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STABILITY ANALYSIS OF REDUCIBLE QUADRATURE METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*)

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Summary. Stability analysis for numerical solutions of Volterra integro-differential equations based on linear multistep methods combined with reducible quadrature rules is presented. The results given are based on the test equation

$$y'(t) = \gamma\, y(t) + \int_0^t (\lambda + \mu t + \nu s)\, y(s)\, \mathrm{d}s$$

and absolute stability is defined in terms of the real parameters γ, λ, μ and ν. Sufficient conditions are illustrated for $(\bar{\theta}; \theta)$ — methods and for combinations of Adams-Moulton and backward differentiation methods.

Keywords: Stability of numerical solution, Volterra integro-differential equations.

1. INTRODUCTION

In this paper we investigate the asymptotic behavior of numerical solutions of Volterra integro-differential equations (VIDEs) based on the class of linear multistep methods combined with reducible quadrature rules. These methods were introduced by Matthys [8] and further investigated by Wolkenfelt [11]. Matthys derived some conditions for A-stability (a concept analogous to that for ordinary differential equations) for these methods based on the test equation

$$y'(t) = \gamma\, y(t) + \int_0^t \lambda\, y(s)\, \mathrm{d}s \tag{1}$$

with γ and λ complex. Wolkenfelt [11] computed via boundary-locus method stability regions with respect to (1) for two classes of methods employing backward differentiation formulas. Stability analysis for some numerical methods based on the test equation (1) was also performed by Brunner/Lambert [5] and Baker et. al. [1] (see also the survey paper by Brunner [4] for more information).

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Stability analysis for reducible quadrature methods for Volterra integral equations of the second kind was performed by Wolkenfelt [12] based on a test equation of convolution type. The analysis is facilitated by taking advantage of the fact that application of reducible multistep methods to this test equation leads to a difference equation of finite order with constant coefficients. Applying the approach of Wolkenfelt to a more general test equation the authors [2] obtained a difference equation of finite order but with variable coefficients. It was observed that this equation is of Poincaré type, and stability analysis was performed with the use of an extension of the classical theorem of Perron. A similar analysis is presented here for VIDEs based on the test equation

$$
\begin{aligned}
(2) \qquad & y'(t) = \gamma\, y(t) + \int_0^t (\lambda + \mu t + \nu s)\, y(s)\, \mathrm{d}s\,, \quad t \geqq 0 \\
& y(0) = 1\,,
\end{aligned}
$$

where $\gamma, \lambda, \mu, \nu$ are real. We are able to characterize the regions in the $(h\gamma, h^2\lambda, h^3\mu, h^3\nu)$-space for which the numerical solutions are bounded. Unfortunately, it is not known to the authors in what region the solutions to (2) are bounded, and in order to judge the quality of numerical methods based on the results presented in this paper such information should be obtained. This seems to be a difficult problem and some progress in this direction has been made by Sanchez [9]. (This problem has recently been solved. See Note added in proof.)

In Section 2 some basic results related to Poincaré type difference equations are presented, and an extension of the classical theorem of Perron [10] is given, which was proved in [3]. These results are used in Section 3 to establish sufficient conditions for absolute stability of the numerical methods considered. Results for $(\bar{\theta}; \theta)$ methods as well as for combinations of Adams-Moulton and backward differentiation methods are obtained in Section 4. The regions of absolute stability are plotted as illustration of our approach.

2. BOUNDEDNESS OF SOLUTIONS OF DIFFERENCE EQUATIONS

In this section we consider the difference equation of order r with variable coefficients

$$
(3) \qquad \sum_{i=0}^{r} \alpha_{i,n} y_{n-i} = 0\,,
$$

$n = r, r+1, \ldots$, where $\alpha_{i,n} = \alpha_i + n^{-1}\beta_i$, $i = 0, 1, \ldots, r$ and α_i and β_i are constants. Define the characteristic polynomials associated with this equation as

$$
\phi(\xi) = \sum_{i=0}^{r} \alpha_i \xi^{r-i}\,, \quad \psi(\xi) = \sum_{i=0}^{r} \beta_i \xi^{r-i}\,.
$$

Equation (3) is a difference equation of Poincaré type (see [10]) and we use results concerning boundedness of solutions of (3). One of the principal facts about such equations is the Perron Theorem, which we state here as given in [10].

