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CONVERGENCE OF MULTISTEP METHODS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH PARAMETERS

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Summary. The author considers the convergence of quasilinear nonstationary multistep methods for systems of ordinary differential equations with parameters. Sufficient conditions for their convergence are given. The new numerical method is tested for two examples and it turns out to be a little better than the Hamming method.

Keywords: multistep method, Hamming method.

1. INTRODUCTION

Consider the ordinary differential vector-equation of the form

$$y'(t) = f(t, y(t), \mu), \quad t \in I = [\alpha, \beta], \tag{1}$$

where $\mu \in R^p$ is a parameter and $f: I \times R^q \times R^p \to R^q$, $(x = (x_1, x_2, \ldots, x_p)^T \in R^p)$. Let $C(I, R^q)$ be the class of continuous functions from I to R^q. By a solution (φ, λ) of $(1-2)$ we mean a function $\varphi \in C(I, R^q)$ and a parameter $\lambda \in R^p$ satisfying the equation (1) and the boundary conditions

$$y(\alpha) = y_p \in R^q, \quad Dy(\beta) = y_k \in R^p, \tag{2}$$

where a constant matrix $D = [d_{ij}]$ of type $p \times q$ and vectors y_p and y_k are given. We can see that this is a problem of terminal control. Many special problems can be reduced to $(1-2)$.

Existence-uniqueness theorems for $(1-2)$ were established in many papers (for example, see [2, 3, 6, 8, 10, 13, 14, 17]). Due to this fact we assume that the problem $(1-2)$ has a solution $(\varphi, \lambda) \in C(I, R^q) \times R^p$. Our task is to find the numerical solution (y_{h_N}, λ_N) for $(1-2)$.

Let d and k be given natural numbers, $d > 1$ and $d > k - 1$. Put $I_{NM} = \{t_{ih_N} = \alpha + ih_N : i \in R_M\}$, $R_M = \{0, 1, \ldots, M\}$, $M \leqq N$ where N is a natural number, $N \geqq d$ and $h_N = (\beta - \alpha)/N$. For given $y_{h_d}(t)$, $t \in I_{d,k-1}$ and λ_{d-1} we can find $y_{h_d}(t)$, $t \in I_{dd} \setminus I_{d,k-1}$. To this end we can apply the quasilinear nonstationary multistep (k-step) method of the form

$$\sum_{i=0}^{k} a_i(t, h_d)\, y_{h_d}(t + ih_d) = h_d\, \mathscr{F}(t, h_d, y_{h_d}, \lambda_{d-1})\,, \quad t \in I_{d,d-k}\,,$$

where

$$\mathscr{F}(t, h, y, v) = F(t, \ldots, t + kh, h, y(t), \ldots, y(t + kh), v)\,.$$

Next, having $y_{h_d}(t)$, $t \in I_{dd}$ we try to find λ_d from the relation

$$\|D[h_d \sum_{i=0}^{d} \gamma_{di} f(t_{ih_d}, y_{h_d}(t_{ih_d}), \lambda_d)] - y_k + Dy_p\| \leqq g(h_d)\,.$$

Generally speaking, the elements (y_{h_N}, λ_N) are found from the relations

$$\sum_{i=0}^{k} a_i(t, h_N)\, y_{h_N}(t + ih_N) = h_N\, \mathscr{F}(t, h_N, y_{h_N}, \lambda_{N-1})\,, \quad t \in I'_N = I_{N,N-k}\,, \quad N \in \mathscr{N} = \{d, d+1, \ldots\}\,. \tag{3}$$

$$\|D[h_N \sum_{i=0}^{N} \gamma_{Ni} f(t_{ih_N}, y_{h_N}(t_{ih_N}), \lambda_N)] - y_k + Dy_p\| \leqq g(h_N)\,, \tag{4}$$

where $\lim_{N \to \infty} g(h_N) = 0$.

It is easy to see that linear methods (stationary and nonstationary) are a special case of the methods of type (3–4).

The purpose of this paper is to establish sufficient conditions for the convergence of the method (3–4) to the solution of (1–2).

Some numerical examples are also presented.

2. CONVERGENCE AND CONSISTENCY AND STABILITY

The following definitions are well known (see [1, 11, 16, 5, 7]).

