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ON MULTIPLE PERIODIC AUTOREGRESSION

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Summary. The model of periodic autoregression is generalized to the multivariate case. The autoregressive matrices are periodic functions of time. The mean value of the process can be a non-vanishing periodic sequence of vectors. Estimators of parameters and tests of statistical hypotheses are based on the Bayes approach. Two main versions of the model are investigated, one with constant variance matrices and the other with periodic variance matrices of the innovation process.

Keywords: Bayes approach, estimating autoregressive matrices, matrixvariate t-distribution, multivariate processes, periodic autoregression, test of periodicity, test of fit.

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1. INTRODUCTION

Let $\{Y_t\}$ be an r-dimensional white noise, i.e. a series of independent random vectors with $\mathsf{E}Y_t = 0$, $\mathsf{Var}\, Y_t = G^{-1}$, where G is a positive definite matrix. If $B_1, \ldots$ $\ldots, B_n$ are $r \times r$ matrices such that

$$\mathsf{Det}\,(z^n I - z^{n-1} B_1 - \ldots - z^0 B_n) \neq 0 \quad \text{for} \quad |z| \geqq 1 ,$$

then the relation

$$X_t = B_1 X_{t-1} + \ldots + B_n X_{t-n} + Y_t$$

determines the classical r-dimensional stationary autoregressive process $\{X_t\}$ with vanishing mean. If we assume that $\mathsf{E}X_t = \eta$, then the autoregressive model can be written in the form

$$X_t - \eta = B_1(X_{t-1} - \eta) + \ldots + B_n(X_{t-n} - \eta) + Y_t ,$$

or equivalently

$$X_t = \xi + B_1 X_{t-1} + \ldots + B_n X_{t-n} + Y_t , \tag{1.1}$$

where

$$\xi = (I - B_1 - \ldots - B_n)\, \eta .$$

In many cases it is known that a real time series $\{X_t\}$ has a seasonal behaviour with a period p. This bears in mind to modify the model (1.1) in such a way that its parameters would be also periodic functions with the period p. It leads to the assumption

$$(1.2) \qquad X_{n+(j-1)p+k} = \mu_k + \sum_{i=1}^{n} U_{ki} X_{n+(j-1)p+k-i} + Y_{n+(j-1)p+k},$$

$k = 1, \ldots, p$, where μ_k are r-dimensional vectors and U_{ki} are $r \times r$ matrices. Usually, it is assumed that the variables $X_1, \ldots, X_n$ are given and (1.2) is used for the construction of X_t for $t > n$. In this case we assume further that the sets $(X_1 \ldots, X_n)$ and $(Y_{n+1}, Y_{n+2}, \ldots)$ of random variables are independent.

There are two main versions of the model (1.2). In the first one it is assumed that $\operatorname{Var} Y_{n+(j-1)p+k} = G^{-1}$ do not depend on k (the model with equal variance matrices), whereas the second one allows a periodic change also here and the assumption reads $\operatorname{Var} Y_{n+(j-1)p+k} = G_k^{-1}$, $k = 1, \ldots, p$ (the model with periodic variance matrices). All the matrices $G_1, \ldots, G_p$ are supposed to be positive definite. In the both cases the model (1.2) is called the multiple periodic autoregression.

Periodic models were introduced by Gladyshev [8]. Their theoretical properties were investigated by Jones and Brelsford [9], Pagano [11], Cleveland and Tiao [6] and by Tiao and Grupe [13]. Anděl [3] and Anděl et al. [4] proposed some methods for a statistical analysis of the periodic autoregression. All the authors dealt only with one-dimensional models. In the present paper we give some methods for statistical analysis of the multiple periodic autoregressive model (1.2). We use the Bayesian approach and generalize the results of [4]. In the one-dimensional case it was possible to derive exact statistical tests. For $r > 1$ we are able to present only asymptotic formulas.

2. PRELIMINARIES

Let $A_1, \ldots, A_m$ be square matrices. Denote $\operatorname{Diag}\{A_1, \ldots, A_m\}$ the block-diagonal matrix with $A_1, \ldots, A_m$ on the diagonal.

Theorem 2.1. *Let $Q_1, \ldots, Q_p$ be $n \times n$ symmetric positive definite matrices. Assume $p \geqq 2$ and put*

$$H = \operatorname{Diag}\{Q_1, \ldots, Q_{p-1}\} - (Q_1, \ldots, Q_{p-1})' \, Q^{-1} (Q_1, \ldots, Q_{p-1}),$$

where $Q = Q_1 + \ldots + Q_p$. Then H is a positive definite matrix.

Proof. See [3], p. 366. □

The fact that a matrix A is positive definite will be denoted by $A > 0$. The symbol c will be used for any constant. Thus the same c in different formulas can represent different constants.

Theorem 2.2. *Let $V = (v_{ij})$, $i = 1, \ldots, r$; $j = 1, \ldots, s$, be a matrix with real elements. Denote by I the unit matrix and introduce the density*

$$t_{r,s,m}(V) = c|I + m^{-1}VV'|^{-m/2}, \quad m \geqq rs + 1 .$$

If $m \to \infty$, then

$$t_{r,s,m}(V) \to (2\pi)^{-rs/2} \exp\{-\tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{s} v_{ij}^2\} .$$

Proof. See [1], p. 238. □

Theorem 2.3. *If random variables V_{ij} $(i = 1, \ldots, r; j = 1, \ldots, s)$ have the density $t_{r,s,m}(V)$, then the random variable $\lambda = \operatorname{Tr} VV'$ has asymptotically (as $m \to \infty$) the χ^2_{rs} distribution.*

Proof. It follows from Theorem 2.2, that V_{ij} are asymptotically independent $N(0, 1)$ variables. Then $\lambda = \sum\sum V_{ij}^2$ has asymptotically the χ^2_{rs} distribution. □

Theorem 2.4. *Let V_{ij} $(i = 1, \ldots, r; j = 1, \ldots, s)$ have the density*

$$T_{r,s,m}(V) = c|I + VV'|^{-m/2}, \quad m \geqq rs + 1 .$$

Assume that $s \geqq 2$ and write V in a block form $V = (A, B)$, where A and B have s_1 and s_2 $(s_1 + s_2 = s)$ columns, respectively. Then the marginal density of the variables V_{ij} for $i = 1, \ldots, r$; $j = 1, \ldots, s_1$ is $T_{r,s_1,m-s_2}(A)$.

Proof. We have

$$T_{r,s,m}(V) = c|I + AA' + BB'|^{-m/2} =$$

$$= c|(I + AA')^{1/2}[I + (I + AA')^{-1/2} BB'(I + AA')^{-1/2}](I + AA')^{1/2}|^{-m/2} =$$

$$= c|I + AA'|^{-m/2} |I + (I + AA')^{-1/2} BB'(I + AA')^{-1/2}|^{-m/2} .$$

The marginal density of the elements of A is given by the formula $\int T_{r,s,m}(V)\, dB$. After the substitution $C = B'(I + AA')^{-1/2}$, the Jacobian of which is $|I + AA'|^{s_2/2}$ (see Anděl [2], p. 60), we get that the desired marginal density is $c|I + AA'|^{-(m-s_2)/2}$. □

Theorem 2.5. *Let $\Omega = \{x_{11}, \ldots, x_{1r}, x_{22}, \ldots, x_{2r}, \ldots, x_{rr}\}$ be such a set in the $r(r + 1)/2$-dimensional Euclidean space $\mathbb{R}_{r(r+1)/2}$, that the matrix*

$$X = (x_{ij}), \quad i = 1, \ldots, r; \quad j = 1, \ldots, r,$$

with $x_{ij} = x_{ji}$ for $i > j$ is positive definite. Let D be an $r \times r$ positive definite matrix. Then for every integer $m > r$ there exists such a positive constant c_{rm} that the function f defined by

$$f(x_{11}, \ldots, x_{rr}) = c_{rm}|D|^{(m-1)/2} |X|^{(m-r-2)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} DX\}$$

$$\text{for} \quad (x_{11}, \ldots, x_{rr}) \in \Omega$$

$$f(x_{11}, \ldots, x_{rr}) = 0 \text{ otherwise}$$

is a density.

