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ON MULTI-PARAMETER ERROR EXPANSIONS IN FINITE DIFFERENCE METHODS FOR LINEAR DIRICHLET PROBLEMS

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Summary. The paper is concerned with the finite difference approximation of the Dirichlet problem for a second order elliptic partial differential equation in an n-dimensional domain.

Considering the simplest finite difference scheme and assuming a sufficient smoothness of the domain, coefficients of the equation, right-hand part, and boundary condition, the author develops a general error expansion formula in which the mesh sizes of an (n-dimensional) rectangular grid in the directions of the individual axes appear as parameters.

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In finite difference methods the one-parameter error expansions have been studied by many authors (cf. for instance [1] and references therein). In this paper we investigate the multi-parameter expansions for solving elliptic linear Dirichlet problems on a multidimensional domain with smooth boundary.

1. THE DIFFERENTIAL PROBLEM

Let R^n be a real n-dimensional Euclidean space. Let Ω be a bounded domain in R^n and Γ its boundary. Denote by $x = (x_1, \ldots, x_n)$ the point in R^n. Let functions of n variables $x_1, \ldots, x_n$: $f(x)$, $p_i(x)$, $q(x)$ on $\bar{\Omega}$ and $g(x)$ on Γ, be given. Consider the differential operator

$$Lu \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p_i \frac{\partial u}{\partial x_i}\right) - qu , \quad x \in \Omega ,$$

The differential problem is

$$Lu = f , \quad x \in \Omega , \tag{1.1}$$

$$u = g , \quad x \in \Gamma . \tag{1.2}$$

Assume that there exist a real number λ $(0 < \lambda < 1)$, and a positive integer m so that (cf. [2])

(1.3) $$\Gamma \in C^{2m+2+\lambda} ;$$

$$p_i \in C^{2m+1+\lambda}(\bar{\Omega}) ; \quad q, f \in C^{2m+\lambda}(\bar{\Omega}) ; \quad g \in C^{2m+2+\lambda}(\Gamma) ;$$

(1.4) $$p_i \geqq \text{const} > 0 ; \quad q \geqq 0 .$$

Then we have ([1])

Lemma 1. *The problem* (1)–(4) *has a unique solution*

(1.5) $$u \in C^{2m+2+\lambda}(\bar{\Omega}) .$$

2. THE GRID

Assume that A_i, B_i, $i = 1, \ldots, n$ are real numbers such that

$$\Omega \subset D = \{x \mid A_i \leqq x_i \leqq B_i\} .$$

Let N_i be given positive integers. We put

$$h_i = (B_i - A_i)/N_i ,$$

$$x_i(j_i) = A_i + j_i h_i ; \quad j_i = 0, 1, 2, \ldots .$$

Then the points $(x_1(j_1), \ldots, x_n(j_n))$, denoted by $(j_1, \ldots, j_n)$, are called grid points in the rectange D, and the grid over Ω, denoted by Ω_h, is defined by

$$\Omega_h = \{(j_1, \ldots, j_n) \mid (j_1, \ldots, j_n) \in \Omega\} .$$

Each point of Ω_h is called an interior grid point. Each interior grid point $(j_1, \ldots, j_n)$ has $2n$ neighbouring points which are

(2.1) $$(j_1, \ldots, j_{k-1}, j_k \pm 1, j_{k+1}, \ldots, j_n) , \quad k = 1, \ldots, n .$$

If all points (2.1) belong to $\bar{\Omega}$ then the point $(j_1, \ldots, j_n)$ is called a regular interior grid point. If at least one point of (2.1) does not belong to $\bar{\Omega}$ then the point $(j_1, \ldots, j_n)$ is an irregular interior grid point. Denote respectively by $\Omega_{h,r}$ and $\Omega_{h,ir}$ the sets of regular and irregular interior grid points. Then we have $\Omega_h = \Omega_{h,r} \cup \Omega_{h,ir}$.

3. THE DISCRETE PROBLEM

3.1. Notation. We introduce the following notation:

1) $i \in I$ iff $i = (i_1, \ldots, i_n)$, $i_k = \text{integer} \geqq 0$.
2) If $i \in I$ then

$$|i| = i_1 + \ldots + i_n ,$$

$$w_{[i]} = w_{i_1 \ldots i_n} ;$$

3) $h = (h_1, \ldots, h_n)$, $h_k = (B_k - A_k)/N_k$, $|h| = \max \{h_1, \ldots, h_n\}$.