Definition 1. *We say the method* (3–4) *is convergent to the solution* (φ, λ) *of the problem* (1–2) *if*

$$\lim_{N \to \infty} \max_{i \in R_N} \|\varphi(t_{ih_N}) - y_{h_N}(t_{ih_N})\| = 0\,, \quad \lim_{N \to \infty} \|\lambda - \lambda_N\| = 0\,.$$

The order of convergence is $v (v > 0)$ *if*

$$\max_{i \in R_N} \|\varphi(t_{ih_N}) - y_{h_N}(t_{ih_N})\| = O(h_N^v)\,, \quad \|\lambda - \lambda_N\| = O(h_N^v)\,.$$

Definition 2. *We say the method* (3–4) *is consistent with the problem* (1–2) *on the solution* (φ, λ) *if there exists a function* ε: $J_{h_N}: H \to R_+ = [0, \infty)$, $J_{h_N} =$

$= [\alpha, \beta - kh_N]$, $H = [0, h_0]$, $h_0 \in (0, \infty)$ *such that*

(i) $$\left\| \sum_{i=0}^{k} a_i(t, h_N)\, \varphi(t + ih_N) - h_N\, \mathscr{F}(t, h_N, \varphi, \lambda) \right\| \leqq \varepsilon(t, h_N)\,,$$

(ii) $$\lim_{N \to \infty} \sum_{i=0}^{N-k} \varepsilon(t_{ih_N}, h_N) = 0\,.$$

Remark 1. Since (φ, λ) is the solution of $(1-2)$ the condition (i) assumes the form

$$\left\| \varphi(t) \sum_{i=0}^{k} a_i(t, h_N) + \sum_{i=1}^{k} a_i(t, h_N) \int_t^{t+ih_N} f(s, \varphi(s), \lambda)\, \mathrm{d}s - h_N\, \mathscr{F}(t, h_N, \varphi, \lambda) \right\| \leqq$$

$$\leqq \varepsilon(t, h_N)\,, \quad t \in J_{h_N}\,.$$

The following theorem deals with the consistency of $(3-4)$.

Theorem 1 (see [7]). *If*

$$1^\circ \quad f: I \times R^q \times R^p \to R^q\,, \quad F: I^{k+1} \times H \times R^{q(k+1)} \times R^p \to R^q\,,$$

$$a_j: I \times H \to R\,, \quad j \in R_{k-1}$$

and f, F and all a_j are bounded, $a_k(t, h) \equiv 1$,

2° *there exists a solution $(\varphi, \lambda) \neq (\theta, \lambda)$ of $(1-2)$ where $\theta = (0, \ldots, 0)^{\mathrm{T}} \in R^q$,*

3° *φ' is Riemann integrable,*

then the method $(3-4)$ is consistent with the problem $(1-2)$ on (φ, λ) provided

(5) $$\lim_{N \to \infty} \sum_{i=0}^{N-k} \left| \sum_{j=0}^{k} a_j(t_{ih_N}, h_N) \right| = 0\,,$$

(6) $$\lim_{N \to \infty} h_N \sum_{i=0}^{N-k} \left\| \sum_{j=1}^{k} j a_j(t_{ih_N}, h_N) f(t_{ih_N}, \varphi(t_{ih_N}), \lambda) - \mathscr{F}(t_{ih_N}, h_N, \varphi, \lambda) \right\| = 0\,.$$

Remark 2 (see [5, 7]). If the functions f and F and a_j are continuous and if for $t \in I$ the conditions

$$\sum_{j=0}^{k} a_j(t, 0) = 0\,, \quad t \in I\,,$$

$$\lim_{N \to \infty} h_N^{-1} \sum_{j=0}^{k} a_j(t, h_N) = 0\,, \quad t \in I\,,$$

$$\sum_{j=1}^{k} j a_j(t, 0) f(t, \varphi(t), \lambda) = F(t, \ldots, t, 0, \varphi(t), \ldots, \varphi(t), \lambda)\,, \quad t \in I\,,$$

are satisfied, then the conditions $(5-6)$ are satisfied as well.