Theorem 1. *Let $q_1, q_2, \ldots, q_s$ be the distinct moduli of the roots of the polynomial ϕ and let l_λ be the number of roots whose modulus is q_λ, multiple roots being counted according to their multiplicity, so that $l_1 + l_2 + \ldots + l_s = r$. Then, if $\alpha_0 \neq 0$, $\alpha_{r,n} \neq$ for all n, the difference equation* (3) *has a fundamental system of solutions, which fall into s classes such that, for the solutions of the λth class and their linear combinations,*

$$\limsup_{n \to \infty} \sqrt[n]{|y_n|} = q_\lambda .$$

The number of solutions of the λth class is l_λ.

We note here that if all roots of ϕ have modulus less than 1 then every solution of (3) is bounded, while if ϕ has a root of modulus greater than 1, there exists an unbounded solution of this equation. In [3] the authors consider the case in which the polynomial ϕ has some roots of modulus 1. This may occur in some applications in numerical analysis, and in Section 4 we give some examples as an illustration. Thus, suppose that ϕ is a simple von Neumann polynomial and denote its essential roots, i.e. those of modulus 1, as $\xi_1, \xi_2, \ldots, \xi_k$. Denote its nonessential roots, i.e. those of modulus less than 1, as $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_r$. We state the result for (3) here as in [3].

Theorem 2. *Assume that $\alpha_0 \neq 0$ and $\alpha_{0,n} \neq 0$ for all n; the polynomials ϕ and ψ have no common factor and ϕ is a simple von Neumann polynomial with essential roots $\xi_1, \xi_2, \ldots, \xi_k$ ($k = 0$ is allowed). Then every solution of* (3) *is bounded, provided that*

$$\frac{\pi}{2} \leqq |\mathrm{Arg}(\xi_i) - \mathrm{Arg}(\gamma_i)| \leqq \frac{3}{2}\pi ,$$

where $\gamma_i = -\psi(\xi_i)/\phi'(\xi_i)$, $i = 1, 2, \ldots, k$. Here, $\mathrm{Arg}(z)$ *stands for the principal value of the argument of the complex number z, i.e.* $\mathrm{Arg}(z) \in (-\pi, \pi]$.

The assumption that ϕ is a simple von Neumann polynomial cannot be relaxed (see Remark 1 in [3]).

3. STABILITY ANALYSIS OF REDUCIBLE QUADRATURE METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Consider the VIDEs of the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \quad t \geqq 0 \\ y(0) &= y_0 , \end{aligned} \tag{4}$$

where

$$z(t) = \int_0^t K(t, s, y(s))\, ds , \tag{5}$$

and f and K are continuous. Let $h > 0$ be a given stepsize and put $t_n = nh$, $n = 0, 1, \ldots$. For the numerical solution of (4)–(5) we will consider the linear multistep method for ODEs with coefficients $\bar{a}_i$, $\bar{b}_i$, $i = 0, 1, \ldots, \bar{k}$, $\bar{a}_0 \neq 0$, $|\bar{a}_{\bar{k}}| + |\bar{b}_{\bar{k}}| \neq 0$, coupled with the quadrature formula

$$\int_0^{t_n} P(t)\,\mathrm{d}t = h \sum_{j=0}^{n} w_{n,j}\, P(t_j)\,.$$

The resulting method is given by

$$\sum_{i=0}^{\bar{k}} \bar{a}_i y_{n-i} = h \sum_{i=0}^{\bar{k}} \bar{b}_i f(t_{n-i}, y_{n-i}, z_{n-i})\,, \tag{6}$$

$i = k, k + 1, \ldots$, where

$$z_{n-i} = h \sum_{j=0}^{n-i} w_{n-i,j} K(t_{n-i}, t_j, y_j)\,. \tag{7}$$

These methods were introduced and convergence theorems were developed by Linz [7]. Following Matthys [8] and Wolkenfelt [11, 12] we will assume that the quadrature rule (7) is reducible to the linear multistep method for ODEs with coefficients a_i, b_i, $i = 0, 1, \ldots, k$, i.e. the following relation holds

$$\sum_{i=0}^{k} a_i w_{n-i,j} = \begin{cases} 0\,, & j = 0, 1, \ldots, n - k - 1\,, \\ b_{n-j}\,, & j = n - k, \ldots, n\,. \end{cases} \tag{8}$$

Denote by $\bar{\varrho}$ and $\bar{\sigma}$ (ϱ and σ) the characteristic polynomials of the linear multistep method with coefficients $\bar{a}_i$ and $\bar{b}_i$, $i = 0, 1, \ldots, \bar{k}$ (a_i, b_i, $i = 0, 1, \ldots, k$), i.e.,

$$\bar{\varrho}(\xi) = \sum_{i=0}^{\bar{k}} \bar{a}_i \xi^{\bar{k}-i}\,, \quad \bar{\sigma}(\xi) = \sum_{i=0}^{\bar{k}} \bar{b}_i \xi^{\bar{k}-i}\,;$$

$$\varrho(\xi) = \sum_{i=0}^{k} a_i \xi^{k-i}\,, \quad \sigma(\xi) = \sum_{i=0}^{k} b_i \xi^{k-i}\,.$$