Proof. See Cramér [7] § 29.5. □

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ an $p \times q$ matrix. The Kronecker product $A \otimes B$ is an $mp \times nq$ matrix defined by

$$A \otimes B = (a_{ij}B)\,.$$

Theorem 2.6. (i) $A_1A_2 \otimes B_1B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2)$.

(ii) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ whenever the inverses exist.

(iii) $(A \otimes B)' = A' \otimes B'$.

Proof. The assertions are well known and can be directly checked (see Rao [12], Chap. 1). □

Let A be an $m \times n$ matrix. Denote by vec A the mn-component vector obtained from A by stacking the columns of A, one on top of the other, in order from left to right. We shall use following properties of the vec operation.

Theorem 2.7. (i) $\operatorname{vec} ABC = (C' \otimes A) \operatorname{vec} B$.

(ii) $\operatorname{Tr} AB = (\operatorname{vec} B')' \operatorname{vec} A = (\operatorname{vec} B)' \operatorname{vec} A'$.

Proof. See Neudecker [10]. □

From here (or from the definition directly) we easily get

$$\operatorname{Tr} A'A = (\operatorname{vec} A)' \operatorname{vec} A = \operatorname{Tr} AA'\,.$$

3. GENERAL ASSUMPTIONS AND NOTATION

We shall assume that $X_1, \ldots, X_n$ are given random vectors which are independent of $Y_{n+1}, Y_{n+2}, \ldots$. Our statistical analysis is based on $X_1, \ldots, X_n, \ldots, X_N$, where N is large enough, since we shall deal with asymptotic distributions for $N \to \infty$. The vectors X_t for $t > n$ are generated by the model (1.2).

Let $U_1, \ldots, U_p$ be $nr \times r$ matrices defined by

$$U_k = (U_{k1}, \ldots, U_{kn})'$$

and let

$$U = (U_1', \ldots, U_p')'\,.$$

The matrix U is of the type $npr \times r$. Analogously we introduce the pr-dimensional vector

$$\mu = (\mu_1', \ldots, \mu_p')'\,.$$

For a given N, we put

$$\alpha_k = \left[\frac{N - n - k}{p}\right] + 1\,, \quad k = 1, \ldots, p\,,$$

where [ ] is the integer part. If $x_1, \ldots, x_N$ is a given realization of our process, we denote

$$x_t^0 = (x'_{t-1}, \ldots, x'_{t-n}) \quad \text{for} \quad t = n+1, \ldots, N .$$

Further, for $k = 1, \ldots, p$ and for $j = 1, \ldots, \alpha_k$ put

$$\bar{x}_k = \alpha_k^{-1} \sum_{j=1}^{\alpha_k} x_{n+k+(j-1)p} , \quad \bar{x}_k^0 = \alpha_k^{-1} \sum_{j=1}^{\alpha_k} x^0_{n+k+(j-1)p} ,$$

$$\Delta_{kj} = x_{n+k+(j-1)p} - \bar{x}_k , \quad \Delta^0_{kj} = x^0_{n+k+(j-1)p} - \bar{x}_k^0 ,$$

$$T_k = \sum_{j=1}^{\alpha_k} \Delta_{kj} \Delta'_{kj} , \quad C_k = \sum_{j=1}^{\alpha_k} \Delta^0_{kj} \Delta'_{kj} , \quad S_k = \sum_{j=1}^{\alpha_k} \Delta^0_{kj} \Delta^{0\prime}_{kj} ,$$

$$\hat{U}_k = S_k^{-1} C_k , \quad \hat{U} = (\hat{U}'_1, \ldots, \hat{U}'_p) , \quad R_k = T_k - \hat{U}'_k S_k \hat{U}_k ,$$

$$R = R_1 + \ldots + R_p , \quad v_k = \mu_k - \bar{x}_k + U'_k \bar{x}_k^0 , \quad v = (v'_1, \ldots, v'_p)' ,$$

$$\hat{\mu}_k = \bar{x}_k - \hat{U}'_k \bar{x}_k^0 , \quad \hat{\mu} = (\hat{\mu}'_1, \ldots, \hat{\mu}'_p)' ,$$

$$q_k = \alpha_k [1 - \alpha_k \bar{x}_k^{0\prime} (S_k + \alpha_k \bar{x}_k^0 \bar{x}_k^{0\prime})^{-1} \bar{x}_k^0] , \quad q = q_1 + \ldots + q_p .$$

Let $\operatorname{Var} Y_{n+(j-1)p+k} = G_k^{-1}$. Then the conditional density of the vectors $X_{n+1}, \ldots, X_N$, given $X_1 = x_1, \ldots, X_n = x_n$ and $U, G_1, \ldots, G_p, \mu$, is

$$(3.1) \qquad f(x_{n+1}, \ldots, x_N \mid x_1, \ldots, x_n, U, G_1, \ldots, G_p, \mu) = (2\pi)^{-r(N-n)/2} \times$$
$$\times |G_1|^{\alpha_1/2} \ldots |G_p|^{\alpha_p/2} \exp \{ -\tfrac{1}{2} \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} [x_{n+k+(j-1)p} - \mu_k -$$
$$- \sum_{i=1}^{n} U_{ki} x_{n+k+(j-1)p-i}]' \, G_k [x_{n+k+(j-1)p} - \mu_k - \sum_{i=1}^{n} U_{ki} x_{n+k+(j-1)p-i}] \} .$$

Let us remark that in the case $G_1 = \ldots = G_p = G$ we have

$$(3.2) \qquad |G_1|^{\alpha_1/2} \ldots |G_p|^{\alpha_k/2} = |G|^{(N-n)/2} .$$

First of all, it is necessary to simplify the formula (3.1).

Theorem 3.1. *The conditional density f can be equivalently expressed by the formula*

$$f(x_{n+1}, \ldots, x_N \mid x_1, \ldots, x_n, U, G_1, \ldots, G_p, \mu) = (2\pi)^{-r(N-n)/2} \times$$
$$\times |G_1|^{\alpha_1/2} \ldots |G_p|^{\alpha_p/2} \exp \{ -\tfrac{1}{2} \sum_{k=1}^{p} \operatorname{Tr} G_k D_k \} ,$$

where

$$D_k = (U_k - \hat{U}_k)' \, S_k (U_k - \hat{U}_k) + \alpha_k v_k v'_k + R_k .$$

Proof. We have

$$z_{kj} = x_{n+k+(j-1)p} - \mu_k - \sum_{i=1}^{n} U_{ki} x_{n+k+(j-1)p-i} =$$
$$= x_{n+k+(j-1)p} - \mu_k - U'_k x^0_{n+k+(j-1)p} = \Delta_{kj} - v_k - U'_k \Delta^0_{kj} .$$

From here we get

$$\sum_{j=1}^{\alpha_k} z'_{kj} G_k z_{kj} = \operatorname{Tr} G_k \sum_{j=1}^{\alpha_k} z_{kj} z'_{kj} =$$

$$= \operatorname{Tr} G_k (T_k - U'_k C_k - C'_k U_k + U'_k S_k U_k + \alpha_k v_k v'_k) = \operatorname{Tr} G_k D_k \,. \qquad \square$$

4. MODEL WITH EQUAL VARIANCE MATRICES

In this Section we assume that $G_1 = \ldots = G_p = G$.