Lemma 1 (see [1]). *If the elements of the sequence $\{z_n\} \subset R_+$ satisfy the recurrent inequality*

$$z_n \leqq a_n + \sum_{i=0}^{n-1} b_i z_i\,, \quad a_n, b_n \in R_+\,, \quad n \in R_{N-k+1}\,,$$

then

$$z_n \leqq a_n + \sum_{i=0}^{n-1} a_i b_i \prod_{j=i+1}^{n-1} (1 + b_j), \quad n \in R_{N-k+1},$$

where

$$\sum_{i=0}^{-1} \ldots = 0, \quad \prod_{j=n}^{n-1} \ldots = 1.$$

Remark 3. If $a_n = a$, $b_n = b$ then we have Squier's result (see [15])

$$z_n \leqq a(1 + b)^n, \quad n \in R_{N-k+1}.$$

Now, let a family of recurrent equations of order k of the form

$$\sum_{i=0}^{k} a_i(t_{nh_N}, h_N)\, z_{n+i}^{h_N} = 0, \quad n \in R_{N-k}, \quad N \in \mathcal{N}, \tag{7}$$

be given. Put $U_n^h = (z_n^h, \ldots, z_{n+k-1}^h)^{\mathrm{T}}$. By $\{U_n^{h_N}(n_0, u^{h_N})\}_{n=n_0}^{N-k+1}$, $N \in \mathcal{N}$ where $n_0 \in R_{N-k+1}$ is fixed we denote the solution of the family (7) such that $U_{n_0}^{h_N}(n_0, u) = (u_0, \ldots, u_{k-1})^{\mathrm{T}}$.

Definition 3. *The trivial solution of the family* (7) *is called uniformly stable if for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for every* $N \in \mathcal{N}$, $n_0 \in R_{N-k+1}$ *the inequality* $\|u^{h_N}\|_* < \delta(\varepsilon)$ implies $\|U_n^{h_N}(n_0, u^{h_N})\|_* < \varepsilon$ *for* $n \in R_N^* = \{n_0, n_0 + 1, \ldots, N - k + 1\}$, *where* $\|\cdot\|_*$ *is some norm of the matrix* A,

$$A(t, h) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ -a_0(t, h) & -a_1(t, h) & -a_2(t, h) & \ldots & -a_{k-1}(t, h) \end{bmatrix}, \quad (t, h) \in I \times H.$$

It is known that the trivial solution of (7) is uniformly stable if

$$\|A(t, h)\|_* \leqq 1 + Mh, \quad M \geqq 0, \quad (t, h) \in I \times H,$$

or, more generally:

Lemma 2 (see [7]). *If there exist a Lebesgue integrable function* $\Lambda: I \to R$ *and a constant* $p \in R_{k-1}$ *such that*

$$\|A(t, h)\|_* \leqq \exp\left(\int_{t+ph}^{t+ph+h} \Lambda(s)\, \mathrm{d}s\right), \quad t + ph + h \in I, \quad h \in H,$$

then the trivial solution of (7) is uniformly stable.

Further, we have

Lemma 3 (see [5, 7, 12]). *If the trivial solution of the family of equations* (7) *is uniformly stable then there exists a constant* $C \geqq 1$ *such that*

$$\max_{s \in R_{k-1}} \|z^{h_N}_{n+s}\| \leqq C[\max_{s \in R_{k-1}} \|u^{h_N}_{n_s}\| + \sum_{s=0}^{n-1} \|c^{h_N}_s\|] , \quad n \in R_{N-k+1} , \quad N \in \mathcal{N} ,$$

where z_{n+s} *satisfies the recurrent equation*

$$\sum_{i=0}^{k} a_i(t_{nh_N}, h_N)\, z^{h_N}_{n+i} = c^{h_N}_n , \quad n \in R_{N-k} , \quad N \in \mathcal{N}$$

We introduce

Definition 4. *The method* $(3-4)$ *is said to be stable if the trivial solution of the family* (7) *associated with it is uniformly stable.*

3. ASSUMPTIONS AND LEMMAS

We introduce the following

Assumption H_1. *Suppose that*

1° $f: I \times R^q \times R^p \to R^q$, *and* f *is continuous;*

2° *there exists a constant* $M \in R_+$ *such that for* $(t, \mu) \in I \times R^p$, $x, \bar{x} \in R^q$ *we have*

$$\|f(t, x, \mu) - f(t, \bar{x}, \mu)\| \leqq M\|x - \bar{x}\| ;$$

3° *there exists a constant* $L > 0$ *such that for a function* $z \in C(I, R^q)$, $z(\alpha) = y_p$, $Dz(\beta) = y_k$ *and* $t \in I$, $\bar{\mu}, \tilde{\mu} \in R^p$ *we have*

$$\left\| D\left[\int_\alpha^\beta f(t, z(t), \bar{\mu})\, dt - \int_\alpha^\beta f(t, z(t), \tilde{\mu})\, dt\right]\right\| \geqq L\|\bar{\mu} - \tilde{\mu}\| ;$$