The methods will be referred to as $(\bar{\varrho}, \bar{\sigma})$ and (ϱ, σ) methods, respectively. It is assumed that:

(i) the $(\bar{\varrho}, \bar{\sigma})$ and (ϱ, σ) methods are consistent, i.e., $\bar{\varrho}(1) = 0$, $\bar{\varrho}'(1) = \bar{\sigma}(1)$; $\varrho(1) = 0$, $\varrho'(1) = \sigma(1)$;

(ii) the $(\bar{\varrho}, \bar{\sigma})$ and (ϱ, σ) methods are zero stable, i.e. $\bar{\varrho}$ and ϱ are simple von Neumann polynomials;

(iii) these methods are implicit, i.e. $\bar{b}_0 \neq 0$ and $b_0 \neq 0$.

Further, it is assumed that ϱ, σ and $\bar{\varrho}$, $\bar{\sigma}$ have no common factors.

It is the purpose of this paper to investigate the boundedness of numerical solutions of VIDEs using (6)–(7) where the weights satisfy (8). As mentioned in the introduction, stability analysis for this method will be based on the test equation

$$(9) \qquad y'(t) = \gamma\, y(t) + \int_0^t (\lambda + \mu t + \nu s)\, y(s)\, \mathrm{d}s\,, \quad t \geqq 0$$
$$y(0) = y_0$$

where γ, λ, μ, ν and y_0 are real. Boundedness properties of solutions to (9) are not known to the authors, however it is conjectured that for $\gamma < 0$, $2\mu + \nu < 0$ and $\mu + \nu < 0$ the solutions of (9) are bounded. (See Note added in proof.) If $\gamma > 0$, $\gamma \geqq 0$ and $\mu + \nu > 0$ it can be shown that solutions of (9) are unbounded.

The method (6)–(7) with $w_{n,j}$ subject to (8) is called a $((\bar{\varrho}, \bar{\sigma}); (\varrho, \sigma))$ method. The investigation of stability properties of these methods will follow the approach of Wolkenfelt [12] and the authors [2], [3], in which application of these methods to the test equation (9) leads to a difference equation of fixed order which characterizes the solution of (9). Thus, the method (6)–(7) applied to (9) takes the form

$$\sum_{i=0}^{\bar{k}} \bar{a}_i y_{n-i} = h \sum_{i=0}^{\bar{k}} \bar{b}_i \Big(\gamma y_{n-i} + h \sum_{j=0}^{n-i} w_{n-i,j} (\lambda + \mu(n-i)\, h + \nu j h)\, y_j\Big)\,,$$

$n = \bar{k}, \bar{k} + 1, \dots$. Taking a weighted sum of these successive equations using the coefficients a_l, $l = 0, 1, \dots, k$, we have

$$\sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} a_l \bar{a}_i y_{n-l-i} = h \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} a_l \bar{b}_i \gamma y_{n-l-i} +$$
$$+ h^2 \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} \sum_{j=0}^{n-l-i} a_l \bar{b}_i w_{n-l-i,j} (\lambda + \mu(n-l-i)\, h + \nu j h)\, y_j\,,$$

$n = k + \bar{k},\ k + \bar{k} + 1, \dots$. Using (8) along with the convention that $w_{n,j} = 0$ for $j > n$ it follows that

$$(10) \qquad \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} a_l \bar{a}_i y_{n-l-i} =$$
$$= h \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} (a_l \bar{b}_i \gamma + h b_l \bar{b}_i (\lambda + \mu(n-i)\, h + \nu(n-l-i)\, h))\, y_{n-l-i} -$$
$$- h^2 \sum_{i=0}^{\bar{k}} \sum_{j=0}^{n-i} \sum_{l=0}^{k} a_l \bar{b}_i w_{n-l-i,j} \mu h l y_j\,,$$

$n = k + \bar{k}, k + \bar{k} + 1, \dots$. Another application of the weighted sum of successive equations (10) using the coefficients a_p, $p = 0, 1, \dots, k$, and condition (8) yields the difference equation of order $2k + \bar{k}$

$$(11) \qquad \sum_{p=0}^{k} \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} (a_p (a_l \bar{a}_i - h a_l \bar{b}_i \gamma -$$
$$- h^2 b_l \bar{b}_i (\lambda + \mu(n - 2p - i)\, h + \nu(n - p - l - i)\, h)\, y_{n-l-i-p} = 0\,.$$

We now define the notion of absolute stability for the $((\bar{\varrho}, \bar{\sigma}); (\varrho, \sigma))$ methods.

