \in H_0^1(I, S(\Omega))$.

Given a domain Ω we can define the *state problem*: find $\sigma \in H_0^1(I, S(\Omega))$ such that

$$\sigma(t) \in K(\Omega; t) \quad \forall t \in I \tag{5}$$

and

$$(\dot{\sigma}(t), \tau - \sigma(t))_\Omega \geqq 0 \tag{6}$$

holds for all $\tau \in K(\Omega; t)$ and almost all $t \in I$.

Throughout the paper, C will denote a positive constant not necessarily the same at each occurence.

Passing to the shape optimization problem, we introduce the following set of admissible design variables

$$U_{ad} = \{v \in C^{(0),1}([0, 1]) \text{ (i.e. Lipschitz functions)}, \quad \alpha \leqq v \leqq \beta\,, \quad |dv/dx_2| \leqq C_1\,, \quad \int_0^1 v \, dx_2 = C_2\}\,,$$

where α, β, C_1 and C_2 are given positive constants.

Throughout the paper, we shall consider a class of domains $\Omega = \Omega(v)$, where $v \in U_{ad}$ and

$$\Omega(v) = \{(x_1, x_2) \mid 0 < x_1 < v(x_2),\, 0 < x_2 < 1\}\,.$$

For any $v \in U_{ad}$, the graph $\Gamma(v)$ of the function v will coincide with the part $\bar{\Gamma}_u$ of $\partial\Omega(v)$. Assume that the constants b_0, b_1 in (3) do not depend on $v \in U_{ad}$.

The function v has to be determined from the following

Optimal Design Problem:

$$\mathscr{J}(\sigma(v)) = \min \tag{7}$$

over the set of $v \in U_{ad}$, where

$$\mathscr{J}(\sigma(v)) = \int_0^T dt \int_{\Omega(v)} f^2(\sigma(v)) \, dx\,, \tag{8}$$

and $\sigma(v)$ is the solution of the state problem (5), (6) on the domain $\Omega \equiv \Omega(v)$.

We first show that the above definition has sense, if we restrict the class of state problems by some assumptions imposed upon the loading forces.

Let $\delta > \beta$, $\Omega_\delta = (0, \delta) \times (0, 1)$, Γ_δ denote the graph of the (constant) function $v = \delta$. Assume that the reference forces $\mathbf{F}^0$ and $\mathbf{g}^0$ are defined on Ω_δ and $\partial\Omega_\delta - \Gamma_\delta$, respectively. Let them be such that there exists a stress field σ^0, satisfying the following conditions:

$$\sigma^0 \in S(\Omega_\delta) \cap [C^{(0),1}(\bar{\Omega}_\delta)]^4\,, \tag{9}$$

$$\langle \sigma^0, \mathbf{e}(\mathbf{w}) \rangle_{\Omega_\delta} = \int_{\Omega_\delta} F_i^0 w_i \, dx + \int_{\partial\Omega_\delta - \Gamma_\delta} g_i^0 w_i \, ds \quad \forall \mathbf{w} \in V(\Omega_\delta)\,; \tag{10}$$

$$\exists \varepsilon > 0 \quad \text{such that} \quad (1 + \varepsilon)\, \gamma(\bar{t})\, \sigma^0 \in P(\Omega_\delta)\,, \tag{11}$$

where $\bar{t} \in I$ is the argument, realizing the maximum of $\gamma(t)$ on the interval I.

We present the existence result, which is based on the paper by C. Johnson [5].

Theorem 1. *Let the assumptions* (9), (10), (11) *be satisfied. Then there exists a unique solution* $\sigma(v)$ *of the state problem* (5), (6) *on* $\Omega(v)$ *for any* $v \in U_{ad}$.

The Proof will not be given here. It is an easy generalization of the proof of C. Johnson, who considered only the case $\partial\Omega = \Gamma_u$. To verify the assumptions of his theorem, we show that

$$\gamma(t)\, \boldsymbol{\sigma}^0|_{\Omega(v)} \in \mathscr{E}(\Omega(v); t) \quad \forall t \in I, \quad \forall v \in U_{ad} \tag{12}$$

and $\gamma(\bar{t})\, \boldsymbol{\sigma}^0|_{\Omega(v)} \in K(\Omega(v); \bar{t})$ follows from (10), (11), (1).

Remark 1. Since f is Lipschitz and (1) holds, we may write

$$|f(\boldsymbol{\sigma})| = |f(\boldsymbol{\sigma}) - f(\mathbf{0})| \leqq C\|\boldsymbol{\sigma}\|,$$

$$\int_\Omega f^2(\boldsymbol{\sigma})\, dx \leqq C \int_\Omega \|\boldsymbol{\sigma}\|^2\, dx = C\|\boldsymbol{\sigma}\|^2_{0,\Omega} < +\infty \quad \forall \boldsymbol{\sigma} \in S(\Omega). \tag{13}$$

Consequently,

$$\mathscr{J}(\boldsymbol{\sigma}(v)) \leqq C \int_0^T \|\boldsymbol{\sigma}(v)\|^2_{0,\Omega(v)}\, dt < +\infty \quad \forall v \in U_{ad}$$

follows from (13), (4) and Theorem 1.

Remark 2. From the condition (10) we derive

$$\operatorname{div} \boldsymbol{\sigma}^0 + \boldsymbol{F}^0 = 0 \quad \text{in} \quad \Omega_\delta,$$
$$\boldsymbol{\sigma}^0 . \boldsymbol{v} = \boldsymbol{g}^0 \quad \text{on} \quad \partial\Omega_\delta - \Gamma_\delta.$$

The latter relation together with (9) implies that $\boldsymbol{g}^0$ is a Lipschitz function on any side of $\partial\Omega_\delta - \Gamma_\delta$. The condition (11) restricts the "magnitude" of the stress field $\boldsymbol{\sigma}^0$ in a certain sense.

2. APPROXIMATIONS BY PIECEWISE CONSTANT STRESS FIELDS

Let N be a positive integer and $h = 1/N$. We denote by Δ_j, $j = 1, \ldots, N$, the subintervals $[(j-1)h, jh]$ and introduce the set

$$U^h_{ad} = \{v_h \in U_{ad} | v_h|_{\Delta_j} \in P_1(\Delta_j) \ \forall j\},$$

where P_k denotes the set of polynomials of k-th degree.

Let Ω_h denote the domain $\Omega(v_h)$, bounded by the graph Γ_h of the function $v_h \in U^h_{ad}$. The domain Ω_h will be carved into triangles as follows.

We choose $\alpha_0 \in (0, \alpha)$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of v_h, if h is fixed.

In the remaining part $\Omega_h - \mathscr{R}$ let the nodal points divide the intervals $[\alpha_0, v_h(jh)]$ into M equal segments, where $M = 1 + [(\beta - \alpha_0) N]$ and the square brackets denote the integer part. Thus we obtain a *regular family of triangulations* $\{\mathscr{T}_h(v_h)\}$, $h \to 0$, $v_h \in U^h_{ad}$. Note that for any $v_h \in U^h_{ad}$ we construct a unique triangulation $\mathscr{T}_h(v_h)$.