Theorem 4.1. *Let the elements of U, G and μ be random variables with a prior density which is proportional to $|G|^{-1/2}$ if G is positive definite and zero otherwise. Let U, G and μ be independent of $X_1, \ldots, X_n$. If $N > n + 1$, then the posterior density of U, G and μ, given $x = (x'_1, \ldots, x'_N)'$, is*

$$g(U, G, \mu \mid x) = c|G|^{(N-n-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} G \sum_{k=1}^{p} D_k\} \,. \tag{4.1}$$

Proof. The assertion follows from Theorem 3.1 and from (3.2) using the Bayes theorem. $\square$

Theorem 4.2. *The marginal posterior densities of G, U and μ are given by the formulas*

(i) $$g_1(G \mid x) = c|G|^{(N-n-p-npr-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} GR\}$$

for $G > 0$ and zero otherwise;

(ii) $$g_2(U \mid x) = c\left|R + \sum_{k=1}^{p} (U_k - \hat{U}_k)' S_k (U_k - \hat{U}_k)\right|^{-(N-n-p+r)/2} ;$$

(iii) $$g_3(\mu \mid x) = c\left|R + \sum_{k=1}^{p} q_k(\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)'\right|^{-(N-n-npr+r)/2} .$$

Proof. (i) Density (4.1) is

$$g(U, G, \mu \mid x) = c|G|^{(N-n-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} GR - \tag{4.2}$$

$$- \tfrac{1}{2} \operatorname{Tr} G \sum_{k=1}^{p} (U_k - \hat{U}_k)' S_k (U_k - \hat{U}_k)\} \times$$

$$\times \exp\{-\tfrac{1}{2} \sum_{k=1}^{p} \alpha_k (\mu_k - \bar{x}_k + U'_k \bar{x}_k^0)' G (\mu_k - \bar{x}_k + U'_k \bar{x}_k^0)\} \,.$$

The simultaneous posterior density of U and G is

$$h_1(U, G \mid x) = \int g(U, G, \mu \mid x) \, d\mu \,.$$

Make the substitution

$$y_k = G^{1/2}(\mu_k - \bar{x}_k + U'_k \bar{x}_k^0) \,, \quad k = 1, \ldots, p \,.$$

The Jacobian is $|G|^{-p/2}$ and thus

$$h_1(U, G \mid x) = c\,|G|^{(N-n-p-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} GR\} \times \tag{4.3}$$

$$\times \exp\{-\tfrac{1}{2} \operatorname{Tr} \sum_{k=1}^{p} G^{1/2}(U_k - \hat{U}_k)' S_k(U_k - \hat{U}_k)\, G^{1/2}\}.$$

From here we get

$$g_1(G \mid x) = \int h_1(U, G \mid x)\, dU.$$

We substitute

$$M_k = G^{1/2}(U_k - \hat{U}_k)', \quad k = 1, \ldots, p,$$

with the Jacobian $|G|^{-npr/2}$ and this gives the formula for the posterior density of G introduced in the theorem.

(ii) The density $g_2(U \mid x)$ can be derived from (4.3) by help of Theorem 2.5.

(iii) From (4.2) and from Theorem 2.5 we get that the simultaneous posterior density of U and μ is

$$h_2(U, \mu \mid x) = c\Big|R + \sum_{k=1}^{p} [\alpha_k(\mu_k - \bar{x}_k + U_k'\bar{x}_k^0)(\mu_k - \bar{x}_k + U_k'\bar{x}_k^0)' +$$

$$+ (U_k - \hat{U}_k)' S_k(U_k - \hat{U}_k)\Big|^{-(N-n+r)/2}.$$

For this part of the proof we denote

$$\begin{gathered} W_k = U_k - \hat{U}_k, \quad v_k = \mu_k - \bar{x}_k + \hat{U}_k'\bar{x}_k^0 = \mu_k - \hat{\mu}_k, \\ U_k^{\#} = \hat{U}_k - \alpha_k(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{-1}\, \bar{x}_k^0 v_k', \\ W_k^{*} = \alpha_k(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{-1}\, \bar{x}_k^0 v_k' = \hat{U}_k - U_k^{\#}. \end{gathered} \tag{4.4}$$

Then we have

$$\begin{aligned} &\alpha_k(\mu_k - \bar{x}_k + U_k'\bar{x}_k^0)(\mu_k - \bar{x}_k + U_k'\bar{x}_k^0)' + (U_k - \hat{U}_k)' S_k(U_k - \hat{U}_k) = \\ &= \alpha_k(v_k + W_k'\bar{x}_k^0)(v_k + W_k'\bar{x}_k^0)' + W_k'S_kW_k = \\ &= (W_k + W_k^{*})'(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})(W_k + W_k^{*}) + q_k v_k v_k' = \\ &= (U_k - U_k^{\#})'(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})(U_k - U_k^{\#}) + q_k(\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)' \end{aligned}$$

and thus

$$h_2(U, \mu \mid x) = c\Big|R + \sum_{k=1}^{p} \{(U_k - U_k^{\#})'(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})(U_k - U_k^{\#}) +$$

$$+ q_k(\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)'\}\Big|^{-(N-n+r)/2}.$$

Now, we make the transformation

$$(U_k - U_k^{\#})'(S_k + \alpha_k\bar{x}_k^0\bar{x}_k^{0\prime})^{1/2} = Q_k, \quad k = 1, \ldots, p,$$

and then we use Theorem 2.4. This leads to the formula for $g_3(\mu \mid x)$. □

Obviously, $g_1(G \mid x)$ is the density of the Wishart distribution $W_r(N - n - p - r - npr, R^{-1})$.

We come to the main point of this Section, which is the problem of estimating parameters and testing statistical hypotheses. The modus of the posterior distribution can serve as a point estimator of the parameters. It is given in the next theorem.

Theorem 4.4. *The modus of the posterior distribution is* $\boldsymbol{U} = \hat{\boldsymbol{U}}$, $\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}$ *and* $\boldsymbol{G} = \hat{\boldsymbol{G}} = (N - n - 1)\,\boldsymbol{R}^{-1}$.

Proof. From (4.2) we obtain the posterior density of $\boldsymbol{U}$, $\boldsymbol{G}$ and $\boldsymbol{\mu}$

$$g_4(\boldsymbol{U}, \boldsymbol{G}, \boldsymbol{\mu} \mid \boldsymbol{x}) = c|\boldsymbol{G}|^{(N-n-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{GR}\} \times$$
$$\times \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{G} \sum_{k=1}^{p} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) -$$
$$- \tfrac{1}{2} \sum_{k=1}^{p} \alpha_k (\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k - \boldsymbol{U}_k' \bar{\boldsymbol{x}}_k^0)' \boldsymbol{G} (\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k - \boldsymbol{U}_k' \bar{\boldsymbol{x}}_k^0)\}\,.$$

It is clear that

$$g_4(\boldsymbol{U}, \boldsymbol{G}, \boldsymbol{\mu} \mid \boldsymbol{x}) \leqq g_4(\hat{\boldsymbol{U}}, \boldsymbol{G}, \hat{\boldsymbol{\mu}} \mid \boldsymbol{x}) = c|\boldsymbol{G}|^{(N-n-1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{GR}\}$$

for every $\boldsymbol{G} > 0$. The function $g_4(\hat{\boldsymbol{U}}, \boldsymbol{G}, \hat{\boldsymbol{\mu}} \mid \boldsymbol{x})$ reaches its maximum for $\boldsymbol{G} = \hat{\boldsymbol{G}}$, which can be proved by the same method as Theorem 12 in [1], p. 241. □

Theorem 4.5. *The posterior distribution of the random variable*

$$\lambda_U = (N - n - p + r) \operatorname{Tr} \boldsymbol{R}^{-1} \sum_{k=1}^{n} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)$$

is asymptotically the $\chi^2_{n^2 r^2}$.