Definition 1. *The $((\bar{\varrho}, \bar{\sigma}); (\varrho, \sigma))$ method is said to be absolutely stable for given $h\gamma$, $h^2\gamma$, $h^3\mu$ and $h^3\nu$ if, for these values, every solution of (11) is bounded.*

Definition 2. *A region $\mathscr{A}$ in $(h\gamma, h^2\lambda, h^3\mu, h^3\nu)$-space is said to be the region of absolute stability for the $((\bar{\varrho}, \bar{\sigma}); (\varrho, \sigma))$ method if the method is absolutely stable for all $(h\gamma, h^2\lambda, h^3\mu, h^3\nu) \in \mathscr{A}$.*

Equation (11) can be written as

$$(12) \qquad \sum_{p=0}^{k} \sum_{l=0}^{k} \sum_{i=0}^{\bar{k}} \{h^3(\mu + \nu)\, a_p b_l \bar{b}_i + n^{-1}(a_p(a_l \bar{b}_i h\gamma - a_l \bar{a}_i - h^3 \mu b_l \bar{b}_i(2p + i) - h^3 \nu b_l \bar{b}_i(p + l + i))\}\, y_{n-l-p-i} = 0$$

which is in the form of (3), and it can easily be established that the characteristic polynomials associated with (12) are given by

$$\phi(\xi) = h^3(\mu + \nu)\, \varrho(\xi)\, \sigma(\xi)\, \bar{\sigma}(\xi)$$

and

$$\psi(\xi) = h\gamma\, \varrho^2(\xi)\, \bar{\sigma}(\xi) - \varrho^2(\xi)\, \bar{\varrho}(\xi) + \varrho(\xi)\, \sigma(\xi)\, \bar{\sigma}(\xi)\, (h^2\lambda - (2k + \bar{k})\, h^3(\mu + \nu)) + \xi h^3 \mu(2\sigma(\xi)\, \bar{\sigma}(\xi)\, \varrho'(\xi) + \sigma(\xi)\, \varrho(\xi)\, \bar{\sigma}'(\xi)) + \xi h^3 \nu \frac{\mathrm{d}}{\mathrm{d}\xi} (\varrho(\xi)\, \sigma(\xi)\, \bar{\sigma}(\xi))\,.$$

Thus, if $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{\bar{p}}$ are the essential roots of $\bar{\sigma}$, $\xi_1 = 1, \xi_2, \ldots, \xi_p$ are the essential roots of ϱ and $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}$ are the essential roots of σ it follows from $\bar{\gamma}_i = -\psi(\bar{\xi}_i)/\phi'(\bar{\xi}_i)$, $i = 1, 2, \ldots, \bar{p}$ and $\gamma_i = -\psi(\xi_i)/\phi'(\xi_i)$, $i = 1, 2, \ldots, p + q$, that

$$(13) \qquad \bar{\gamma}_i = \frac{\varrho(\bar{\xi}_i)\, \varrho(\bar{\xi}_i)}{h^3(\mu + \nu)\, \sigma(\bar{\xi}_i)\, \bar{\sigma}'(\bar{\xi}_i)} - \bar{\xi}_i\,,$$

$i = 1, 2, \ldots, \bar{p}$,

$$(14) \qquad \gamma_i = -\frac{\xi_i(2\mu + \nu)}{\mu + \nu}\,,$$

$i = 1, 2, \ldots, p$, and

$$(15) \qquad \gamma_i = \frac{\varrho(\xi_i)\, \bar{\varrho}(\xi_i)}{h^3(\mu + \nu)\, \bar{\sigma}(\xi_i)\, \sigma'(\xi_i)} - \frac{\gamma\, \varrho(\xi_i)}{h^2(\mu + \nu)\, \sigma'(\xi_i)} - \frac{\nu \xi_i}{\mu + \nu}$$

$i = p + 1, p + 2, \ldots, p + q$.

Since the difference equation (12) is of Poincaré type, we have the following theorem as a consequence of Theorem 2.