Denoting the triangles of $\mathscr{T}_h(v_h)$ by $\mathscr{T}$, we define the finite element spaces

$$V_h(\Omega_h) = \{\mathbf{w}_h \in V(\Omega_h) \,|\mathbf{w}_h|_{\mathscr{T}} \in [P_1(\mathscr{T})]^2 \quad \forall \mathscr{T} \in \mathscr{T}_h(v_h)\} ,$$
$$\mathscr{H}_h(\Omega_h) = \{\tau \in S(\Omega_h) \,|\tau|_{\mathscr{T}} \in [P_0(\mathscr{T})]^4 \ \forall \mathscr{T} \in \mathscr{T}_h(v_h)\}$$

and external approximations of the set $\mathscr{E}(\Omega_h; t)$:

$$\mathscr{E}_h(\Omega_h; t) = \{\tau_h \in \mathscr{H}_h(\Omega_h) \mid \langle \tau_h, \mathbf{e}(\mathbf{w}_h)\rangle_{\Omega_h} = L_{\Omega_h}(\mathbf{w}_h, t) \ \forall \mathbf{w}_h \in V_h(\Omega_h)\} .$$

Let n be a positive integer, $k = T/n$,

$$\partial\sigma^m = (\sigma^m - \sigma^{m-1})/k\,, \quad m = 1, 2, \ldots, n\,,$$
$$t^m = mk\,, \quad \sigma^m = \sigma(t^m)\,,$$
$$K_h(\Omega_h; t^m) = \mathscr{E}_h(\Omega_h; t^m) \cap P(\Omega_h)\,.$$

Assume that γ is such that a uniform partion D^0 of I exists such that γ is monotone in every subinterval $[t^{m-1}, t^m]$ of D^0. Henceforth we consider only partitions, refining the partition D^0.

We define the *approximate state problem*:

find the array $\{\sigma_{hk}^1, \sigma_{hk}^2, \ldots, \sigma_{hk}^n\}$

such that for $m = 1, 2, \ldots, n$

(14) $$\sigma_{hk}^m \in K_h(\Omega_h; t^m)\,, \quad \sigma_{hk}^0 = 0\,,$$

(15) $$(\partial\sigma_{hk}^m, \tau - \sigma_{hk}^m)_{\Omega_h} \geqq 0 \quad \forall \tau \in K_h(\Omega_h; t^m)\,.$$

Lemma 1. *Assume that* (10), (11) *hold.*

Then the approximate state problem has a unique solution.

Proof. 1^0 We show that $K_h(\Omega_h, t^m) \neq \emptyset \ \forall m$. In fact, (12) yields $\gamma(t^m)\,\sigma^0|_{\Omega_h} \in \mathscr{E}(\Omega_h, t^m)$. From (11), the convexity of f and $f(\mathbf{0}) = 0$ we easily deduce that $\gamma(t^m)\,\sigma^0|_{\Omega_h} \in P(\Omega_h)$.

Let us introduce a projection mapping r_h: $S(\Omega_h) \to \mathscr{H}_h(\Omega_h)$ by means of the relation

(16) $$\langle \tau - r_h\tau, \sigma_h\rangle_{\Omega_h} = 0 \quad \forall \sigma_h \in \mathscr{H}_h(\Omega_h)\,.$$

Let us write for simplicity $\sigma^0|_{\Omega_h} \equiv \sigma^0$ and show that

(17) $$\gamma(t^m)\, r_h\sigma^0 = r_h(\gamma(t^m)\,\sigma^0) \in \mathscr{E}_h(\Omega_h, t^m)\,.$$

In fact, given a function $\mathbf{w}_h \in V_h(\Omega_h)$, we have $\mathbf{e}(\mathbf{w}_h) \in \mathscr{H}_h(\Omega_h)$ and $\mathbf{w}_h \in V(\Omega_h)$, so that

$$\langle \mathbf{e}(\mathbf{w}_h), \gamma(t^m)\, r_h\sigma^0\rangle_{\Omega_h} = \langle \mathbf{e}(\mathbf{w}_h), \gamma(t^m)\,\sigma^0\rangle_{\Omega_h} = L_{\Omega_h}(\mathbf{w}_h, t^m)\,.$$

Consequently, (17) holds.

Furthermore, we have

$$r_h\sigma^0 = (\operatorname{mes} \mathscr{T})^{-1} \int_{\mathscr{T}} \sigma^0 \, \mathrm{d}x \quad \forall \mathscr{T} \in \mathscr{T}_h(v_h)$$

and the convexity of f implies

(18) $$\gamma(t^m)\,\sigma^0 \in \mathcal{B} \text{ a.e.} \Rightarrow \gamma(t^m)\,r_h\sigma^0 \in \mathcal{B} \text{ a.e.},$$

where

$$\mathcal{B} = \{\tau \in \mathbb{R}_\sigma \mid f(\tau) \leqq 1\}\,.$$

Consequently, $\gamma(t^m)\,r_h\sigma^0 \in P(\Omega_h)$ follows from (11) and (18). Thus we obtain $\gamma(t^m)\,r_h\sigma^0 \in K_h(\Omega_h;\, t^m)$.

Every inequality (15) is equivalent with the minimization of the quadratic functional

$$\Phi(\tau) \equiv \tfrac{1}{2}\|\tau\|^2_{\Omega_h} - (\sigma^{m-1}_{hk}, \tau)_{\Omega_h}$$

over the set $K_h(\Omega_h;\, t^m)$. The functional is strictly convex, the set $K_h(\Omega_h;\, t^m)$ is non-empty, convex and closed in $S(\Omega_h)$. Hence the existence and uniqueness of σ^m_{hk} follows by induction scheme for $m = 1, 2, \ldots, n$.

Proposition 1. *Assume that* (9), (10), (11) *holds. Let* $\{v_h\}$, $h = 2^{-j}$, $j = 1, 2, \ldots$, *be a sequence of* $v_h \in U^h_{\text{ad}}$, *such that* $v_h \to v$ *in* $C([0, 1])$. *Let* $\{E\sigma^m_{hk}\}^n_{m=1}$ *be the solution of the problem* (14), (15), *extended by zero to the domain* $\Omega_\delta - \Omega_h$.

Then for any $m = 1, 2, \ldots, n$

(19) $$\lim_{h\to 0} E\sigma^m_{hk} = \sigma^m_k \quad in \quad S(\Omega_\delta)$$

and the functions σ^m_k *satisfy the following conditions:*

(20) $$\sigma^m_k = 0 \quad \text{on} \quad \Omega_\delta - \Omega(v)\,,$$

(21) $$\sigma^m_k|_{\Omega(v)} \in K(\Omega(v);\, t^m)\,,$$

(22) $$(\partial\sigma^m_k, \tau - \sigma^m_k)_{\Omega(v)} \geqq 0 \quad \forall \tau \in K(\Omega(v);\, t^m)\,,$$

with $\sigma^0_k = 0$.

Proof. 1° By an induction scheme, we prove that $\{E\sigma^m_{hk}\}_{h\to 0}$ is bounded. In fact, we may substitute $\tau = \gamma(t^m)\,r_h\sigma^0$ into the inequality (15). For $m = 1$ we obtain (cf. (3))

$$\|\sigma^1_{hk}\|_{\Omega_h} \leqq \gamma(\bar{t})\,.\,\|r_h\sigma^0\|_{\Omega_h} \leqq \gamma(\bar{t})\,.\,b_1^{1/2}\|r_h\sigma^0\|_{0,\Omega_h} \leqq C\|\sigma^0\|_{0,\Omega_\delta} = \tilde{C}\,.$$

Consequently

$$\|E\sigma^1_{hk}\|_{0,\Omega_\delta} \leqq b_0^{-1/2}\tilde{C}.$$

Assume that $E\sigma^{m-1}_{hk}$ are bounded by a constant, independent of h. Then we may write for $m > 1$ (dropping the indices hk for the time being)

$$\|\sigma^m\|^2_{\Omega_h} \leqq (\sigma^{m-1}, \sigma^m)_{\Omega_h} + (\tau, \sigma^m - \sigma^{m-1})_{\Omega_h} \leqq \|\sigma^m\|_{\Omega_h}\,\|\sigma^{m-1}\|_{\Omega_h} + \tilde{C}(\|\sigma^m\|_{\Omega_h} + \|\sigma^{m-1}\|_{\Omega_h}) \leqq C\|\sigma^m\|_{\Omega_h} + C_0\,.$$

We conclude that

(23) $$\|\sigma^m_{hk}\|_{\Omega_h} \leqq C\,, \quad m = 1, \ldots, n\,,$$

where C is independent of h.