Proof. From Theorem 4.2 (iii) we derive

$$g_3(\boldsymbol{U} \mid \boldsymbol{x}) = c|\boldsymbol{I} + \sum_{k=1}^{p} \boldsymbol{R}^{-1/2} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) \boldsymbol{R}^{-1/2}|^{-(N-n-p+r)/2}\,.$$

Put

$$\boldsymbol{P}_k = (N - n - p + r)^{1/2} \boldsymbol{R}^{-1/2} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k^{1/2}\,, \quad k = 1, \ldots, p\,,$$
$$\boldsymbol{P} = (\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n)\,.$$

The Jacobian of this transformation is a constant, and since

$$\boldsymbol{P}_1 \boldsymbol{P}_1' + \ldots + \boldsymbol{P}_n \boldsymbol{P}_n' = \boldsymbol{PP}'\,,$$

we get the posterior density of $\boldsymbol{P}$ in the form

$$h_3(\boldsymbol{P} \mid \boldsymbol{x}) = c|\boldsymbol{I} + (N - n - p + r)^{-1} \boldsymbol{PP}'|^{-(N-n-p+r)/2}\,.$$

Theorem 2.3 implies that the asymptotic distribution of $\operatorname{Tr} \boldsymbol{PP}'$ is the $\chi^2_{n^2 r^2}$. However,

$$\operatorname{Tr} \boldsymbol{PP}' = \operatorname{Tr} \sum_{k=1}^{p} \boldsymbol{P}_k \boldsymbol{P}_k' = \operatorname{Tr} \sum_{k=1}^{p} (N - n - p + r) \boldsymbol{R}^{-1/2} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \times$$
$$\times \boldsymbol{S}_k (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) \boldsymbol{R}^{-1/2} = (N - n - p + r) \operatorname{Tr} \boldsymbol{R}^{-1} \sum_{k=1}^{p} (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)\,. \;\square$$

Theorem 4.6. *The posterior distribution of the random variable*

$$\lambda_\mu = (N - n - npr + r) \sum_{k=1}^{p} q_k(\mu_k - \hat{\mu}_k)' R^{-1}(\mu_k - \hat{\mu}_k)$$

is asymptotically the χ^2_{pr}.

Proof. Let $m = N - n - npr + r$. The density $g_5(\mu \mid x)$ is

$$g_5(\mu \mid x) = c\left|I + \sum_{k=1}^{p} q_k R^{-1/2}(\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)' R^{-1/2}\right|^{-m/2}.$$

Define

$$t_k = (mq_k)^{1/2} R^{-1/2}(\mu_k - \hat{\mu}_k), \quad k = 1, \ldots, p, \quad T = (t_1, \ldots, t_p).$$

Then the posterior density of T is

$$h_4(T \mid x) = c\left|I + m^{-1} \sum_{k=1}^{p} t_k t_k'\right|^{-m/2} = c\left|I + m^{-1} TT'\right|^{-m/2}.$$

From Theorem 2.3 we have that $\mathrm{Tr}\, TT'$ has asymptotically the χ^2_{pr} distribution. Further we obtain

$$\mathrm{Tr}\, TT' = \mathrm{Tr}\, T'T = \sum_{k=1}^{p} t_k' t_k = m \sum_{k=1}^{p} q_k(\mu_k - \hat{\mu}_k)' R^{-1}(\mu_k - \hat{\mu}_k). \qquad \square$$

Theorems 4.5 and 4.6 can be used for testing that U and μ have given values (e.g. as tests of quality of a simulation, because in such a case U and μ are known). If λ_U or λ_μ exceed critical values of the corresponding χ^2 distributions, we reject the hypotheses that U or μ are the true values. However, it is more important to know if U_k and μ_k really depend on k. If not, we can come back to the classical autoregressive process. Procedure for the decision whether the data should be described by the classical or by the periodic model, are based on the two following theorems.

Theorem 4.7. *Denote*

$$H = \mathrm{Diag}\{q_1, \ldots, q_{p-1}\} - q^{-1}(q_1, \ldots, q_{p-1})'(q_1, \ldots, q_{p-1}),$$

$$\Delta_k = \mu_k - \mu_p - (\hat{\mu}_k - \hat{\mu}_p) \quad \text{for} \quad k = 1, \ldots, p-1, \quad \Delta_p = \mu_p - \hat{\mu}_p,$$

$$\Delta = (\Delta_1, \ldots, \Delta_{p-1}).$$

Then the posterior distribution of the random variable

$$\pi_\mu = (N - n - npr + r - 1)\, \mathrm{Tr}\, R^{-1} \Delta H \Delta'$$

is asymptotically the $\chi^2_{(p-1)r}$.

Proof. The matrix H is positive definite according to Theorem 2.1. Put $m = N - n - npr + r$. From Theorem 4.3 (iii) after the transformation $(\mu_1, \ldots, \mu_p) \rightarrow (\Delta_1, \ldots, \Delta_p)$ we get the posterior density of $\Delta^* = (\Delta_1, \ldots, \Delta_p)$ in the form

$$h_5(\Delta^* \mid x) = c\left|R + q_p\Delta_p\Delta_p' + \sum_{k=1}^{p-1} q_k(\Delta_k + \Delta_p)(\Delta_k + \Delta_p)'\right|^{-m/2} =$$

$$= c\left|R + q(\Delta_p + q^{-1}\sum_{k=1}^{p-1} q_k\Delta_k)(\Delta_p + q^{-1}\sum_{k=1}^{p-1} q_k\Delta_k)' + \right.$$

$$\left. + \sum_{k=1}^{p-1} q_k\Delta_k\Delta_k' - q^{-1}\sum_{k=1}^{p-1} q_k\Delta_k \sum_{j=1}^{p-1} q_j\Delta_j'\right|^{-m/2} .$$

Put

$$\boldsymbol{\delta}_k = R^{-1/2}\Delta_k\,, \quad k = 1, \ldots, p\,, \quad \boldsymbol{\delta}^* = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_p)\,, \quad \boldsymbol{\delta} = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{p-1})\,.$$

The posterior density of $\boldsymbol{\delta}^*$ is

$$h_6(\boldsymbol{\delta}^* \mid x) = c\left|I + q(\boldsymbol{\delta}_p + q^{-1}\sum_{k=1}^{p-1} q_k\boldsymbol{\delta}_k)(\boldsymbol{\delta}_p + q^{-1}\sum_{k=1}^{p-1} q_k\boldsymbol{\delta}_k)' + M\right|^{-m/2} ,$$

where

$$M = \sum_{k=1}^{p-1} q_k\boldsymbol{\delta}_k\boldsymbol{\delta}_k' - q^{-1}\sum_{j=1}^{p-1} q_j\boldsymbol{\delta}_j \sum_{k=1}^{p-1} q_k\boldsymbol{\delta}_k' = \boldsymbol{\delta} H \boldsymbol{\delta}' .$$