Theorem 3. *Assume that conditions* (i)–(iii) *hold for $(\bar{\varrho}, \bar{\sigma})$ and (ϱ, σ) and that ϕ is a simple von Neumann polynomial. Then the region of absolute stability $\mathscr{A}$ for the $((\bar{\varrho}, \bar{\sigma}); (\varrho, \sigma))$ method is given by*

$$\mathscr{A} = \{(h\gamma, h\gamma, h\mu, h\nu) : \frac{\pi}{2} \leqq |\mathrm{Arg}(\bar{\xi}_i) - \mathrm{Arg}(\bar{\gamma}_i)| \leqq \frac{3}{2}\pi\,, \quad i = 1, 2, \ldots, \bar{p}\,, \quad \textit{and}$$

$$\frac{\pi}{2} \leqq |\mathrm{Arg}(\xi_i) - \mathrm{Arg}(\gamma_i)| \leqq \frac{3}{2}\pi\,, \quad i = 1, 2, \ldots, p + q\}\,,$$

where $\bar{\gamma}_i$ *and* γ_i *are defined as in* (13), (14) *and* (15).

4. EXAMPLES

As a special case we may consider the $(\bar{\theta};\, \theta)$-methods in which the characteristic polynomials are given by

$$\bar{\varrho}(\xi) = \xi - 1\,, \quad \bar{\sigma}(\xi) = \bar{\theta}\xi + 1 - \bar{\theta}\,,$$
$$\varrho(\xi) = \xi - 1\,, \quad \sigma(\xi) = \theta\xi + 1 - \theta\,,$$

where $0 \leqq \bar{\theta},\ \theta \leqq 1$.

We first observe that if $\bar{\theta} < \frac{1}{2}$ or $\theta < \frac{1}{2}$, then $\bar{\sigma}$ or σ has a root with modulus greater than 1, hence by Theorem 1 there exists an unbounded solution of (11) so the region of absolute stability is empty.

If $\bar{\theta} > \frac{1}{2}$ and $\theta > \frac{1}{2}$ then the only essential root of $\phi(\xi)$ is $\xi_1 = 1$. In this case $\gamma_1 = -(2\mu + \nu)/(\mu + \nu)$ and $\mathscr{A}$ consists of all $(h\gamma, h^2\gamma, h^3\mu, h^3\nu)$ such that $2\mu + \nu > 0$ and $\mu + \nu > 0$ or $2\mu + \nu < 0$ and $\mu + \nu < 0$.

If $\bar{\theta} = \frac{1}{2}$ and $\theta > \frac{1}{2}$ then the essential roots of ϕ are $\bar{\xi}_1 = -1$ and $\xi_1 = 1$. Thus $\bar{\gamma}_1 = (8 + h^3(\mu + \nu)(1 - 2\theta))/(h^3(\mu + \nu)(1 - 2\theta))$, and from Theorem 3, $\bar{\gamma}_1 > 0$. The region of absolute stability in this case consists of all $(h\gamma, h^2\lambda, h^3\mu, h^3\nu)$ such that $2\mu + \nu < 0$ and $\mu + \nu < 0$, or $2\mu + \nu > 0$ and $\mu + \nu > 8/(h^3(2\theta - 1))$.

If $\bar{\theta} > \frac{1}{2}$ and $\theta = \frac{1}{2}$, then $\xi_1 = 1$, $\xi_2 = -1$, γ_1 is defined as before and $\gamma_2 = 8/(h^3(\mu + \nu)(1 - 2\bar{\theta})) + 4\gamma/(h^2(\mu + \nu)) + \nu/(\mu + \nu)$. Theorem 3 requires that $\gamma_2 > 0$ for this method to be absolutely stable, from which it follows that $\mathscr{A} = \mathscr{B} \cup \mathscr{C}$ where

$$\mathscr{B} = \{(h\gamma, h^2\lambda, h^3\mu, h^3\nu){:}\ 2\mu + \nu < 0,\ \mu + \nu < 0,\ \nu < -8/(h^3(1 - 2\bar{\theta})) - 4\gamma/h^2\}$$
$$\mathscr{C} = \ (h\gamma, h^2\lambda, h^3\mu, h^3\nu){:}\ 2\mu + \nu > 0,\ \mu + \nu > 0,\ \nu > -8/(h^3(1 - 2\bar{\theta})) - 4\gamma/h^2\}\,.$$

If $\bar{\theta} = \theta = \frac{1}{2}$, Theorem 3 cannot be applied since ϕ is not a simple von Neumann polynomial. The authors conjecture in this case that the region of absolute stability is empty (see Remark 1 in [3]).