Consequently, there exist a subsequence of $\{E\sigma_{hk}^m\}$, $h \to 0$, (we shall denote it by the same symbol) and a function $\sigma_k^m \in S(\Omega_\delta)$ such that

$$E\sigma_{hk}^m \to \sigma_k^m \quad \text{(weakly) in} \quad S(\Omega_\delta)\,, \quad \forall m\,. \tag{24}$$

2° We show that $\sigma_k^m = 0$ a.e. in $\Omega_\delta - \Omega(v)$. In fact, let $\sigma_k^m \neq 0$ on a set $M \subset \Omega_\delta \dot{-} \Omega(v)$, mes $M > 0$. Introducing the characteristic function χ_M of M, we obtain for $h \to 0$

$$\langle E\sigma_{hk}^m, \chi_M \sigma_k^m \rangle_{\Omega_\delta} \to \langle \sigma_k^m, \chi_M \sigma_k^m \rangle_{\Omega_\delta} = \|\sigma_k^m\|_{0,M}^2 > 0$$

by virtue of (24).

On the other hand,

$$|\langle E\sigma_{hk}^m, \chi_M \sigma_k^m \rangle_{\Omega_\delta}| = |\langle E\sigma_{hk}^m, \sigma_k^m \rangle_{\Omega_h \cap M}| \leqq$$
$$\leqq \|E\sigma_{hk}^m\|_{0,\Omega_\delta} \cdot \|\sigma_k^m\|_{0,\Omega_h \cap M} \to 0$$

follows from (23), if we realize that

$$\lim_{h \to 0} (\text{mes}\,(\Omega_h \cap M)) = 0\,.$$

Thus we arrive at a contradiction.

3° We show that

$$\sigma_k^m|_{\Omega(v)} \in K(\Omega(v); t^m) \quad \forall m\,. \tag{25}$$

Let $\boldsymbol{w} \in V(\Omega(v))$ and denote its extension by zero into $\Omega_\delta - \Omega(v)$ by $E\boldsymbol{w}$. There exists a sequence $\{\boldsymbol{w}_\varkappa\}$, $\varkappa \to 0$, such that

$$\boldsymbol{w}_\varkappa \in [C^\infty(\bar{\Omega}_\delta)]^2\,, \quad \boldsymbol{w}_\varkappa = 0 \quad \text{in} \quad \bar{\Omega}_\delta - \Omega(v)\,, \quad \text{supp} \quad \boldsymbol{w}_\varkappa \cap \Gamma(v) = \emptyset\,,$$
$$\boldsymbol{w}_\varkappa \to E\boldsymbol{w} \quad \text{in} \quad [H^1(\Omega_\delta)]^2 \quad \text{for} \quad \varkappa \to 0\,. \tag{26}$$

Obviously, $\boldsymbol{w}_\varkappa|_{\Omega_h} \in V(\Omega_h)$ for all sufficiently small h. Let us consider the interpolates $\pi_h \boldsymbol{w}_\varkappa \in V_h(\Omega_h)$ and denote their extensions by zero to $\Omega_\delta - \Omega_h$ by the same symbol. By definition of $\mathscr{E}_h(\Omega_h; t^m)$ we have

$$\langle \sigma_{hk}^m, \mathbf{e}(\pi_h \boldsymbol{w}_\varkappa) \rangle_{\Omega_h} = L_{\Omega_h}(\pi_h \boldsymbol{w}_\varkappa, t^m)\,,$$

which can be rewritten as follows

$$\langle E\sigma_{hk}^m, \mathbf{e}(\pi_h \boldsymbol{w}_\varkappa) \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\pi_h \boldsymbol{w}_\varkappa, t^m)\,. \tag{27}$$

Since

$$\pi_h \boldsymbol{w}_\varkappa \to \boldsymbol{w}_\varkappa \quad \text{for} \quad h \to 0 \quad \text{in} \quad [H^1(\Omega_\delta)]^2\,,$$

we have

$$\mathbf{e}(\pi_h \boldsymbol{w}_\varkappa) \to \mathbf{e}(\boldsymbol{w}_\varkappa) \quad \text{in} \quad S(\Omega_\delta)\,.$$

Passing to the limit with $h \to 0$ in (27) and using (24), we obtain

$$\langle \sigma_k^m, \mathbf{e}(\boldsymbol{w}_\varkappa) \rangle_{\Omega_\delta} = L_{\Omega_\delta}(\boldsymbol{w}_\varkappa, t^m)\,.$$

Passing to the limit with $\varkappa \to 0$ and using (26), we arrive at

$$\langle \sigma_k^m, \mathbf{e}(E\mathbf{w}) \rangle_{\Omega_\delta} = \langle \sigma_k^m, \mathbf{e}(\mathbf{w}) \rangle_{\Omega(v)} = L_{\Omega_\delta}(E\mathbf{w}, t^m) = L_{\Omega(v)}(\mathbf{w}, t^m) .$$

Consequently, $\sigma_k^m \in \mathscr{E}(\Omega(v); t^m)$.

Since $P(\Omega_\delta)$ is closed and convex in $S(\Omega_\delta)$, it is weakly closed.

Any $E\sigma_{hk}^m$ belongs to $P(\Omega_\delta)$ and hence the weak limit $\sigma_k^m \in P(\Omega_\delta)$. Then $\sigma_k^m|_{\Omega(v)} \in \in P(\Omega(v))$.

4° We show that the restrictions of σ_k^m solve the inequalities (22). Let $\tau \in K(\Omega(v); t^m)$ be given. First we construct a "shifted" function τ^λ on the domain $\Omega_\lambda \equiv \Omega(v + \lambda)$, where λ is a small positive constant.

Let us define:

$$\omega = \tau - \gamma(t^m)\, \sigma^0 ,$$

denote by $E\omega$ the extension of ω by zero to the negative half-plane $(x_1 < 0)$ and

$$\omega^\lambda(x_1, x_2) = E\omega(x_1 - \lambda, x_2) , \quad \mathbf{x} \in \Omega_\lambda .$$

We can show that

$$\langle \omega^\lambda, \mathbf{e}(\mathbf{w}) \rangle_{\Omega_\lambda} = 0 \quad \forall \mathbf{w} \in V(\Omega_\lambda) . \tag{28}$$

In fact, we use the coordinates

$$y_1 = x_1 - \lambda , \quad y_2 = x_2 \tag{29}$$

and define $\hat{\mathbf{w}}(\mathbf{y}) = \mathbf{w}(y_1 + \lambda, y_2) = \mathbf{w}(\mathbf{x})$. Then

$$\langle \omega^\lambda, \mathbf{e}(\mathbf{w}) \rangle_{\Omega_\lambda} = \int_{\Omega_\lambda} E\omega(x_1 - \lambda, x_2)\, \mathbf{e}(\mathbf{w}(\mathbf{x}))\, \mathrm{d}x =$$

$$= \int_{\Omega_\lambda *} E\omega(\mathbf{y})\, \mathbf{e}(\hat{\mathbf{w}}(\mathbf{y}))\, \mathrm{d}y = \int_{\Omega(v)} \omega(\mathbf{y})\, \mathbf{e}(\hat{\mathbf{w}}(\mathbf{y}))\, \mathrm{d}y = 0 .$$

Here we used the fact that $\hat{\mathbf{w}} \in V(\Omega(v))$, $\omega = \tau - \gamma(t^m)\, \sigma^0$, τ and $\gamma(t^m)\, \sigma^0$ belong to $\mathscr{E}(\Omega(v); t^m)$.