The density

$$h_7(\boldsymbol{\delta} \mid x) = \int h_6(\boldsymbol{\delta}^* \mid x)\, \mathrm{d}\boldsymbol{\delta}_p$$

can be calculated using the substitution

$$w = \boldsymbol{\delta}_p + q^{-1}\sum_{k=1}^{p-1} q_k\boldsymbol{\delta}_k .$$

We get similarly as in the proof of Theorem 2.4

$$h_7(\boldsymbol{\delta} \mid x) = c \int \left|I + qww' + M\right|^{-m/2} \mathrm{d}w =$$

$$= c\left|I + M\right|^{-m/2} \int \left|I + (I + M)^{-1/2}\, ww'(I + M)^{-1/2}\right|^{-m/2} \mathrm{d}w .$$

After the substitution $v = (I + M)^{-1/2}\, w$, the Jacobian of which is $\left|I + M\right|^{1/2}$, we get

$$h_7(\boldsymbol{\delta} \mid x) = c\left|I + M\right|^{-m/2+1/2} \int \left|I + vv'\right|^{-m/2} \mathrm{d}v = c\left|I + M\right|^{-(m-1)/2} =$$

$$= c\left| I + \frac{(m-1)^{1/2}\, \boldsymbol{\delta} H^{1/2} H^{1/2}\, \boldsymbol{\delta}'(m-1)^{1/2}}{m-1}\right|^{(m-1)/2} .$$

From Theorem 2.2 we get that the random variable $(m - 1)\,\mathrm{Tr}\,\boldsymbol{\delta} H\boldsymbol{\delta}'$ has asymptotically the $\chi^2_{(p-1)r}$ distribution. Because

$$(m-1)\,\mathrm{Tr}\,\boldsymbol{\delta} H\boldsymbol{\delta}' = (m-1)\,\mathrm{Tr}\,R^{-1/2}\Delta H\Delta' R^{-1/2} = (m-1)\,\mathrm{Tr}\,R^{-1}\Delta H\Delta' ,$$

the assertion is proved. □

Theorem 4.7 can be used for testing the hypothesis that $\boldsymbol{\mu}_1 = \ldots = \boldsymbol{\mu}_p$. We put $\Delta_k = \hat{\boldsymbol{\mu}}_p - \hat{\boldsymbol{\mu}}_k$ and if π_μ exceeds the critical value $\chi^2_{(p-1)r}(\alpha)$, we reject the hypothesis on a level which is asymptotically equal to α.

Theorem 4.8. *Put*

$$S = S_1 + \ldots + S_p ,$$

$$J = \mathrm{Diag}\,\{S_1, \ldots, S_{p-1}\} - (S_1, \ldots, S_{p-1})'\, S^{-1}(S_1, \ldots, S_{p-1}) .$$

$$\Gamma_k = U_k - U_p - (\hat{U}_k - \hat{U}_p) \quad \text{for} \quad k = 1, \ldots, p-1 ,$$

$$\Gamma_p = U_p - \hat{U}_p , \quad \Gamma = (\Gamma'_1, \ldots, \Gamma'_{p-1}) .$$

Then the posterior distribution of the random variable

$$\pi_U = (N - n - p + r - nr)\,\mathrm{Tr}\, R^{-1}\Gamma J \Gamma'$$

is asymptotically the $\chi^2_{n(p-1)r^2}$.

Proof is analogous to that of the previous theorem. From Theorem 2.1 we know that J is positive definite. Put

$$\Gamma^* = (\Gamma'_1, \ldots, \Gamma'_p) , \quad m = N - n - p + r .$$

The posterior density of U is given in Theorem 4.2 (iii). It follows from here that Γ^* has the posterior density

$$h_8(\Gamma^* \mid x) = c\Big|R + \Gamma'_p S_p \Gamma_p + \sum_{k=1}^{p-1} (\Gamma_k - \Gamma_p)'\, S_k(\Gamma_k - \Gamma_p)\Big|^{-m/2} =$$

$$= c\Big|R + (\Gamma_p - S^{-1}\sum_{k=1}^{p-1} S_k\Gamma_k)'\, S(\Gamma_p - S^{-1}\sum_{k=1}^{p-1} S_k\Gamma_k) +$$

$$+ \sum_{k=1}^{p-1} \Gamma'_k S_k \Gamma_k - (\sum_{k=1}^{p-1} S_k\Gamma_k)'\, S^{-1}(\sum_{k=1}^{p-1} S_k\Gamma_k)\Big|^{-m/2} .$$

Put

$$\gamma_k = \Gamma_k R^{-1/2} , \quad \gamma^* = (\gamma'_1, \ldots, \gamma'_p) , \quad \gamma = (\gamma'_1, \ldots, \gamma'_{p-1}) .$$

Then the posterior density of γ^* is

$$h_9(\gamma^* \mid x) = c\Big|I + (\gamma_p - S^{-1}\sum_{k=1}^{p-1} S_k\gamma_k)'\, S(\gamma_p - S^{-1}\sum_{k=1}^{p-1} S_k\gamma_k) + M\Big|^{-m/2} ,$$

where

$$M = \sum_{k=1}^{p-1} \gamma'_k S_k \gamma_k - (\sum_{k=1}^{p-1} S_k\gamma_k)'\, S^{-1}(\sum_{k=1}^{p-1} S_k\gamma_k) = \gamma J \gamma' .$$

The posterior density of γ can be calculated from

$$h_{10}(\gamma \mid x) = \int h_9(\gamma^* \mid x)\, \mathrm{d}\gamma_p .$$

After the substitution

$$w = \gamma_p - S^{-1}\sum_{k=1}^{p-1} S_k\gamma_k$$

we have

$$h_{10}(\gamma \mid x) = c \int |I + wSw' + M^{-m/2}|\, \mathrm{d}w =$$

$$= c|I + M|^{-m/2} \int |I + (I + M)^{-1/2}\, wS^{1/2}S^{1/2}w'(I + M)^{-1/2}|^{-m/2}\, \mathrm{d}w .$$

The next substitution is $v = (I + M)^{-1/2} wS^{1/2}$ and it has the Jacobian $|S|^{-r/2} \times$ $\times |I + M|^{nr/2}$. Thus

$$h_{10}(\gamma \mid x) = c|I + M|^{-(m-nr)/2} =$$
$$= c\left| I + \frac{(m - nr)^{1/2}\gamma H^{1/2}H^{1/2}\gamma'(m - nr)^{1/2}}{m - nr} \right|^{-(m-nr)/2} .$$

Using Theorem 2.3 we come to the conclusion that the posterior distribution of $\mathrm{Tr}(m - nr)\,\gamma J\gamma'$ is asymptotically the $\chi^2_{n(p-1)r^2}$. Since

$$\mathrm{Tr}\,(m - nr)\,\gamma J\gamma' = (m - nr)\,\mathrm{Tr}\,R^{-1/2}\Gamma J\Gamma' R^{-1/2} =$$
$$= (m - nr)\,\mathrm{Tr}\,R^{-1}\Gamma J\Gamma' ,$$

the proof is finished. □

A test of the hypothesis $U_1 = \ldots = U_p$ can be based on Theorem 4.8. Under the hypothesis we have $\Gamma_k = \hat{U}_p - \hat{U}_k$. If $\pi_U \geqq \chi^2_{n(p-1)r^2}(\alpha)$, we reject the hypothesis.