4° *there exist* $\lambda_N \in R^p$ *and* $g: H \to R_+$, $\lim_{N\to\infty} g(h_N) = 0$, *such that the condition* (4) *is fulfilled;*

5° *there exist constants* γ_{Ni}, $|\gamma_{Ni}| \leqq \Gamma$, $i \in R_N$, $N \in \mathcal{N}$, *and a function* $c: H \to R_+$, $\lim_{N\to\infty} c(h_N) = 0$, *such that the function* $z \in C(I, R^q)$ *from* 3° satisfies the inequality

$$\left\|\int_\alpha^\beta f(t, z(t), \lambda_N)\, dt - h_N \sum_{i=0}^{N} \gamma_{Ni} f(t_{ih_N}, z(t_{ih_N}), \lambda_N)\right\| \leqq c(h_N) , \quad N \in \mathcal{N} .$$

Remark 4. Let $p = 1$ and $f(t, y, v) = vf(t, y)$. Then the condition 3° is satisfied with

$$0 < L \leqq \inf\left\{D\int_\alpha^\beta f(t, z(t))\, dt : z \in C(I, R^q),\ z(\alpha) = y_p,\ Dz(\beta) = y_k\right\} .$$

Moreover, if

$$d_{1i} f_i(t, z) \geqq m_i > 0 , \quad i = 1, 2, \ldots, q ,$$

then

$$0 < L \leqq (\beta - \alpha) \sum_{i=1}^{q} m_i .$$

Remark 5. The condition 3° guarantees that if the problem $(1-2)$ has a solution (φ, λ) then the $\lambda \in R^p$ satisfying $(1-2)$ is unique. To prove it we suppose that there exist $\lambda_1, \lambda_2 \in R^p$, $\lambda_1 \neq \lambda_2$. Then we have

$$0 = \left\| D\left[\int_\alpha^\beta f(t, \varphi(t), \lambda_1)\,\mathrm{d}t - \int_\alpha^\beta f(t, \varphi(t), \lambda_2)\,\mathrm{d}t\right]\right\| \geqq L\|\lambda_1 - \lambda_2\|,$$

i.e. $\lambda_1 = \lambda_2$.

Remark 6. Let $w\colon H \to R^p$ be a function such that $\|w(h_N)\| \leqq g(h_N)$. Now the condition 4° leads to finding a solution λ_N of the equation

$$G(\lambda) = 0\,, \quad G\colon R^p \to R^p\,, \tag{8}$$

where

$$G(\lambda) = D[h_N \sum_{i=0}^{N} \gamma_{Ni} f(t_{ih_N}, y_{h_N}(t_{ihN}), \lambda)] - y_k + Dy_p - w(h_N)\,.$$

Let the solution of (8) belong to $\Omega \subset R^p$. This solution can be found by iteration methods. Suppose $\lambda^{(0)}$ is a point lying in Ω such that the Jacobi matrix $W(\lambda) = [\partial G_i/\partial\lambda_j]$ has the inverse $W^{-1}(\lambda^{(0)})$. Then by the assumptions of the Kantorovich theorem the Newton process

$$\lambda^{(n+1)} = \lambda^{(n)} - W^{-1}(\lambda^{(0)})\, G(\lambda^{(n)})\,, \quad n = 0, 1, \ldots, \tag{9}$$

converges to a solution λ_N of the system (8).

For $p = 1$ the system (8) is a scalar equation. If there exist constants m and M such that

$$0 < m \leqq h_N \sum_{j=1}^{q} d_{1j} \sum_{i=0}^{N} \gamma_{Ni} \frac{\partial f_j}{\partial\lambda}(t_{ih_N}, y_{h_N}(t_{ih_N}), \lambda) \leqq M\,,$$

then the method (9) assumes the form

$$\lambda^{(n+1)} = \lambda^{(n)} - \frac{1}{M} G(\lambda^{(n)})\,, \quad \lambda^{(0)} \in \Omega\,, \quad n = 0, 1, \ldots. \tag{10}$$

This method is convergent to the solution λ_N of (8).