If we define

$$\tau^\lambda = \gamma(t^m)\, \sigma^0 + \varrho(\lambda)\, \omega^\lambda ,$$

where

$$\varrho(\lambda) = (1 - \sqrt{(\lambda)}/\varepsilon)/(1 + \sqrt{(\lambda)}) ,$$

then $\tau^\lambda \in \mathscr{E}(\Omega_\lambda; t^m)$ follows from (10) and (28).

Next we prove that $\tau^\lambda \in P(\Omega_\lambda)$. To this end we introduce

$$\sigma^\lambda(\mathbf{y}) = \gamma(t^m)\, \sigma^0(\mathbf{y}) + \varrho(\lambda)\, \omega(\mathbf{y}) , \quad \mathbf{y} \in \Omega(v) ,$$

where $\mathbf{y}$ is defined as a shift of $\mathbf{x}$ by (29).

For all $\mathbf{x} \in \Omega_\lambda \dot{-} (0, \lambda) \times (0, 1)$ we may write

$$\|\tau^\lambda(\mathbf{x}) - \sigma^\lambda(\mathbf{y})\| = \|\gamma(t^m)(\sigma^0(\mathbf{x}) - \sigma^0(\mathbf{y})\| \leqq \leqq \gamma(\bar{t}) \|\sigma^0(\mathbf{x}) - \sigma^0(\mathbf{y})\| \leqq C\|\mathbf{x} - \mathbf{y}\| = C\lambda, \tag{30}$$

making use of the assumption (9).

One can prove that

$$f((1 + \sqrt{\lambda})\, \sigma^\lambda(\mathbf{y})) = (1 + \sqrt{\lambda})\, f(\sigma^\lambda(\mathbf{y})) \leqq 1 \tag{31}$$

holds for sufficiently small λ and for almost all $\mathbf{y} \in \Omega(v)$. In fact, we have

$$(1 + \sqrt{\lambda})\, \sigma^\lambda = (1 + \sqrt{\lambda}) [\gamma(t^m)\, \sigma^0 + \varrho(\tau - \gamma(t^m)\, \sigma^0)] =$$

$$= \gamma(t^m)\, \sigma^0\, (1 + \varepsilon) \frac{\sqrt{\lambda}}{\varepsilon} + \tau \left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right)$$

and (31) holds for $\sqrt{(\lambda)} < \varepsilon$, since both $\gamma(t^m)(1 + \varepsilon)\, \sigma^0$ and τ belong to the set $P(\Omega(v))$.

Since f is Lipschitz, we may use (31) and (30) to derive

$$f(\tau^\lambda(\mathbf{x})) \leqq f(\sigma^\lambda(\mathbf{y})) + C\|\tau^\lambda(\mathbf{x}) - \sigma^\lambda(\mathbf{y})\| \leqq (1 + \sqrt{\lambda})^{-1} + \tilde{C}\lambda \leqq 1$$

for sufficiently small λ and for almost all $\mathbf{x} \in \Omega_\lambda \dot{-} (0, \lambda) \times (0, 1)$. In the strip $(0, \lambda) \times (0, 1)$ we have

$$\tau^\lambda = \gamma(t^m)\, \sigma^0$$

since $\omega^\lambda(\mathbf{x})$ vanishes. Then

$$f(\tau^\lambda(\mathbf{x})) = f(\gamma(t^m)\, \sigma^0(\mathbf{x})) \leqq 1$$

follows from the assumption (11). Thus we obtain

$$\tau^\lambda \in P(\Omega_\lambda) \cap \mathscr{E}(\Omega_\lambda; t^m) = K(\Omega_\lambda; t^m) \tag{32}$$

for λ sufficiently small.

Besides, we may write for $\lambda \to 0$

$$\|\tau^\lambda - \tau\|_{0,\Omega(v)} = \|\varrho(\lambda)\, \omega^\lambda - \omega\|_{0,\Omega(v)} \leqq \leqq \varrho(\lambda) \|\omega^\lambda - \omega\|_{0,\Omega(v)} + |\varrho(\lambda) - 1| \|\omega\|_{0,\Omega(v)} \to 0, \tag{33}$$

since

$$\lim \varrho(\lambda) = 1$$

and

$$\lim \|\omega^\lambda - \omega\|_{0,\Omega(v)} = 0$$

(cf. [7] – Theorem 1.1).

The function τ^λ will now be used to the construction of test functions in the approximate problem (15). It is obvious that $\Omega_h \subset \Omega_\lambda$ for all $h < h_0(\lambda)$. Then

$$\tau^\lambda|_{\Omega_h} \in \mathscr{E}(\Omega_h; t^m) \cap P(\Omega_h) = K(\Omega_h; t^m)$$

and

(34) $$r_h\tau^\lambda \in \mathscr{E}_h(\Omega_h; t^m) \cap P(\Omega_h) = K_h(\Omega_h; t^m)$$

(cf. the proof of Lemma 1).

Let $\Omega_{\lambda H}$ be a polygonal domain inscribed into Ω_λ (i.e., $\Omega_{\lambda H} \subset \Omega_\lambda$) and such that the following two conditions

(i) $$\Omega_h \subset \Omega_{\lambda H} ,$$

(ii) the partitions D_h of the interval $[0, 1]$ refine the partition D_H, (i.e., H is a multiple of h),

hold for the sequence of h under consideration, provided $h < h_1(\lambda)$.

Let us consider extended regular triangulations

$$\mathscr{T}_{hH} \supset \mathscr{T}_h(v_h)$$

of the domain $\Omega_{\lambda H}$ and the projection mapping

$$r_h^{\lambda H}: S(\Omega_{\lambda H}) \to \mathscr{H}_h(\Omega_{\lambda H})$$

defined on $\mathscr{T}_{hH}$ by means of the relation parallel to (16). Obviously, $r_h^{\lambda H}\tau^\lambda$ is an extension of $r_h\tau^\lambda$ onto $\Omega_{\lambda H}$.