5. MODEL WITH PERIODIC VARIANCE MATRICES

This case is described in Theorem 3.1. Some proofs are analogous to those given in Section 4, and so we shall sketch them only. On the other side, the model with periodic variance matrices leads also to new mathematical problems, which will be analyzed in detail.

Theorem 5.1. *Let the prior density of G, U and μ be proportional to $|G_1|^{-1/2} \ldots$ $\ldots |G_p|^{-1/2}$ for positive definite matrices G_k independently of $X_1, \ldots, X_n$ and let the prior density be zero otherwise. Then the posterior density of U, G and μ is for positive definite matrices G_k given by the formula*

$$g(U, G, \mu \mid x) = c \prod_{k=1}^{p} |G_k|^{(\alpha_k - 1)/2} \exp\left\{-\tfrac{1}{2}\,\mathrm{Tr}\,G_k D_k\right\} ,$$

where the matrices D_k are introduced in Theorem 3.1.

Proof follows immediately from the Bayes theorem and from Theorem 3.1. □

Theorem 5.2. *The marginal posterior densities of G, U and μ are given by the formulas*

(i) $$g_1(G \mid x) = c \prod_{k=1}^{p} |G_k|^{(\alpha_k - 2 - nr)/2} \exp\left\{-\tfrac{1}{2}\,\mathrm{Tr}\,G_k R_k\right\}$$

for positive definite $G_1, \ldots, G_p$ and zero otherwise;

(ii) $$g_2(U \mid x) = c \prod_{k=1}^{p} |R_k + (U_k - \hat{U}_k)'\,S_k(U_k - \hat{U}_k)|^{-(\alpha_k + r - 1)/2} ;$$

(iii) $$g_3(\boldsymbol{\mu} \mid \boldsymbol{x}) = c \prod_{k=1}^{p} \left|\boldsymbol{R}_k + q_k(\boldsymbol{\mu}_k - \hat{\boldsymbol{\mu}}_k)(\boldsymbol{\mu}_k - \hat{\boldsymbol{\mu}}_k)'\right|^{-(\alpha_k + r - nr)/2} .$$

Proof. (i) Since

(5.1) $$g(\boldsymbol{U}, \boldsymbol{G}, \boldsymbol{\mu} \mid \boldsymbol{x}) = c \prod_{k=1}^{p} |\boldsymbol{G}_k|^{(\alpha_k - 1)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{G}_k \boldsymbol{R}_k -$$
$$- \tfrac{1}{2} \operatorname{Tr} \boldsymbol{G}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) - \tfrac{1}{2}\alpha_k(\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k + \boldsymbol{U}_k'\bar{\boldsymbol{x}}_k^0)' \boldsymbol{G}_k(\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k + \boldsymbol{U}_k'\bar{\boldsymbol{x}}_k^0)\},$$
the density
$$h_1(\boldsymbol{U}, \boldsymbol{G} \mid \boldsymbol{x}) = \int g(\boldsymbol{U}, \boldsymbol{G}, \boldsymbol{\mu} \mid \boldsymbol{x}) \, \mathrm{d}\boldsymbol{\mu}$$
can be calculated using the substitution
$$\boldsymbol{y}_k = \boldsymbol{G}_k^{1/2}(\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k + \boldsymbol{U}_k'\bar{\boldsymbol{x}}_k^0), \quad k = 1, \dots, p .$$
The Jacobian is $|\boldsymbol{G}_1|^{-1/2} \dots |\boldsymbol{G}_p|^{-1/2}$, and thus

(5.2) $$h_1(\boldsymbol{U}, \boldsymbol{G} \mid \boldsymbol{x}) = c \prod_{k=1}^{p} |\boldsymbol{G}_k|^{(\alpha_k - 2)/2} \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{G}_k \boldsymbol{R}_k\} \times$$
$$\times \exp\{-\tfrac{1}{2} \operatorname{Tr} \boldsymbol{G}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)\} .$$
Obviously, $g_1(\boldsymbol{G} \mid \boldsymbol{x}) = \int h_1(\boldsymbol{U}, \boldsymbol{G} \mid \boldsymbol{x}) \, \mathrm{d}\boldsymbol{U}$. We come to the result putting
$$\boldsymbol{M}_k = \boldsymbol{G}_k^{1/2}(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)', \quad k = 1, \dots, p,$$
the Jacobian of which is $\prod_{k=1}^{p} |\boldsymbol{G}_k|^{-nr/2}$.

(ii) The density $g_2(\boldsymbol{U} \mid \boldsymbol{x})$ follows from (5.2) and from Theorem 2.5.

(iii) The density $h_2(\boldsymbol{U}, \boldsymbol{\mu} \mid \boldsymbol{x}) = \int g(\boldsymbol{U}, \boldsymbol{G}, \boldsymbol{\mu} \mid \boldsymbol{x}) \, \mathrm{d}\boldsymbol{G}$ can be calculated from (5.1) using Theorem 2.5. The result is
$$h_2(\boldsymbol{U}, \boldsymbol{\mu} \mid \boldsymbol{x}) = c \prod_{k=1}^{p} |\boldsymbol{R}_k + (\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) +$$
$$+ \alpha_k(\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k + \boldsymbol{U}_k'\bar{\boldsymbol{x}}_k^0)(\boldsymbol{\mu}_k - \bar{\boldsymbol{x}}_k + \boldsymbol{U}_k'\bar{\boldsymbol{x}}_k^0)'|^{-(\alpha_k + r)/2} =$$
$$= c \prod_{k=1}^{p} |\boldsymbol{R}_k + (\boldsymbol{U}_k - \boldsymbol{U}_k^{\#})'(\boldsymbol{S}_k + \alpha_k \bar{\boldsymbol{x}}_k^0 \bar{\boldsymbol{x}}_k^{0\prime})(\boldsymbol{U}_k - \boldsymbol{U}_k^{\#}) +$$
$$+ q_k(\boldsymbol{\mu}_k - \hat{\boldsymbol{\mu}}_k)(\boldsymbol{\mu}_k - \hat{\boldsymbol{\mu}}_k)'|^{-(\alpha_k + r)/2},$$
where $\boldsymbol{U}_k^{\#}$ are defined in (4.4). Put
$$\boldsymbol{Q}_k = (\boldsymbol{U}_k - \boldsymbol{U}_k^{\#})'(\boldsymbol{S}_k + \alpha_k \bar{\boldsymbol{x}}_k^0 \bar{\boldsymbol{x}}_k^{0\prime})^{1/2}, \quad k = 1, \dots, p .$$
The corresponding Jacobian is
$$\prod_{k=1}^{p} |\boldsymbol{S}_k + \alpha_k \bar{\boldsymbol{x}}_k^0 \bar{\boldsymbol{x}}_k^{0\prime}|^{1/2} .$$

If we denote briefly $Q = (Q_1, ,.., Q_p)'$, then the posterior density of Q and μ is

$$h_3(Q, \mu \mid x) = c \prod_{k=1}^{p} \left|R_k + Q_k Q_k' + q_k(\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)'\right|^{-(\alpha_k + r)/2} .$$

We use Theorem 2.4 for calculating $g_3(\mu \mid x)$ and the proof is finished. □

Theorem 5.3. *The modus of the posterior distribution is* $U = \hat{U}$, $\mu = \hat{\mu}$ and $G_k = \hat{G}_k = (\alpha_k - 2 - nr) R_k^{-1}$.