As an illustration, the regions of absolute stability for these cases of the $(\bar{\theta};\, \theta)$-methods are plotted in Figure 1, where

$$c_\theta = 8/(h^3(2\theta - 1))\,, \quad \bar{\nu}_{\bar{\theta}} = -8/(h^3(1 - 2\bar{\theta})) - 4\gamma/h^2\,.$$

As a second example consider the combinations of Adams-Moulton (AM) and backward differentiation (BD) methods. For the Adams-Moulton method with $k \geqq 2$, the characteristic polynomial ϕ has a root of modulus greater than 1 (see Gladwin/Jeltsch [6]). In this case the regions of absolute stability for the methods (AM; $\cdot$), $\bar{k} \geqq 2$ and ($\cdot$; AM), $k \geqq 2$ are all empty.

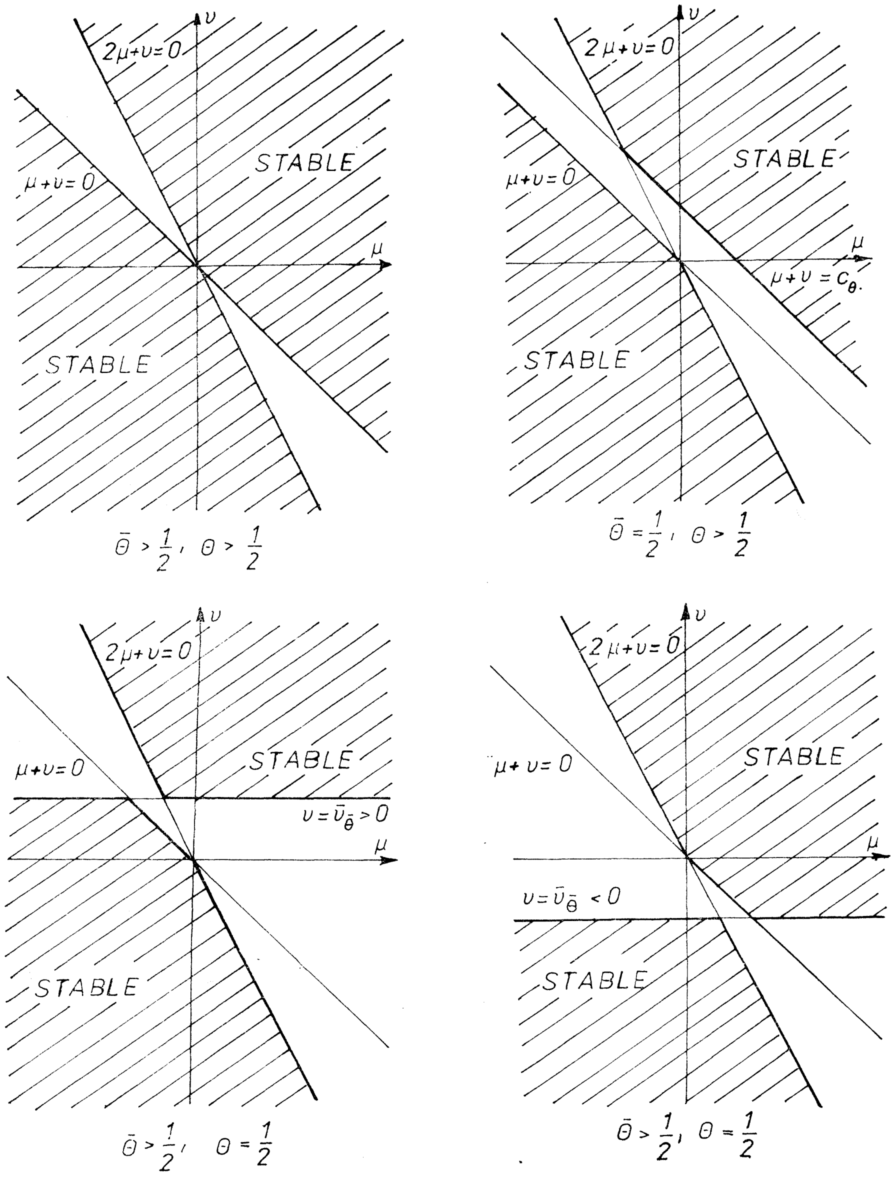

Fig. 1. Stability regions for $(\bar{\theta}, \theta)$-methods.