By definition (15) and using (34), we may write

$$(\partial\sigma_{hk}^m, r_h\tau^\lambda - \sigma_{hk}^m)_{\Omega_h} \geqq 0 , \quad m = 1, 2, \ldots, n ,$$

which is equivalent to

(35) $$(\sigma_{hk}^m, r_h\tau^\lambda)_{\Omega_h} - (\sigma_{hk}^{m-1}, r_h\tau^\lambda)_{\Omega_h} - \|\sigma_{hk}^m\|_{\Omega_h}^2 + (\sigma_{hk}^{m-1}, \sigma_{hk}^m)_{\Omega_h} \geqq 0 .$$

First let us consider $m = 1$. Since $\sigma_{hk}^0 = 0$, we obtain

(36) $$(\sigma_{hk}^1, r_h\tau^\lambda)_{\Omega_h} \geqq \|\sigma_{hk}^1\|_{\Omega_h}^2 .$$

Passing to the limit with $h \to 0$, using (20), (24) and

(37) $$\lim_{h\to 0} \|r_h^{\lambda H}\tau^\lambda - \tau^\lambda\|_{0,\Omega_{\lambda H}} = 0 ,$$

we deduce that

(38) $$(\sigma_{hk}^1, r_h\tau^\lambda)_{\Omega_h} = (E\sigma_{hk}^1, r_h^{\lambda H}\tau^\lambda)_{\Omega_{\lambda H}} \to (\sigma_k^1, \tau^\lambda)_{\Omega_{\lambda H}} = (\sigma_k^1, \tau^\lambda)_{\Omega(v)} .$$

The weak convergence (24) and (20) imply

(39) $$\liminf_{h\to 0} \|\sigma_{hk}^1\|_{\Omega_h}^2 \geqq \|\sigma_k^1\|_{\Omega(v)}^2 .$$

To prove the strong convergence (19), we insert $\tau = \sigma_k^1 \in K(\Omega(v); t^1)$ (cf. (25)) into the previous argument. We obtain – as in (36) –

$$\|E\sigma_{hk}^1\|_{\Omega_\delta}^2 \leqq (\sigma_{hk}^1, r_h(\sigma_k^1)^\lambda)_{\Omega_h} .$$

Passing to the limit with $h \to 0$, we arrive at

$$\limsup_{h \to 0} \|E\sigma_{hk}^1\|_{\Omega_\delta}^2 \leqq (\sigma_k^1, (\sigma_k^1)^\lambda)_{\Omega(v)}\,.$$

Passing to the limit with $\lambda \to 0$ and using (33), we obtain

$$\limsup_{h \to 0} \|E\sigma_{hk}^1\|_{\Omega_\delta}^2 \leqq \|\sigma_k^1\|_{\Omega(v)}^2\,.$$

Combining this result with (39), we are led to

$$\lim_{h \to 0} \|E\sigma_{hk}^1\|_{\Omega_\delta}^2 = \|\sigma_k^1\|_{\Omega_\delta}^2\,.$$

Together with the weak convergence (24) and the equivalence of norms we thus obtain that

$$E\sigma_{hk}^1 \to \sigma_k^1$$

(strongly) in $S(\Omega_\delta)$ for $h \to 0$.

Let us assume that for $m > 1$

$$\lim_{h \to 0} E\sigma_{hk}^{m-1} = \sigma_k^{m-1} \quad \text{in} \quad S(\Omega_\delta) \quad (\text{strongly})\,. \tag{40}$$

Passing to the limit with $h \to 0$ in (35), we obtain

$$(\sigma_{hk}^m, r_h\tau^\lambda)_{\Omega_h} = (E\sigma_{hk}^m, r_h^{\lambda H}\tau^\lambda)_{\Omega_{hH}} \to (\sigma_k^m, \tau^\lambda)_{\Omega(v)} \tag{41}$$

using also (24) and (20). The same argument yields

$$(\sigma_{hk}^{m-1}, r_h\tau^\lambda)_{\Omega_h} \to (\sigma_k^{m-1}, \tau^\lambda)_{\Omega(v)}\,. \tag{42}$$

Moreover, from (40) and (24), (20) we get

$$(\sigma_{hk}^{m-1}, \sigma_{hk}^m)_{\Omega_h} = (E\sigma_{hk}^{m-1}, E\sigma_{hk}^m)_{\Omega_\delta} \to (\sigma_k^{m-1}, \sigma_k^m)_{\Omega(v)}\,. \tag{43}$$

The weak convergence (24) implies

$$\liminf_{h \to 0} \|\sigma_{hk}^m\|_{\Omega_h}^2 \geqq \|\sigma_k^m\|_{\Omega(v)}^2\,. \tag{44}$$

Combining (41), (42), (43) and (44), we arrive at

$$(\sigma_k^m, \tau^\lambda)_{\Omega(v)} - (\sigma_k^{m-1}, \tau^\lambda)_{\Omega(v)} + (\sigma_k^{m-1}, \sigma_k^m)_{\Omega(v)} \geqq \|\sigma_k^m\|_{\Omega(v)}^2\,.$$

To prove the strong convergence (19), we insert $\tau = \sigma_k^m$ into the previous argument. We thus obtain, on the basis of (35),

$$(\sigma_{hk}^m, r_h(\sigma_k^m)^\lambda)_{\Omega_h} + (\sigma_{hk}^{m-1}, \sigma_{hk}^m - r_h(\sigma_k^m)^\lambda)_{\Omega_h} \geqq \|\sigma_{hk}^m\|_{\Omega_h}^2\,.$$

Passing to the limit with $h \to 0$ and using (41), (42), (43), we may write

$$\limsup_{h \to 0} \|E\sigma_{hk}^m\|_{\Omega_\delta}^2 \leqq (\sigma_k^m, (\sigma_k^m)^\lambda)_{\Omega(v)} + (\sigma_k^{m-1}, \sigma_k^m - (\sigma_k^m)^\lambda)_{\Omega(v)}\,.$$

Passing to the limit with $\lambda \to 0$ and using (33), we obtain

$$\limsup \|E\sigma_{hk}^m\|_{\Omega_\delta}^2 \leqq \|\sigma_k^m\|_{\Omega(v)}^2 .$$

Combining this result with (44), we arrive at

$$\lim_{h\to 0} \|E\sigma_{hk}^m\|_{\Omega_\delta}^2 = \|\sigma_k^m\|_{\Omega(v)}^2 . \tag{45}$$

From the weak convergence (24) and the convergence of norms we deduce the strong convergence

$$E\sigma_{hk}^m \to \sigma_k^m \tag{46}$$

in $S(\Omega_\delta)$ for $h \to 0$. By induction we thus conclude that (46) holds for all $m = 1, 2, \ldots, n$.

Finally, let us write (35) in the form

$$0 \leqq (E\sigma_{hk}^m - E\sigma_{hk}^{m-1}, r_h^{\lambda H}\tau^\lambda - E\sigma_{hk}^m)_{\Omega_{\lambda H}}$$

and pass to the limit with $h \to 0$. By virtue of (46) and (37)

$$0 \leqq (\sigma_k^m - \sigma_k^{m-1}, \tau^\lambda - \sigma_k^m)_{\Omega(v)} .$$

Passing to the limit with $\lambda \to 0$, on the basis of (33) we obtain the inequality (22).

It is easy to show that there exists a unique array $\{\sigma_k^1, \sigma_k^2, \ldots, \sigma_k^n\}$, satisfying (21), (22). In fact, every inequality (22) is equivalent with minimization of the following strictly convex functional

$$\Phi(\sigma) \equiv \tfrac{1}{2}\|\sigma\|_{\Omega(v)}^2 - (\sigma_k^{m-1}, \sigma)_{\Omega(v)}$$

on a convex closed and non-empty set $K(\Omega(v); t^m)$. From the uniqueness we conclude that the *whole* original sequence $\{E\sigma_{hk}^m\}$, $h \to 0$, tends to σ_k^m in $S(\Omega_\delta)$, for any $m = 1, 2, \ldots, n$. Q.E.D.

Proposition 2. *Assume that* (9), (10), (11) *holds. Let σ be the solution of the state problem* (5), (6) *and let the array $\{\sigma_k^m\}_{m=1}^n$ be the solution of the semi-discrete problem* (21), (22) *on the domain $\Omega \leqq \Omega(v)$.*

Then

$$\max_{1\leqq m\leqq n} \|\sigma_k^m - \sigma(t^m)\|_{0,\Omega} \leqq Ck^{1/2}$$

holds for sufficiently small time-steps k.