Assumption H_2. *Let*

1° $F\colon I^{k+1} \times H \times R^{q(k+1)} \times R^p \to R^q$,

2° *there exist constants* $p_i \in R_+$, $i \in R_k$ *and functions* $r\colon I \times H \to R_+$, $\varepsilon^*\colon I \times H \to R_+$ *such that for* x_i, $\bar{x}_i \in R^q$, $i \in R_k$, $\mu, \bar{\mu} \in R^p$, $s_i \in I$, $i \in R_k$, $s_{\min} = \min\{s_i\colon i \in R_k\}$ *we have*

$$\|F(s_0, \ldots, s_k, h, x_0, \ldots, x_k, \mu) - F(s_0, \ldots, s_k, h, \bar{x}_0, \ldots, \bar{x}_k, \bar{\mu})\| \leqq$$
$$\leqq \sum_{i=0}^{k} p_i\|x_i - \bar{x}_i\| + r(s_{\min}, h)\,\|\mu - \bar{\mu}\| + \varepsilon^*(s_{\min}, h)\,,$$
$$\lim_{N\to\infty} h_N \sum_{i=0}^{N-k} \varepsilon^*(t_{ih_N}, h_N) = 0\,.$$

Lemma 4. *If Assumption* H_1 *is satisfied and the problem* $(1-2)$ *has the solution* (φ, λ) *then we have the estimate*

$$\|\lambda - \lambda_N\| \leqq \frac{1}{L}\{g(h_N) + \|D\| [c(h_N) + Q \max_{j\in R_N} v_{h_N}(t_{jh_N})]\}, \quad N \in \mathcal{N},$$

where

$$v_h(t) = \|\varphi(t) - y_h(t)\|, \quad Q = \Gamma M(\beta - \alpha)(1 + d^{-1}).$$

Proof. It is easy to see that

$$D\left[\int_\alpha^\beta f(t, \varphi(t), \lambda_N)\,\mathrm{d}t - \int_\alpha^\beta f(t, \varphi(t), \lambda)\,\mathrm{d}t\right] =$$

$$= D\left[\int_\alpha^\beta f(t, \varphi(t), \lambda_N)\,\mathrm{d}t - h_N \sum_{i=0}^N \gamma_{Ni} f(t_{ih_N}, \varphi(t_{ihN}), \lambda_N)\right] +$$

$$+ D[h_N \sum_{i=0}^N \gamma_{Ni} f(t_{ih_N}, \varphi(t_{ih_N}), \lambda_N) - h_N \sum_{i=0}^N \gamma_{Ni} f(t_{ih_N}, y_{h_N}(t_{ih_N}), \lambda_N)] +$$

$$+ D[h_N \sum_{i=0}^N \gamma_{Ni} f(t_{ih_N}, y_{h_N}(t_{ih_N}), \lambda_N)] - y_k + Dy_p.$$

Now, using Assumption H_1 we have the desired estimate.

4. CONVERGENCE OF THE METHOD (3–4)

Let

$$P = \sum_{j=0}^k p_j,$$

$$C_1 = C(1 - Cp_k(\beta - \alpha)\,d^{-1})^{-1} \quad \text{where} \quad C \geqq 1,$$

$$\delta^*(t, h) = \varepsilon(t, h) + h\,\varepsilon^*(t, h),$$

$$u_N = \max_{j\in R_N} v_{h_N}(t_{jh_N}),$$

$$T_N = C_1\{e_{Oh_N} + L^{-1} h_N \sum_{j=0}^{N-k} r(t_{jh_N}, h_N)\,[g(h_{N-1}) + \|D\|\,(c(h_{N-1}) + Qu_{N-1})] + \sum_{j=0}^{N-k} \delta^*(t_{jh_N}, h_N)\},$$

$$C_2 = C_1 \exp(C_1 P(\beta - \alpha)),$$

$$S_N = C_2\{e_{Oh_N} + L^{-1} h_N \sum_{j=0}^{N-k} r(t_{jh_N}, h_N)\,[g(h_{N-1}) + \|D\|\,c(h_{N-1})] + \sum_{j=0}^{N-k} \delta^*(t_{jh_N}, h_N)\},$$

$$P_N = C_2 L^{-1} Q \|D\| h_N \sum_{j=0}^{N-k} r(t_{jh_N}, h_N) .$$

Now we can formulate the theorem on the convergence of the method (3−4).