Proof is the same at that of Theorem 4.4. □

Theorem 5.4. *Random variables*

$$\lambda_{U,k} = (\alpha_k + r - 1) \operatorname{Tr} R_k^{-1}(U_k - \hat{U}_k)' S_k(U_k - \hat{U}_k)$$

for $k = 1, \ldots, p$ *are independent and each of them has asymptotically the* $\chi^2_{n^2r^2}$ *distribution. The variable*

$$\lambda_U = \lambda_{U,1} + \ldots + \lambda_{U,p}$$

has asymptotically the $\chi^2_{pn^2r^2}$ *distribution.*

Proof is the same as that of Theorem 4.5. □

Theorem 5.5. *Random variables*

$$\lambda_{\mu,k} = (\alpha_k + r - nr)\, q_k(\mu_k - \hat{\mu}_k)' R_k^{-1}(\mu_k - \hat{\mu}_k)$$

for $k = 1, \ldots, p$ *are independent and each of them has asymptotically the* χ^2_{pr} *distribution. The variable*

$$\lambda_\mu = \lambda_{\mu,1} + \ldots + \lambda_{\mu,p}$$

has asymptotically the $\chi^2_{p^2r}$ *distribution.*

Proof is the same as that of Theorem 4.6. □

Theorem 5.6. *If* $U_1 = \ldots = U_p$, *then the asymptotic posterior distribution of the variable*

$$\lambda_U^* = \sum_{k=1}^{p-1} (\alpha_k + r - 1) \operatorname{Tr} R_k^{-1}(\hat{U}_k - \hat{U}_p)' S_k(\hat{U}_k - \hat{U}_p) +$$

$$+ \left\{\sum_{k=1}^{p-1} (\alpha_k + r - 1) \operatorname{vec}\left[R_k^{-1}(\hat{U}_k - \hat{U}_p)' S_k\right]\right\}' \left[\sum_{k=1}^{p} (\alpha_k + r - 1)(S_k \otimes R_k^{-1})\right]^{-1} \times$$

$$\times \left\{\sum_{k=1}^{p-1} (\alpha_k + r - 1) \operatorname{vec}\left[R_k^{-1}(\hat{U}_k - \hat{U}_p)' S_k\right]\right\}$$

is the $\chi^2_{n(p-1)r^2}$.

Proof. From Theorem 5.2 (iii) we have that

$$g_3(\boldsymbol{U} \mid \boldsymbol{x}) = c \prod_{k=1}^{p} |\boldsymbol{I} + \boldsymbol{R}_k^{-1/2}(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k) \boldsymbol{R}_k^{-1/2}|^{-(\alpha_k+r-1)/2} .$$

It is clear that $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_p$ are conditionally, given $\boldsymbol{x}$, independent. Introduce the random matrices

$$\boldsymbol{V}_k = (\alpha_k + r - 1)^{1/2} \boldsymbol{R}_k^{-1/2}(\boldsymbol{U}_k - \hat{\boldsymbol{U}}_k)' \boldsymbol{S}_k^{1/2} , \quad k = 1, \ldots, p .$$

Denote

$$\boldsymbol{v}_k = \operatorname{vec} \boldsymbol{V}_k , \quad \boldsymbol{A}_k = (\alpha_k + r - 1)^{1/2} (\boldsymbol{S}_k^{1/2} \otimes \boldsymbol{R}_k^{-1/2}) ,$$

$$\boldsymbol{u}_k = \operatorname{vec} \boldsymbol{U}_k , \quad \hat{\boldsymbol{u}}_k = \operatorname{vec} \hat{\boldsymbol{U}}_k , \quad \boldsymbol{F}_k = \boldsymbol{A}_k'\boldsymbol{A}_k , \quad \boldsymbol{F} = \boldsymbol{F}_1 + \ldots + \boldsymbol{F}_p ,$$

$$\boldsymbol{J} = \operatorname{Diag} \{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_{p-1}\} - (\boldsymbol{F}_1, \ldots, \boldsymbol{F}_{p-1})' \boldsymbol{F}^{-1}(\boldsymbol{F}_1, \ldots, \boldsymbol{F}_{p-1}) .$$

Using Theorem 2.7 we have $\boldsymbol{v}_k = \boldsymbol{A}_k(\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k)$. According to Theorem 2.2, $\boldsymbol{v}_k$ has asymptotically $N(\boldsymbol{0}, \boldsymbol{I})$ distribution. Then $\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k$ has asymptotically $N(\boldsymbol{0}, \boldsymbol{F}_k^{-1})$ distribution. The density of the asymptotic prosterior distribution of the vector $\boldsymbol{u} = (\boldsymbol{u}_1', \ldots, \boldsymbol{u}_p')'$ is

$$c \exp \{ -\tfrac{1}{2} \sum_{k=1}^{p} (\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k)' \boldsymbol{F}_k(\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k)\} .$$

Let

$$\boldsymbol{\Delta}_k = (\boldsymbol{u}_k - \boldsymbol{u}_p) - (\hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}_p) , \quad k = 1, \ldots, p - 1 , \quad \boldsymbol{\Delta}_p = \boldsymbol{u}_p - \hat{\boldsymbol{u}}_p ,$$

$$\boldsymbol{\Delta}^* = (\boldsymbol{\Delta}_1', \ldots, \boldsymbol{\Delta}_p')' , \quad \boldsymbol{\Delta} = (\boldsymbol{\Delta}_1', \ldots, \boldsymbol{\Delta}_{p-1}')' .$$

The asymptotic posterior distribution of $\boldsymbol{\Delta}^*$ has the density

$$h_6(\boldsymbol{\Delta}^* \mid \boldsymbol{x}) = c \exp \{ -\tfrac{1}{2}[\boldsymbol{\Delta}_p'\boldsymbol{F}_p\boldsymbol{\Delta}_p + \sum_{k=1}^{p-1} (\boldsymbol{\Delta}_k + \boldsymbol{\Delta}_p)' \boldsymbol{F}_k(\boldsymbol{\Delta}_k + \boldsymbol{\Delta}_p)]\} =$$

$$= c \exp \{ -\tfrac{1}{2}[(\boldsymbol{\Delta}_p - \boldsymbol{F}^{-1} \sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k)' \boldsymbol{F}(\boldsymbol{\Delta}_p - \boldsymbol{F}^{-1} \sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k) +$$

$$+ \sum_{k=1}^{p-1} \boldsymbol{\Delta}_k'\boldsymbol{F}_k\boldsymbol{\Delta}_k - (\sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k)' \boldsymbol{F}^{-1}(\sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k)]\} .$$

From here we obtain the marginal asymptotic posterior density

$$h_7(\boldsymbol{\Delta} \mid \boldsymbol{x}) = c \exp \{ -\tfrac{1}{2}[\sum_{k=1}^{p-1} \boldsymbol{\Delta}_k'\boldsymbol{F}_k\boldsymbol{\Delta}_k - (\sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k)' \boldsymbol{F}^{-1}(\sum_{k=1}^{p-1} \boldsymbol{F}_k\boldsymbol{\Delta}_k)]\} =$$

$$= c \exp \{ -\tfrac{1}{2}\boldsymbol{\Delta}'\boldsymbol{J}\boldsymbol{\Delta}\} .$$

Then the variable

$$\lambda_U^* = \boldsymbol{\Delta}'\boldsymbol{J}\boldsymbol{\Delta}$$

has the asymptotic posterior distribution $\chi^2_{n(p-1)r^2}$. If $\boldsymbol{U}_1 = \ldots = \boldsymbol{U}_p$, then $\boldsymbol{\Delta}_k =$