Proof. We shall follow some ideas of C. Johnson [5], [6], (cf. also [8]). The key role in the proof is played by the following

Lemma 2. *Let the assumptions of Proposition 2 be fulfilled. Then there exist positive constants C and k_0 such that*

$$\sum_{m=1}^{n} k\|\partial\sigma_k^m\|_{0,\Omega}^2 \leqq C \quad \forall k < k_0 .$$

To prove the lemma, we shall introduce a penalized semi-discrete problem first, as follows. We define the projection mapping $\pi\colon \mathbb{R}_\sigma \to \mathscr{B}$ onto the convex set $\mathscr{B}$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ and the penalty functional

$$J_\mu(\tau) = (2\mu)^{-1}\|\tau - \pi\tau\|^2_{0,\Omega}\,, \quad \tau \in S(\Omega)\,, \quad \mu > 0\,.$$

We shall consider the penalized semi-discrete problem: find the array $\{\sigma^1_{k\mu}, \sigma^2_{k\mu}, \ldots, \sigma^n_{k\mu}\}$ such that $\sigma^m_{k\mu} \in \mathscr{E}(\Omega; t^m)$,

$$(47) \qquad (\partial\sigma^m_{k\mu}, \tau)_\Omega + \langle J'_\mu(\sigma^m_{k\mu}), \tau\rangle_\Omega = 0 \quad \forall\tau \in \mathscr{E}_0\,, \quad m = 1, \ldots, n\,,$$

where $\sigma^0_{k\mu} = 0$,

$$\mathscr{E}_0 = \{\tau \in S(\Omega) \mid \langle\tau, \mathbf{e}(\mathbf{w})\rangle_\Omega = 0 \ \forall \mathbf{w} \in V(\Omega)\}$$

and

$$J'_\mu(\tau) = \frac{1}{\mu}(\tau - \pi\tau)$$

is the Gâteau derivative of J_μ; note that J'_μ is monotone and J_μ is convex.

The problem (47) has a unique solution for every m, since $\sigma^m_{k\mu}$ minimizes the strictly convex, coercive and continuous functional

$$\Phi(\sigma) = \tfrac{1}{2}\|\sigma\|^2_\Omega + k\,J_\mu(\sigma) - (\sigma^{m-1}, \sigma)_\Omega$$

on the set $\mathscr{E}(\Omega; t^m)$, which is closed and convex in $S(\Omega)$. By the technique of C. Johnson [5] the following a priori estimates for $\sigma^m_{k\mu}$ can be proven. Positive constants C and k_0 exist such that

$$(48) \qquad \max_{1\leqq m\leqq n}\|\sigma^m_{k\mu}\|_\Omega \leqq C\,,$$

$$(49) \qquad \sum_{m=1}^{n} k\,J_\mu(\sigma^m_{k\mu}) \leqq C\,,$$

$$(50) \qquad \sum_{m=1}^{n} k\|J'_\mu(\sigma^m_{k\mu})\|_{L^1(\Omega)} \leqq C\,,$$

hold for all $k \leqq k_0$ and any $\mu > 0$. Here

$$\|Z\|_{L^1(\Omega)} = \int_\Omega \|Z\|\,\mathrm{d}x\,.$$

Making use of (50), one derives that positive constants C and k_1 exist such that

$$(51) \qquad \sum_{m=1}^{n} k\|\partial\sigma^m_{k\mu}\|^2_{0,\Omega} \leqq C$$

holds for all $k \leqq k_1$ and all $\mu > 0$.

Let us consider a sequence $\mu \to 0$, $\mu > 0$. From (48) it follows that

$$(52)\qquad \|\sigma_{k\mu}\|_{l^2} \equiv \Big(\sum_{m=1}^{n} k\|\sigma_{k\mu}^m\|_{0,\Omega}^2\Big)^{1/2} \leqq C$$

for all $k \leqq k_0$ and $\mu > 0$.

Hence a subsequence of $\{\mu\}$ exists such that

$$(53)\qquad \sigma_{k\mu} \rightharpoonup \bar{\sigma}_k \quad \text{(weakly) for} \quad \mu \to 0$$

in the space $l^2(S)$ of n-arrays, equipped with the norm $\|\cdot\|_{l^2}$, introduced in (52).

Similarly, (51) yields the existence of a subsequence such that

$$(54)\qquad \partial\sigma_{k\mu} \rightharpoonup S_k \quad \text{(weakly) for} \quad \mu \to 0 \quad \text{in} \quad l^2(S)\,.$$

It is easy to verify that $S_k = \partial\bar{\sigma}_k$.

Next we show that $\bar{\sigma}_k$ is a solution of the semi-discrete problem (21), (22). Since J_μ is convex, we have

$$(55)\qquad J_\mu(\tau^m) \geqq J_\mu(\sigma_{k\mu}^m) + \langle J'_\mu(\sigma_{k\mu}^m), \tau^m - \sigma_{k\mu}^m\rangle_\Omega\,.$$

If $\tau^m \in \mathscr{E}(\Omega; t^m)$, then $\tau^m - \sigma_{k\mu}^m \in \mathscr{E}_0$ and we may use the equation (47) to obtain (we again drop the subscripts)

$$\langle J'_\mu(\sigma^m), \tau^m - \sigma^m\rangle_\Omega = -(\partial\sigma^m, \tau^m - \sigma^m)_\Omega\,.$$

Combining this with (55), we may write

$$(\partial\sigma^m, \tau^m - \sigma^m)_\Omega + J_\mu(\tau^m) - J_\mu(\sigma^m) \geqq 0\,.$$

Let us consider $\tau^m \in K(\Omega, t^m)$, so that $J_\mu(\tau^m) = 0$ and

$$(\partial\sigma_{k\mu}^m, \tau^m - \sigma_{k\mu}^m)_\Omega \geqq J_\mu(\sigma_{k\mu}^m) \geqq 0 \quad \forall\tau^m \in K(\Omega; t^m)\,.$$

On the basis of (53), (54) we deduce for any $M = 1, 2, \ldots, n$

$$(56)\qquad 0 \leqq \limsup_{\mu\to 0}\Big[-\sum_{m=1}^{M} k(\partial\sigma_{k\mu}^m, \sigma_{k\mu}^m)_\Omega + \sum_{m=1}^{M} k(\partial\sigma_{k\mu}^m, \tau^m)_\Omega\Big] =$$

$$= \limsup_{\mu\to 0}\Big[-\tfrac{1}{2}\|\sigma_{k\mu}^M\|_\Omega^2 - \tfrac{1}{2}\sum_{m=1}^{M}\|\sigma_{k\mu}^m - \sigma_{k\mu}^{m-1}\|_\Omega^2 + \sum_{m=1}^{M} k(\partial\sigma_{k\mu}^m, \tau^m)_\Omega\Big] \leqq$$

$$\leqq -\tfrac{1}{2}\|\bar{\sigma}_k^M\|_\Omega^2 - \tfrac{1}{2}\sum_{m=1}^{M}\|\bar{\sigma}_k^m - \bar{\sigma}_k^{m-1}\|_\Omega^2 + \sum_{m=1}^{M} k(\partial\bar{\sigma}_k^m, \tau^m)_\Omega =$$

$$= \sum_{m=1}^{M} k(\partial\bar{\sigma}_k^m, \tau^m - \bar{\sigma}_k^m)_\Omega\,.$$

We can show that $\bar{\sigma}_k^m \in K(\Omega; t^m)$ $\forall m$. Recall that $\sigma_{k\mu}^m \in \mathscr{E}(\Omega; t^m)$ and $\mathscr{E}(\Omega; t^m)$ is weakly closed in $S(\Omega)$. From (53) it follows easily that $\sigma_{k\mu}^m \rightharpoonup \bar{\sigma}_k^m$ (weakly) in $S(\Omega)$. Consequently, $\bar{\sigma}_k^m \in \mathscr{E}(\Omega; t^m)$.