Theorem 2. *If Assumptions* H_1 *and* H_2 *are satisfied and if*

1° *there exists a solution* (φ, λ) *of* (1−2),

2° *the method* (3−4) *is stable and consistent with* (1−2) *on the solution* (φ, λ),

3° $d > C(\beta - \alpha) p_k$ *where* $C \geqq 1$ *is the constant from Lemma* 3,

4° $\lim_{N \to \infty} \|\varphi(t) - y_{h_N}(t)\| = 0 , \quad t \in I_{N,k-1}$,

then

$$(11) \qquad \begin{cases} u_N \leqq u_{d-1} \prod_{i=d}^{N} P_i + \sum_{i=d}^{N} S_i \prod_{j=i+1}^{N} P_j \equiv \tilde{u}_N , \\ \|\lambda - \lambda_N\| \leqq L^{-1}\{g(h_N) + \|D\| [c(h_N) + Q u_N]\} , \quad N \in \mathcal{N} . \end{cases}$$

Moreover, if $\tilde{u}_N \to 0$ *then the method* (3−4) *is convergent to the solution* (φ, λ) *of the problem* (1−2).

Proof. For $t \in I'_N$, $n \in R_{N-k}$ we have

$$\sum_{i=0}^{k} a_i(t, h_N) [y_{hN}(t + ih_N) - \varphi(t + ih_N)] =$$

$$= h_N \mathcal{F}(t, h_N, y_{h_N}, \lambda_{N-1}) - h_N \mathcal{F}(t, h_N, \varphi, \lambda) +$$

$$+ h_N \mathcal{F}(t, h_N, \varphi, \lambda) - \sum_{i=0}^{k} a_i(t, h_N) \varphi(t + ih_N) .$$

Using Lemma 3 we get

$$e_{nh_N} = \max_{s \in R_{k-1}} v_{h_N}(t_{s+n,h_N}) \leqq$$

$$\leqq C\{e_{Oh_N} + h_N \sum_{j=0}^{n-1} [\sum_{i=0}^{k} p_i v_{h_N}(t_{j+i,h_N}) + r(t_{jh_N}, h_N) \|\lambda_{N-1} - \lambda\|] +$$

$$+ \sum_{j=0}^{n-1} [h_N \varepsilon^*(t_{jh_N}, h_N) + \varepsilon(t_{jh_N}, h_N)]\} , \quad n \in R_{N-k+1}, \quad N \in \mathcal{N} ,$$

and

$$e_{nh_N} \leqq T_N + C_1 P h_N \sum_{j=0}^{n-1} e_{jh_N} , \quad n \in R_{N-k+1} , \quad N \in \mathcal{N} .$$

From Remark 3 we obtain

$$e_{nhN} \leqq T_N(1 + C_1 P h_N)^n \leqq S_N + P_N u_{N-1} , \quad n \in N - k + 1 ,$$

and hence

$$u_N \leqq S_N + P_N u_{N-1}\,, \quad N \in \mathcal{N}\,.$$

Now we have the estimate (11) and the proof of Theorem 2 is complete.

Remark 7. Theorem 2 and Lemma 4 remain true if the conditions 3° and 5° of Assumption H_1 are satisfied only for $z = \varphi$.

Remark 8. If

1° there exists a constant $s \in [0, 1]$ such that

$$\bigwedge_{N\in\mathcal{N}} h_N^s \sum_{j=0}^{N-k} r(t_{jh_N}, h_N) \leqq W(s) < \infty\,, \quad C_2\, Q\|D\|\, W(0) \leqq L\,,$$

and $C_3 = L^{-1} C_2\, Q\|D\|\, W(s) \leqq 1$ if $s \in [0, 1)$ while $C_3 < 1$ if $s = 1$,

2° $\bigwedge_{N\in\mathcal{N}} \sum_{i=d}^{N} S_i < \infty\,,$

then the method (3–4) is convergent to the solution (φ, λ) of (1–2) and

$$P_i \leqq B_1(s)\, h_i^{1-s}\,, \quad B_1(s) = C_2 L^{-1}\, Q\|D\|\, W(s)\,, \quad i \in \mathcal{N}\,.$$

Now, if there exists a constant $v > 0$ such that for $i \in \mathcal{N}$

$$e_{Oh_i} = O(h_i^{1+v})\,,$$

$$\sum_{j=0}^{i-k} \delta^*(t_{jh_i}, h_i) = O(h_i^{1+v})\,,$$

$$h_i^{1-s}[c(h_i) + g(h_i)] = O(h_i^{1+v})\,,$$

then the condition 2° is satisfied.

Indeed, it is easy to see that

$$\lim_{N\to\infty} \prod_{i=d}^{N} P_i = 0\,.$$

Further, we have

$$(12) \qquad u_N \leqq \prod_{i=d}^{N} P_i\{u_{d-1} + \sum_{i=d}^{N} S_i\}\,, \quad N \in \mathcal{N}\,,$$

and hence see that the method (3–4) is convergent to (φ, λ).