The polynomial ϕ associated with the backward differentiation method for $k = 1, 2, \ldots, 6$ has only one essential root, namely $\xi_1 = 1$. The other characteristic polynomial is of the form $\sigma(\xi) = b_k \xi^k$. Thus, for the methods (BD; BD), $\bar{k}, k =$

= 1, 2, ..., 6, the polynomial ϕ associated with the difference equation (12) has only one essential root, and the hypotheses of Theorem 3 are satisfied. The region of absolute stability for these methods is given by $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$, where

$$\mathscr{A}_1 = \{(h\gamma, h^2\lambda, h^3\mu, h^2\nu): \mu + \nu < 0,\ 2\mu + \nu < 0\}\,,$$
$$\mathscr{A}_2 = \{(h\gamma, h^2\lambda, h^3\mu, h^3\nu): \mu + \nu > 0,\ 2\mu + \nu > 0\}\,.$$

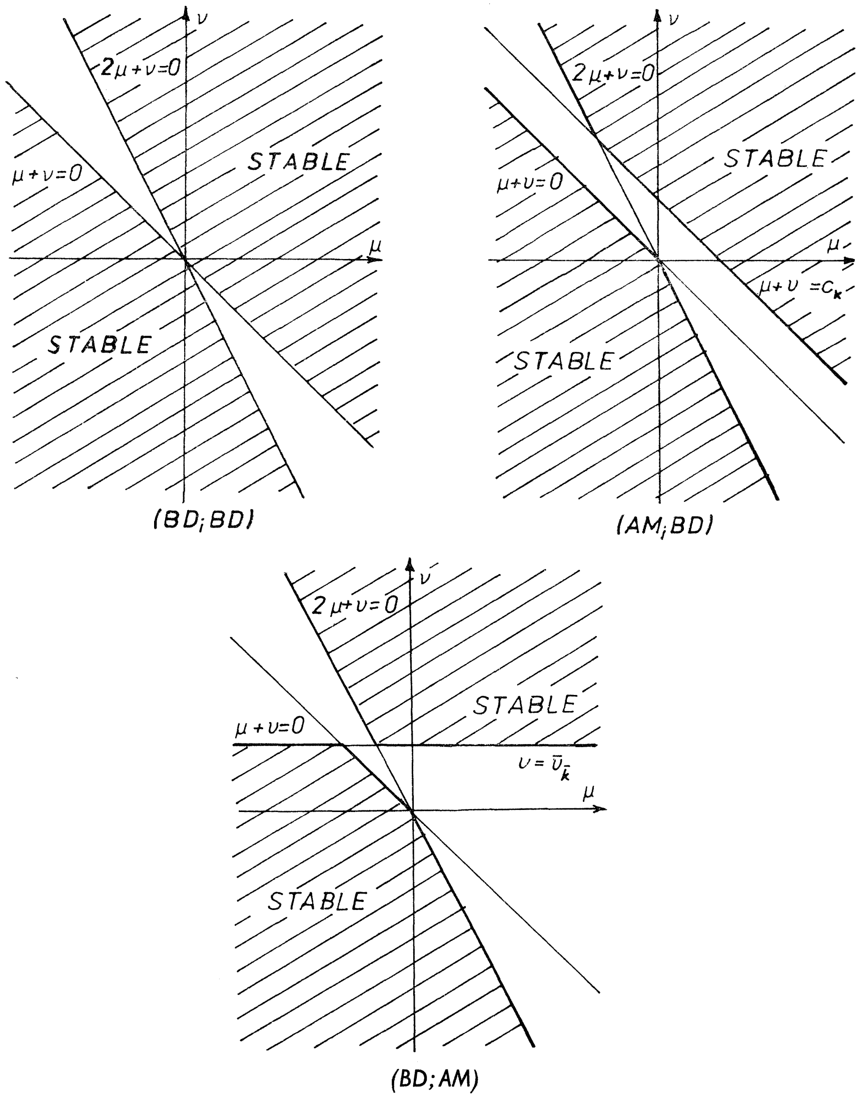

Fig. 2. Stability regions for (BD; BD), (AM; BD) methods.

If we combine BD with AM, the only AM method necessary to consider is for $k = 1$, since for $k = 0$ this method is equivalent to BD with $k = 1$, and if $k \geqq 2$, as mentioned above, the polynomial $\bar{\sigma}$ has a root with modulus greater than 1. The characteristic polynomials associated with AM, $k = 1$, are given by

$$\varrho(\xi) = \xi - 1\,, \quad \sigma(\xi) = \tfrac{1}{2}(\xi + 1)\,.$$

If we consider (AM; BD), $\bar{k} = 1$, $1 \leqq k \leqq 6$, the hypotheses of Theorem 3 are satisfied, and the characteristic polynomial ϕ associated with (12) has the essential roots $\bar{\xi}_1 = -1$ and $\xi_1 = 1$. From (13) we have