Making use of the estimate (49), we obtain

$$C \geqq k\, J_\mu(\sigma_{k\mu}^m) = (2\mu)^{-1}\, k\|\sigma_{k\mu}^m - \pi\sigma_{k\mu}^m\|_{0,\Omega}^2\,.$$

Then

$$\|\bar{\sigma}_k^m - \pi\bar{\sigma}_k^m\|_{0,\Omega}^2 \leqq \liminf_{\mu\to 0} \|\sigma_{k\mu}^m - \pi\sigma_{k\mu}^m\|_{0,\Omega}^2 \leqq \lim 2\mu C k^{-1} = 0$$

and consequently, $\bar{\sigma}_k^m \in P(\Omega)$.

We may thus insert $\tau^m = \bar{\sigma}_k^m$ into (56) for all $m < M$, if $M > 1$, to obtain

$$(\partial\bar{\sigma}_k^M, \tau - \bar{\sigma}_k^M)_\Omega \geqq 0 \quad \forall\tau \in K(\Omega, t^M), \quad M > 1 .$$

The case $M = 1$ follows from (56) immediately.

Since the problem (21), (22) is uniquely solvable, we have $\bar{\sigma}_k^m = \sigma_k^m \ \forall m$.

Finally, making use of (54), we arrive at

$$\sum_{m=1}^{n} k\|\partial\sigma_k^m\|_{0,\Omega}^2 \leqq \liminf_{\mu\to 0} \sum_{m=1}^{n} k\|\partial\sigma_{k\mu}^m\|_{0,\Omega}^2 \leqq C .$$

Thus the Lemma 2 is proved. Modifying slightly the argument of C. Johnson ([6] – Theorem 1.) we are now able to prove the Proposition 2. Q.E.D.

Theorem 2. *Let the assumptions of Proposition* 1 *be satisfied. Let the array* $\sigma_{hk} \equiv \{\sigma_{hk}^1, \sigma_{hk}^2, \ldots, \sigma_{hk}^n\}$ *be the solution of the approximate state problem* (14), (15).

Let us define

$$\mathscr{J}_{hk}(\sigma_{hk}) = k \sum_{j=1}^{n} c_j \int_{\Omega_h} f^2(\sigma_{hk}^j)\,\mathrm{d}x ,$$

where c_j are the coefficients of the trapezoidal or Simpson's rule.

Then an increasing positive function $H(k)$ exists such that $\lim_{k\to 0} H(k) = 0$,

$$\lim_{\substack{k\to 0\\ h\leqq H(k)}} \mathscr{J}_{hk}(\sigma_{hk}) = \mathscr{J}(\sigma(v)) ,$$

where $\sigma(v)$ is the solution of the state problem (5), (6), *on the domain $\Omega(v)$.*

Proof. Using the extensions $E\sigma_{hk}$ and $\sigma = \sigma(v)$ by zero to the domain $\Omega_\delta \dot{-} \Omega_h$ and $\Omega_\delta \dot{-} \Omega(v)$, respectively, we may write

$$\mathscr{J}(\sigma) = \int_0^T F(t)\,\mathrm{d}t , \quad F(t) = \int_{\Omega_\delta} f^2(\sigma(t)\,\mathrm{d}x ,$$

$$\mathscr{J}_{hk}(\sigma_{hk}) = k \sum_{j=1}^{n} c_j F_{hk}^j , \quad F_{hk}^j = \int_{\Omega_\delta} f^2(E\sigma_{hk}^j)\,\mathrm{d}x .$$

Denoting $\sigma(t^j)$ by σ^j, we also have

$$|f^2(E\sigma_{hk}^j) - f^2(\sigma^j)| \leqq |f(E\sigma_{hk}^j) - f(\sigma^j)|\,|f(E\sigma_{hk}^j) + f(\sigma^j)| \leqq$$
$$\leqq C\|E\sigma_{hk}^j - \sigma^j\|\,(2|f(\sigma^j)| + C\|E\sigma_{hk}^j - \sigma^j\|) .$$

Since $f(\sigma^j) \leqq 1$ a.e. in Ω_δ, we obtain

$$|F^j_{hk} - F(t^j)| \leqq \int_{\Omega_\delta} |f^2(E\sigma^j_{hk}) - f^2(\sigma^j)| \,\mathrm{d}x \leqq$$

$$\leqq C\left[\int_{\Omega_\delta} \|E\sigma^j_{hk} - \sigma^j\| \,\mathrm{d}x + \int_{\Omega_\delta} \|E\sigma^j_{hk} - \sigma^j\|^2 \,\mathrm{d}x\right] \leqq$$

$$\leqq C[\|E\sigma^j_{hk} - \sigma^j\|_{0,\Omega_\delta} + \|E\sigma^j_{hk} - \sigma^j\|^2_{0,\Omega_\delta}] .$$

Making use of Proposition 1 and 2, we may write

$$\|E\sigma^j_{hk} - \sigma^j\|_{0,\Omega_\delta} \leqq \|E\sigma^j_{hk} - \sigma^j_k\|_{0,\Omega_\delta} + \|\sigma^j_k - \sigma^j\|_{0,\Omega_\delta} \to 0 \tag{57}$$

for $k \to 0$, $h \leqq H(k) \to 0$.

Consequently,

$$\lim_{\substack{k\to 0\\ h\to 0, h\leqq H(k)}} |F^j_{hk} - F(t^j)| = 0 \tag{58}$$

holds uniformly with respect to j.

Next we show that the function F is continuous on I. In fact, let $t, s \in I$. Then

$$|F(s) - F(t)| \leqq C[\|\sigma(s) - \sigma(t)\|_{0,\Omega_\delta} + \|\sigma(s) - \sigma(t)\|^2_{0,\Omega_\delta}]$$

and the continuity is a consequence of (4).

For any $F \in C(I)$ it holds

$$\lim_{k\to 0} \left|\int_0^T F(t) \,\mathrm{d}t - k\sum_{j=0}^n c_j F(t^j)\right| = 0 . \tag{59}$$

Finally, we write

$$|\mathscr{J}(\sigma(v)) - \mathscr{J}_{hk}(\sigma_{hk})| \leqq \left|\int_0^T F(t) \,\mathrm{d}t - k\sum_{j=1}^n c_j F(t^j)\right| + \left|k\sum_{j=1}^n c_j(F(t^j) - F^j_{hk})\right| . \tag{60}$$

On the basis of (58), for the last term we have the upper bound

$$k\sum_{j=1}^n c_j|F(t^j) - F^j_{hk}| \leqq Cnk\varepsilon_1 = CT\varepsilon_1 \quad \forall\varepsilon_1 > 0 , \tag{61}$$

if k and $h = h(k)$ are small enough.

Combining (59) and (61), the assertion of the theorem follows from (60). Q.E.D.