3.2. **Approximation of the differential operator.** Let v be a function defined on $\Omega_h \cup \Gamma$. Then its value at a point P is denoted by $v(P)$ or $v(x_1(P), \ldots, x_n(P))$, $x_k(P)$ being the k-coordinate of P. Now at $P \in \Omega_{h,r}$ we consider the discrete operator

$$L_h v \equiv \sum_{i=1}^{n} (a_i v_{\bar{x}_i})_{x_i} - qv$$

where

$$(a_i v_{\bar{x}_i})_{x_i} = h_i^{-2}[a_i^{(+i)}(P)(v^{(+i)}(P) - v(P)) - a_i^{(-i)}(P)(v(P) - v^{(-i)}(P))],$$
$$a_i^{(\pm i)}(P) = p_i(x_1(P), \ldots, x_{i-1}(P), x_i(P) \pm 0{\cdot}5h_i, x_{i+1}(P), \ldots, x_n(P)),$$
$$v_i^{(\pm i)}(P) = v(x_1(P), \ldots, x_{i-1}(P), x_i(P) \pm h_i, x_{i+1}(P), \ldots, x_n(P)).$$

It is obvious that we have

Lemma 2. *The discrete operator L_h satisfies the maximum principle.*

Now by applying Taylor's formula we obtain

Lemma 3. *For any function $w \in C^{2l+2+\lambda}(\bar{\Omega})$ we have*

$$L_h w = Lw + \sum_{i=1}^{n} \sum_{k=1}^{l} h_i^{2k} F_{ik}(w) + r_1,$$

where $F_{ik}(w)$ depend only on w and on the derivatives of w up to order $2k + 2$, and $|r_1| \leqq$ const. $|h|^{2l+\lambda}$.

Lemma 4. *For any $w_{[j]} \in C^{2m-2|j|+2+\lambda}(\bar{\Omega})$, $j \in I$, we have*

$$L_h(u + S_m) = Lu + \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} \ldots h_n^{2j_n}(Lw_{[j]} + G_{[j]}(u, \ldots, w_{[i]}, \ldots)) + r_2$$

where u satisfies (1.5),

$$S_m = \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} \ldots h_n^{2j_n} w_{[j]}, \tag{3.1}$$

$G_{[j]}$ depends only on u and $w_{[i]}$ up to $|i| < |j|$, and $|r_2| \leqq$ const. $|h|^{2m+\lambda}$.

Proof. We have

$$L_h(u + S_m) = L_h u + \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} \ldots h_n^{2j_n} L_h w_{[j]}.$$

Then the application of Lemma 3 to $L_h u$ and $L_h w_{[j]}$ completes the proof.

Lemma 5. *Under the assumptions* (1.3) (1.4) *there exist functions $w_{[j]} \in C^{2m-2k+2+\lambda}(\bar{\Omega})$, $|j| = k$, $k = 1, \ldots, m$, independent of h so that*

$$L_h(u + S_m) = Lu + r_3$$

where S_m has the form (3.1) *and $|r_3| \leqq$ const. $|h|^{2m+\lambda}$.*

Proof. We can write the conditions that make the coefficients of $h_1^{2j_1} \dots h_n^{2j_n}$ in Lemma 4 equal to zero:

$$Lw_{[j]} = -G_{[j]}, \quad x \in \Omega ; \quad w_{[j]} = 0, \quad x \in \Gamma .$$

Then, according to Lemma 1, the functions $w_{[j]}$ are successively determined for $|j| = 1$ to $|j| = m$ and belong to $C^{2m-2|j|+2+\lambda}(\bar{\Omega})$.