$= \hat{u}_p - \hat{u}_k$ for $k = 1, \ldots, p-1$. In this case

$$\lambda_U^* = \sum_{k=1}^{p-1} (\hat{u}_k - \hat{u}_p)' F_k(\hat{u}_k - \hat{u}_p) - [\sum_{k=1}^{p-1} F_k(\hat{u}_k - \hat{u}_p)]' F^{-1} [\sum_{k=1}^{p-1} F_k(\hat{u}_k - \hat{u}_p)] .$$

In the next formulas we use Theorem 2.7. Since

$$A_k(\hat{u}_k - \hat{u}_p) = (\alpha_k + r - 1)^{1/2} [(S_k^{1/2} \otimes R_k^{-1/2}) (\text{vec}\, \hat{U}_k' - \text{vec}\, \hat{U}_p')] =$$
$$= (\alpha_k + r - 1)^{1/2}\, \text{vec}\, R_k^{-1/2} (\hat{U}_k - \hat{U}_p)' S_k^{1/2} ,$$

we have

$$(\hat{u}_k - \hat{u}_p)' F_k(\hat{u}_k - \hat{u}_p) =$$
$$= (\alpha_k + r - 1)\, \text{Tr}\, R_k^{-1/2} (\hat{U}_k - \hat{U}_p)' S_k^{1/2} S_k^{1/2} (\hat{U}_k - \hat{U}_p) R_k^{-1/2} =$$
$$= (\alpha_k + r - 1)\, \text{Tr}\, R_k^{-1} (\hat{U}_k - \hat{U}_p)' S_k (\hat{U}_k - \hat{U}_p) .$$

Further we get

$$F_k(\hat{u}_k - \hat{u}_p) = (\alpha_\kappa + r - 1) (S_k^{1/2} \otimes R_k^{-1/2}) (S_k^{1/2} \otimes R_k^{-1/2})\, \text{vec}\, (\hat{U}_k - \hat{U}_p)' =$$
$$= (\alpha_k + r - 1) (S_k \otimes R_k^{-1})\, \text{vec}\, (\hat{U}_k - \hat{U}_p)' =$$
$$= (\alpha_k + r - 1)\, \text{vec}\, [R_k^{-1} (\hat{U}_k - \hat{U}_p)' S_k]$$

and

$$F = \sum_{k=1}^{p} F_k = \sum_{k=1}^{p} A_k' A_k = \sum_{k=1}^{p} (\alpha_k + r - 1) (S_k^{1/2} \otimes R_k^{-1/2}) (S_k^{1/2} \otimes R_k^{-1/2}) =$$
$$= \sum_{k=1}^{p} (\alpha_k + r - 1) (S_k \otimes R_k^{-1}) . \ \square$$

Theorem 5.6 can be used for testing the hypothesis $U_1 = \ldots = U_p$. If $\lambda_U^* \geqq \chi^2_{n(p-1)r^2}(\alpha)$, we reject the hypothesis.

It remains to solve the problem how to test if the process is generated by a model with periodic variance matrices or by a model with constant variance matrices. From Theorem 5.2 (i) we can see that, given x, the random matrices $G_1, \ldots, G_p$ are independent and the density of G_k is

$$c |R_k|^{(\alpha_k - nr + r - 1)/2} |G_k|^{(\alpha_k - nr - 2)/2} \exp \{-\tfrac{1}{2} \text{Tr}\, R_k G_k\} .$$

This means that

$$G_k \sim W_r[\alpha_k - (n-1) r - 1, R_k^{-1}] , \quad k = 1, \ldots, p ,$$

where $W_r(m, V)$ is a Wishart distribution with the density

$$c |V|^{-m/2} |X|^{(m-r-1)/2} \exp \{-\tfrac{1}{2} \text{Tr}\, V^{-1} X\}$$

for $r \times r$ positive definite matrices X.

Let us remark that if $X_1, \ldots, X_N$ is a sample from $N_r(\mu, V)$, then

$$A = \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})' \sim W_r(N - 1, V)$$

(see Rao [12], Chap. 8b(IV)). If there are p independent samples such that kth sample is from $N_r(\boldsymbol{\mu}_k, \boldsymbol{V}_k)$ and its size is N_k, we can calculate matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_p$ and $\boldsymbol{A}_k \sim W_r(N_k - 1, \boldsymbol{V}_k)$. Anderson [5] describes a procedure based on $\boldsymbol{A}_1, \dots, \boldsymbol{A}_p$ for testing the hypothesis $\boldsymbol{V}_1 = \dots = \boldsymbol{V}_p$. The same procedure can be used also in our case to test $\boldsymbol{G}_1 = \dots = \boldsymbol{G}_p$. Denote

$$n_k = \alpha_k - nr - 1\,, \quad n_0 = n_1 + \dots + n_p = N - n - p(nr - 1)\,,$$

$$V_1 = |\boldsymbol{R}|^{-n_0/2} \prod_{k=1}^{p} |\boldsymbol{R}_k|^{n_k/2}\,, \quad W_1 = V_1 n_0^{rn_0/2} \prod_{k=1}^{p} n_k^{-rn_k/2}\,,$$

$$f = \tfrac{1}{2}(p - 1)\, r(r + 1)\,, \quad \varrho = 1 - \left(\sum \frac{1}{n_k} - \frac{1}{n_0}\right) \frac{2r^2 + 3r - 1}{6(r + 1)(p - 1)}\,,$$

$$\omega_2 = \frac{r(r + 1)}{48\varrho^2} \left[(r - 1)(r + 2)\left(\sum \frac{1}{n_k^2} - \frac{1}{n_0^2}\right) - 6(p - 1)(1 - \varrho^2)\right].$$

For a given z put

$$Q(z) = \mathsf{P}\{\chi_f^2 \leqq z\} + \omega_2[\mathsf{P}\{\chi_{f+4}^2 \leqq z\} - \mathsf{P}\{\chi_f^2 \leqq z\}]\,.$$

Then

$$\mathsf{P}\{-\varrho \ln W_1 \leqq z\} = Q(z) + O(n_0^{-3})\,.$$

If $Q(-\varrho \ln W_1) \geqq 1 - \alpha$, we reject the hypothesis $\boldsymbol{G}_1 = \dots = \boldsymbol{G}_p$ on a level which is approximately equal to α.

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Souhrn

O MNOHOROZMĚRNÉ PERIODICKÉ AUTOREGRESI

JIŘÍ ANDĚL

Model periodické autoregrese je v tomto článku zobecněn na mnohorozměrný případ. Autoregresní matice jsou periodickými funkcemi času. Střední hodnota procesu může být nenulovou periodickou posloupností vektorů. Odhady parametrů a testy statistických hypotéz jsou založeny na bayesovském přístupu. Jsou vyšetřovány dvě hlavní varianty modelu. Jedna se týká případu, kdy inovační proces má konstantní varianční matice, druhá připouští možnost periodických variančních matic.

Резюме

О МНОГОМЕРНОЙ ПЕРИОДИЧЕСКОЙ АВТОРЕГРЕССИИ

JIŘÍ ANDĚL

В статье дается обобщение модели периодической авторегрессии на многомерный случай. Матрицы авторегрессии являются периодическими функциями времени. Среднее значение процесса выражается в виде периодической последовательности векторов. Оценки параметров и проверки гипотез основаны на принципе Баеса. Изучаются модели с постоянными и с периодическими ковариационными матрицами белого шума.

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