We define the *Approximate Optimal Design Problem*: find $u_h^{(k)} \in U^h_{\mathrm{ad}}$ such that

$$\mathscr{J}_{hk}(\sigma_{hk}(u_h^{(k)})) \leqq \mathscr{J}_{hk}(\sigma_{hk}(v_h)) \quad \forall v_h \in U^h_{\mathrm{ad}} , \tag{62}$$

where $\sigma_{hk}(u_h^{(k)})$ denotes the solution of the approximate state problem (14), (15) on the domain $\Omega_h \equiv \Omega(u_h^{(k)})$.

Lemma 3. *Assume that* (10) *and* (11) *hold. Then the Approximate Optimal Design Problem* (62) *has a solution for any* $h = 1/N$ *and* $k = T/n$.

Proof. Denoting by $\boldsymbol{a} \in \mathbb{R}^{N+1}$ the vector of nodal values

$$v_h(ih) = a_i\,, \quad i = 0, 1, \ldots, N\,,$$

it is readily seen that

$$v_h \in U^h_{\text{ad}} \Leftrightarrow \boldsymbol{a} \in \mathscr{A}\,,$$

where $\mathscr{A}$ is a compact set.

One can prove that the function

$$\boldsymbol{a} \mapsto \mathscr{I}_{hk}(\sigma_{hk}(\boldsymbol{a}))$$

is continuous on the set $\mathscr{A}$. In fact, the conditions

$$\langle \mathbf{e}(\mathbf{w}_h), \sigma^m_{hk} \rangle_{\Omega_h(\boldsymbol{a})} = L_{\Omega_h(\boldsymbol{a})}(\mathbf{w}_h, t^m) \quad \forall \mathbf{w}_h \in V_h(\Omega_h(\boldsymbol{a}))$$

are equivalent with linear systems

$$A^m(\boldsymbol{a})\mathbf{s}^m = F^m(\boldsymbol{a})\,, \quad m = 1, 2, \ldots, n\,, \tag{63}$$

where $\mathbf{s}^m$ denotes the vector of values of σ^m_{hk} in the triangles $\mathscr{T} \in \mathscr{T}_h(\boldsymbol{a})$ and the functions $\boldsymbol{a} \mapsto A^m(\boldsymbol{a})$, $\boldsymbol{a} \mapsto F^m(\boldsymbol{a})$ are continuous. The conditions $\sigma^m_{hk} \in P(\Omega_h(\boldsymbol{a}))$ are equivalent with the following system of inequalities

$$f(\sigma^m_{hk}|_{\mathscr{T}}) \leqq 1 \quad \forall \mathscr{T} \in \mathscr{T}_h(\boldsymbol{a})\,, \tag{64}$$

which are independent of $\boldsymbol{a}$.

The coefficients of the quadratic functional

$$\Phi^*(\mathbf{s}^m) = \Phi(\tau) = \tfrac{1}{2}\|\tau\|^2_{\Omega_h(\boldsymbol{a})} - (\sigma^{m-1}_{hk}, \tau)_{\Omega_h(\boldsymbol{a})}$$

depend continuously on $\boldsymbol{a}$. The minimizer $\mathbf{s}^m(\boldsymbol{a})$ of $\Phi^*(\mathbf{s}^m)$ with the constraints (63), (64) exists by virtue of Lemma 1 for any $\boldsymbol{a} \in \mathscr{A}$. Consequently, we can prove that the functions $\boldsymbol{a} \mapsto \mathbf{s}^m(\boldsymbol{a})$ are continuous. The continuity of $\mathscr{I}_{hk}(\sigma_{hk}(\boldsymbol{a}))$ then follows easily from the properties of the yield function f.

Theorem 3. *Assume that* (9), (10) *and* (11) *hold. Let* $\{u^{(k)}_h\}$, $k \to 0$, $h \to 0$ *be a sequence of solutions of the Approximate Optimal Design Problem* (62), *such that* $h \leqq H(k)$ (*i.e., h is sufficiently small with respect to k*), $h = 2^{-j}$, $j = 1, 2, \ldots$, and $k = T/n$.

Then a subsequence $\{u^{(\tilde{k})}_{\tilde{h}}\}$ *exists such that*

$$u^{(\tilde{k})}_{\tilde{h}} \to u \quad in \quad C([0, 1])\,, \tag{66}$$

$$\max_{1 \leqq m \leqq T/\tilde{k}} \|E\sigma^m_{\tilde{h}\tilde{k}}(u^{(\tilde{k})}_{\tilde{h}}) - \sigma^m(u)\|_{0,\Omega_\delta} \to 0 \tag{67}$$

$$for \quad \tilde{k} \to 0\,, \quad \tilde{h} \leqq H(\tilde{k}) \to 0\,,$$

where u is a solution of the Optimal Design Problem (7), $\sigma^m(u)$ *is the solution of* (5), (6) *at* $t = t^m$, *extended by zero to* $\Omega_\delta \dot{-} \Omega(u)$.

Proof. Let $v \in U_{\text{ad}}$ be given. There exists a sequence $\{v_h\}$, $h \to 0$, such that $v_h \in U^h_{\text{ad}}$, v_h converge uniformly to v on the interval $[0, 1]$ (cf. [9] – Lemma 7.1).

Since U_{ad} is compact in $C([0, 1])$, a subsequence $\{u_{\tilde{h}}^{(\tilde{k})}\}$ exists such that (66) holds and $u \in U_{ad}$. By the definition (62) we have

$$\mathscr{J}_{\tilde{h}\tilde{k}}(\sigma_{\tilde{h}\tilde{k}}(u_{\tilde{h}}^{(\tilde{k})})) \leqq \mathscr{J}_{\tilde{h}\tilde{k}}(\sigma_{\tilde{h}\tilde{k}}(v_{\tilde{h}})) .$$

Passing to the limit with $\tilde{k} \to 0$ and $\tilde{h} \leqq H(\tilde{k}) \to 0$, on the basis of Theorem 2 we obtain

$$\mathscr{J}(\sigma(u)) \leqq \mathscr{J}(\sigma(v)) ,$$

so that u is a solution of the problem (7). The convergence (67) follows from the estimate (57), by virtue of Propositions 1 and 2.

Corollary. *Let* (9), (10) *and* (11) *hold. Then there exists at least one solution of the Optimal Design Problem* (7).

Proof is an immediate consequence of Lemma 3 and Theorem 3.

Remark. The limit of any uniformly convergent subsequence of $\{u_h^{(k)}\}$ represents a solution of (7) and (67) holds for the corresponding stress fields.

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Souhrn

OPTIMALIZACE TVARU PRUŽNĚ-DOKONALE PLASTICKÉHO TĚLESA

IVAN HLAVÁČEK

Minimalizuje se účelový funkcionál vzhledem k části hranice, na níž je (dvojrozměrné) těleso upevněno. Kritériem optimality je integrál z čtverce funkce plasticity. V rámci Prandtlova-Reussova modelu je stavová úloha zformulována v napětích pomocí evoluční variační nerovnice. Pomocí metody konečných prvků se definuje přibližné řešení a dokazuje se konvergence k řešení původní optimalizační úlohy.

Резюме

ОПТИМИЗАЦИЯ ФОРМЫ УПРУГО-ПЛАСТИЧНОГО ТЕЛА

IVAN HLAVÁČEK

Минимизируется целевой функционал относительно части границы, на которой (двумерное) тело фиксировано. Критерием оптимальности служит интеграл функции пластичности. В рамках модели Прандтла-Ройса задача состояния формулирована в напряжениях посредством эволюционного вариационного неравенства. При помощи метода конечных элементов определяется приближенное решение и доказывается сходимость к решению проблемы оптимизации.

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