Remark 9. If there exist a Lebesgue integrable function $\Lambda: I \to R_+$ and constants $v > 0$, $0 \leqq c < 1$, $\delta_0 \geqq 0$, $p \in R_{k-1}$ such that

$$\delta^*(t, h) = h^{1+v} \int_{t+ph}^{t+ph+h} \Lambda(\tau)\, d\tau$$

or

$$\delta^*(t, h) = \delta^* h^{1+v} c^{(t-\alpha)/h}$$

then

$$\sum_{j=0}^{i-k} \delta^*(t_{jhi}, h_i) = O(h_i^{1+\nu}) .$$

Moreover, if the function Λ is bounded then $\delta^*(t, h) = O(h^{2+\nu})$.
Now we can give sufficient conditions for the method $(3-4)$ of the order ν.

Theorem 3. *If the assumptions of Theorem 2 are satisfied and if*

1° $\bigwedge_{N \in \mathcal{N}} \sum_{i=d}^{N} S_i < \infty$,

2° *there exists a constant* $\nu > 0$ *such that*

$$P_i \leqq h_i^\nu (i-1)^\nu (\beta - \alpha)^{-\nu} , \quad i \in \mathcal{N} ,$$

3° $c(h_i) = g(h_i) = O(h_i^\nu)$, $i \in \mathcal{N}$,

then the order of convergence of the method $(3-4)$ *is* ν.

Proof. Using 2° we get

$$\prod_{i=d}^{N} P_i \leqq \left(\frac{d-1}{\beta - \alpha}\right)^\nu h_N^\nu , \quad N \in \mathcal{N} .$$

Now the assertion of the theorem follows from (11) and (12).

Remark 10. If

1° the condition 2° from Remark 8 is satisfied,

2° there exist a function $\bar{r}: I \times H \to R_+$ and constant $B_0, \nu > 0$ such that

$$r(t, h) = \left[\frac{\beta - \alpha}{h} - 1\right]^\nu h^{\nu - 1} \bar{r}(t, h) ,$$

$$\bigwedge_{i \in \mathcal{N}} \sum_{j=0}^{i-k} \bar{r}(t_{jh_i}, h_i) \leqq B_0 ,$$

3° $C_2 \, Q \|D\| \, B_0 (\beta - \alpha)^\nu \leqq L$,

then the condition 2° of Theorem 3. is satisfied.

Remark 11. It is easy to see that if

$$\bar{r}(t, h) = \int_{t+ph}^{t+ph+h} \Lambda(\tau) \, d\tau , \quad p \in R_{k-1} ,$$

or

$$\bar{r}(t, h) = \bar{Q} c^{(t-\alpha)/h} .$$

where $\Lambda: I \; R_+$ is a Lebesgue integrable function and $\bar{Q} \geqq 0$, $0 \leqq c < 1$, then

$$B_0 = \int_\alpha^\beta \Lambda(\tau) \, d\tau \quad \text{or} \quad B_0 = \bar{Q}(1 - c)^{-1} .$$

Remark 12. We considered our problem with the condition $Dy(\beta) = y_k$ where D was a constant matrix. This condition may be replaced by

$$\psi(y(\alpha), y'(\alpha), y(\beta), y'(\beta), \lambda) = \theta$$

where $\psi: R^{4q} \times R^p \to R^p$. Then the equation (8) assumes a more complicated form.

5. NUMERICAL RESULTS

In this section we report on numerical experiments with the new linear method of the form

$$(13) \qquad \sum_{i=0}^{3} a_i(t, h_N)\, y_{h_N}(t + ih_N) = h_N \sum_{i=0}^{2} b_i(t, h_N)\, f(t + ih_N, y_{h_N}(t + ih_N), \lambda_{N-1}),$$

where

$$\begin{aligned} a_0(t, h) &= -1, & b_0(t, h) &= \cdot 5, \\ a_1(t, h) &= -\cdot 5(\sqrt{(t + h)} - \sqrt{t}), & b_1(t, h) &= \cdot 5 + a_2(t, h), \\ a_2(t, h) &= -a_1(t, h), & b_2(t, h) &= 2, \\ a_3(t, h) &= 1. \end{aligned}$$

This method is stable and consistent (see [7]). The iterative scheme for λ_N will be described later. We try to compare this method with the Hamming method (described in [4]) on two problems.