3.3. **Approximation of the boundary condition.** Now let $P \in \Omega_{h,ir}$. We shall calculate the value $v(P)$ with the help of Lagrange's interpolating polynomials starting with the values of v on the boundary Γ and at some points of $\Omega_{h,r}$ ([1]). First, in a way analogous to [1] consider the quantity

$$B(d) = \sum_{k=1}^{2m} \frac{(2m)!}{k!\,(2m-k)!} \cdot \frac{d}{d+k}, \quad d > 0 .$$

We observe that $B(d)$ decreases when d decreases and tends to zero when d tends to zero. So there exists $\delta > 0$ such that

$$B(d) \leqq B(\delta) < 1 , \quad d < \delta .$$

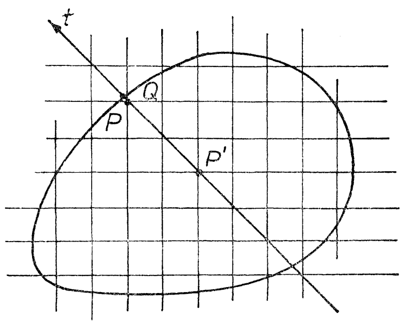

Fig. 1.

Let $P \in \Omega_{h,ir}$. Consider a fixed point P' (fig. 1) of $\Omega_{h,r}$. As the grid is uniform along each coordinate direction, the line PP', which can but need not be parallel to a coordinate direction, passes through many equally spaced grid points of $\Omega_{h,r}$. Let η be the distance between these equally spaced points. Denote by Pt the axis obtained by orienting the line PP' from the origin P to the exterior of Ω. Let Q be the intersection of Pt with the boundary Γ. Let $PQ = \sigma\eta$ with some positive σ. Let μ be the smallest positive integer satisfying $\mu \geqq \sigma/\delta$ and $H = \mu\eta$. Then $PQ = dH$ with $d = \sigma/\mu \leqq \delta$. Consider the points on Pt with the abscissae

$$(3.2) \qquad -2mH, \quad -(2m-1)H, \dots, -2H, -H, \mathrm{d}H ,$$

under the assumption that all these points belong to $\bar{\Omega}$. This assumption is satisfied when h is small enough. Then these points belong to $\Omega_{h,r} \cup \Gamma$.

Now let $w(t)$ be a smooth enough function on $[-2mH, dH]$. Consider the interpolating polynomial $P_{2m}(t)$ of degree $2m$ at the nodes (3.2), so that

$$P_{2m}(-kH) = w(-kH)\,, \quad k = 1, \ldots, 2m\;; \quad P_{2m}(dH) = w(dH)\,.$$

Then we get

$$w(P) = w(0) = J_d\, w(0) + \Lambda_d\, w(dH) + R(0)\,,$$

where

$$J_d\, w(0) = \sum_{k=1}^{2m} (-1)^k \frac{(2m)!}{k!\,(2m-k)!} \cdot \frac{d}{d+k} \cdot w(-kH)\,,$$

$$\Lambda_d\, w(dH) = \Lambda_d\, w(Q) = \sum_{k=1}^{2m} \frac{k}{d+k}\,.$$

(The above formulae for the operators J_d, Λ_d have been introduced in [1].) Concerning the remaining term $R(0)$ we have

Lemma 6. *If* $w(t) \in C^{M+1}[-2mH, dH]$, $M \leqq 2m$, *then*

$$|R(0)| \leqq H^{M+1} \frac{d}{M+1} \max_{t\in[-2mH,dH]} |w^{(M+1)}(t)|\,.$$

The proof can be done by repeated application of Rolle's theorem.

If $P \in \Omega_{h,ir}$ we put, analogously to [1]:

$$v(P) = J_d\, v(P) + \Lambda_d\, v(Q)\,.$$

Then Lemma 6 yields

Lemma 7. *If* $w \in C^{M+1}(\bar{\Omega})$, $M \leqq 2m$ *then*

$$w(P) - J_d\, w(P) - \Lambda_d\, w(Q) = H^{M+1} r_4\,,$$

where $|r_4| \leqq$ const (*independent of* h).