$$\bar{\gamma}_1 = (-1)^{k+1}\, 4\varrho(-1)/(h^3(\mu + \nu)\, b_k) + 1$$

and from (14),

$$\gamma_1 = -\frac{2\mu + \nu}{\mu + \nu}\,.$$

From Theorem 3 it follows that we we must have $\bar{\gamma}_1 > 0$ and $\gamma_1 < 0$ for stability. Define the region

$$\tilde{\mathscr{A}} = \{(h\gamma, h^2\lambda, h^3\mu, h^3\nu): (-1)^{k+1}\, 4\varrho(-1)/(h^3(\mu + \nu)\, b_k) + 1 > 0\}\,.$$

Then the region of absolute stability for the methods (AM; BD), $\bar{k} = 1$, $1 \leqq k/\leqq 6$ is given by $\mathscr{A} \cap \tilde{\mathscr{A}}$.

If we consider the methods (BD; AM), $1 \leqq \bar{k} \leqq 6$, $k = 1$, the polynomial ϕ has the essential roots $\xi_1 = 1$ and $\xi_2 = -1$. Again, the hypotheses of Theorem 3 are satisfied and if we define the region

$$\tilde{\mathscr{A}} = \{(h\gamma, h^2\lambda, h^3\mu, h^3\nu): (-1)^{\bar{k}+1}\, 4\bar{\varrho}(-1)/(h^3(\mu + \nu)\, \bar{b}_{\bar{k}}) + \\ + 4\gamma/(h^2(\mu + \nu)) + \nu/(\mu + \nu) \geqq 0\}\,,$$

then the region of absolute stability for these methods is given by $\mathscr{A} \cap \tilde{\mathscr{A}}$.

The only method in this class left to consider is (AM; AM), $\bar{k} = k = 1$. This corresponds to $(\bar{\theta}, \theta)$ methods with $\bar{\theta} = \theta = \frac{1}{2}$.

The above regions are illustrated in Figure 2, where

$$c_k = (-1)^k\, 4\varrho(-1)/(h^3 b_k)\,, \quad \bar{v}_{\bar{k}} = (-1)^{\bar{k}}\, 4\bar{\varrho}(-1)/(h^3 \bar{b}_{\bar{k}}) - 4\gamma/h^2$$

Note added in proof. The authors recently shown that for $2\mu + \nu < 0$ and $\mu + \nu < 0$ the solutions of the test equation (2) are bounded. The approach is to consider the associated third order differential equation. If one makes a substitution of the form $t = s^\alpha$, $0 < \alpha < 1$, the resulting equation has bounded coefficients and as a consequence we can use classical stability criteria to investigate boundedness of solutions. A complete proof will appear in a subsequent paper.

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Souhrn

ANALÝZA STABILITY REDUCIBILNÍCH METOD ŘEŠENÍ VOLTERROVÝCH INTEGRODIFERENCIÁLNÍCH ROVNIC

V. L. BAKKE, Z. JACKIEWICZ

Je podána analýza stability numerického řešení Volterrových integrodiferenciálních rovnic založená na lineárních vícekrokových metodách kombinovaných s reducibilními metodami kvadratury. Výsledky jsou založeny na testovací rovnici

$$y'(t) = \gamma\, y(t) + \int_0^t (\lambda + \mu t + \nu s)\, y(s)\, \mathrm{d}s$$

a absolutní stabilita je definována pomocí reálných parametrů γ, λ, μ a ν. Pro $(\bar{\theta}; \theta)$-metody a pro kombinace Adamsových-Moultonových metod s metodami zpětného diferencování jsou ilustrovány postačující podmínky.

Резюме

АНАЛИЗ УСТОЙЧИВОСТИ ПРИВОДИМЫХ МЕТОДОВ РЕШЕНИЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВОЛЬТЕРРА

V. L. BAKKE, Z. JACKIEWICZ

В статье проведён анализ устойчивости численного решения интегро-дифференциальных уравнеий Вольтерра, основанного на линейных многошаговых методах комбинированных с приводимыми методами квадратуры. Результаты основаны на испытательном уравнении

$$y'(t) = \gamma\, y(t) + \int_0^t (\lambda + \mu t + \nu s)\, y(s)\, \mathrm{d}s$$

и абсолютная устойчивость определена при помощи действительных параметров γ, λ, μ и ν. Для $(\bar{\theta}; \theta)$-методов и для метода Адамса-Мултона, комбинированного с методами обратного дифференцирования, уллюстрированы достаточные условия.

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