Example 1. Our first problem

$$\begin{aligned} y'(t) &= \lambda \sin t + t - \cdot 5\, y(t), \\ y(0) &= -4, \\ y(\pi) &= -4, \end{aligned}$$

has an exact solution (φ, λ) of the form

$$\begin{aligned} \varphi(t; \lambda) &= 2t - 4 + \cdot 4\lambda(\sin t - 2\cos t + 2\exp(-\cdot 5t)) \\ \lambda &= -2\cdot 5\pi/(1 + \exp(-\cdot 5\pi)). \end{aligned}$$

We get the numerical solution (y_{h_N}, λ_N) of our problem by (3–4). Here (4) assumes the form

$$\lambda_N = -2 - \cdot 25\pi^2 + \cdot 25 E_{h_N}(\pi) \quad \text{with} \quad \lambda_2 = 0,$$

where $E_{h_N}(\pi)$ is an approximate value of $\int_0^\pi y_{h_N}(s)\, ds$ using the method [9].

Example 2. Now we consider the second problem

$$\begin{aligned} y'(t) &= \lambda \sin t + t - \cdot 5\, y(t) + \cdot 1 \sin(t + \lambda), \\ y(0) &= -4, \\ y(4\cdot 25) &= \cdot 5. \end{aligned}$$

It has an exact solution of the form

$$\varphi(t; \lambda) = C \exp(-\cdot 5t) + \cdot 4\lambda(\sin t - 2\cos t) + 2t - 4 + \\ + \cdot 04(\sin(t + \lambda) - 2\cos(t + \lambda)),$$

where

$$C = \cdot 8\lambda + \cdot 04(-\sin\lambda + 2\cos\lambda)$$

and λ is a solution of the equation

$$(14) \qquad \lambda = 1\cdot 25\{\cdot 04(\sin\lambda - 2\cos\lambda) - \exp(2\cdot 125)[\cdot 4\lambda(\sin 4\cdot 25 - 2\cos 4\cdot 25) + \\ + \cdot 04(\sin(4\cdot 25 + \lambda) - 2\cos(4\cdot 25 + \lambda)) + 4]\}.$$

Here $\lambda_2 = 0$ and λ_N is a solution of the equation

$$(15) \qquad \lambda = (1 - \cos 4\cdot 25)^{-1}\{4\cdot 5 - \cdot 5(4\cdot 25)^2 + \cdot 5E_{h_N}(4\cdot 25) + \\ + \cdot 1[\cos(4\cdot 25 + \lambda) - \cos\lambda)]\}.$$

We use the method of successive approximations to find solutions of (14) and (15).

In Tables 1 and 2 we compare absolute errors between exact and computed solutions. The computations were carried out on the Polish computer ODRA 1305.

Table 1 Example 1 $h_N = \pi/100$

	$\max\limits_{j\in R_{100}} \lvert\varphi(t_{jh_N}) - y_{h_N}(t_{jh_N})\rvert$	$\lvert\lambda - \lambda_N\rvert$	number of iterations for (y_{h_N}, λ_N)
method (13)	·000345	·000026	15
Hamming method	·000701	·000326	15

Table 2 Example 2 $h_N = 4\cdot 25/125$

	$\max\limits_{j\in R_{125}} \lvert\varphi(t_{jh_N}) - y_{h_N}(t_{jh_N})\rvert$	$\lvert\lambda - \lambda_N\rvert$	number of iterations for (y_{hN}, λ_N)
method (13)	·063803	·063776	68
Hamming method	·086634	·083214	68

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Souhrn

KONVERGENCE VÍCEKROKOVÝCH METOD PRO SYSTÉMY DIFERENCIÁLNÍCH ROVNIC S PARAMETRY

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V článku jsou zkoumány kvazilineární nestacionární vícekrokové metody pro systémy obyčejných diferenciálních rovnic s parametry. Jsou odvozeny postačující podmínky pro jejich konvergenci. Nová numerická metoda je testována na dvou příkladech, které ukazují, že je poněkud výhodnější než Hammingova metoda.

Резюме

СХОДИМОСТЬ МНОГОШАГОВЫХ МЕТОДОВ ДЛЯ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПАРАМЕТРАМИ

Tadeusz Jankowski

В статье исследуются квазилинейные нестационарные многошаговые методы для систем обыкновенных дифференциальных уравнений с параметрами. Выведены достаточные условия для их сходимости. Новый численный метод проверяется на двух примерах, которые показывают, что он немного выгоднее метода Хамминга.

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