3.4. **The discrete problem.** We introduce the following discrete problem:

$$L_h\, v(P) = f(P)\,, \quad P \in \Omega_{h,r}\,, \tag{3.3}$$

$$v(P) = J_d\, v(P) + \Lambda_d\, v(Q)\,, \quad P \in \Omega_{h,ir}\,, \tag{3.4}$$

$$v(P) = g(P)\,, \quad P \in \Gamma\,. \tag{3.5}$$

4. THE ASYMPTOTIC ERROR EXPANSION

4.1. **Theorem 1.** *The discrete problem* (3.3)–(3.5) *has a unique solution* v *which is the limit of* $v^{(\nu)}$ *calculated by the iterations*

$$L_h\, v^{(\nu)} = f(P)\,, \quad P \in \Omega_{h,r}\,,$$

$$v^{(\nu)} = J_d\, v^{(\nu-1)}(P) + \Lambda_d\, v^{(\nu-1)}(Q)\,, \quad P \in \Omega_{h,ir}\,,$$
$$v^{(\nu)} = g(P)\,, \quad P \in \Gamma\,.$$

Proof. We have

$$L_h(v^{(\nu+1)} - v^{(\nu)}) = 0\,, \quad P \in \Omega_{h,r}\,, \tag{4.1}$$

$$v^{(\nu+1)} - v^{(\nu)} = J_d(v^{(\nu)} - v^{(\nu-1)})\,, \quad P \in \Omega_{h,ir}\,. \tag{4.2}$$

We define the norms

$$\|w\|_h = \max_{P\in\Omega_h} |w(P)|\,, \quad \|w\|_{h,ir} = \max_{P\in\Omega_{h,ir}} |w(P)|\,.$$

By virtue of the maximum principle (Lemma 2) we deduce from (4.1), (4.2)

$$\|v^{(\nu+1)} - v^{(\nu)}\|_h \leqq \|v^{(\nu+1)} - v^{(\nu)}\|_{h,ir} =$$
$$= \|J_d(v^{(\nu)} - v^{(\nu-1)})\|_{h,ir} \leqq B(\delta)\|v^{(\nu)} - v^{(\nu-1)}\|_h\,.$$

Therefore

$$\|v^{(\nu+1)} - v^{(\nu)}\|_h \leqq \varrho\|v^{(\nu)} - v^{(\nu-1)}\|_h \tag{4.3}$$

where $\varrho = B(\delta) < 1$.

Hence the discrete problem (3.3)–(3.5) has a unique solution which is the limit when $\nu \to \infty$ of $v^{(\nu)}$ for any $v^{(0)}$.

4.2. **Theorem 2.** *There exist functions* $w_{[j]} \in C^{2m-2k+2+\lambda}(\bar{\Omega})$, $j \in I$, $|j| = k$, $k = 1, \ldots, m$, *independent of* h, *so that we have the asymptotic error expansion*

$$v(P) = u(P) + S_m + r_5\,,$$

where v *and* u *are solutions of the discrete and differential problems, respectively,* S_m *has the form* (3.1) *and* $|r_5| \leqq$ const (*independent of* h). $|h|^{2m+\lambda}$.

Proof. From (4.3) we deduce

$$\|v^{(\nu+1)} - v^{(\nu)}\|_h \leqq \varrho^\nu\|v^{(1)} - v^{(0)}\|_h\,,$$

hence

$$\|v^{(\nu)} - v^{(0)}\|_h \leqq \frac{1}{1-\varrho}\, \| v^{(1)} - v^{(0)}\|_h\,.$$

Therefore

$$\|v - v^{(0)}\|_h \leqq \frac{1}{1-\varrho}\, \| v^{(1)} - v^{(0)}\|_h$$

and we choose

$$v^{(0)} = u + S_m = u + \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} \ldots h_n^{2j_n} w_{[j]}$$

where $w_{[j]}$ are determined in Lemma 5 in which u is the solution of the differential problem.

In order to evaluate $\|v^{(1)} - v^{(0)}\|_h$ we write

$$L_h v^{(1)} = f(P), \quad P \in \Omega_{h,r},$$
$$v^{(1)} = J_d v^{(0)}(P) + \Lambda_d v^{(0)}(Q), \quad P \in \Omega_{h,ir}.$$

On the other hand, by Lemma 5 we have

$$L_h v^{(0)} = L_h(u + S_m) = Lu + r_3 .$$

So putting $v^{(1)} - v^{(0)} = z$ we have

$$L_h z = -r_3, \quad P \in \Omega_{h,r},$$
$$z = J_d v^{(0)}(P) + \Lambda_d v^{(0)}(Q) - v^{(0)}(P), \quad P \in \Omega_{h,ir}.$$

Since $v^{(0)} = u + S_m$ we have at $P \in \Omega_{h,ir}$

$$z = J_d u(P) + \Lambda_d u(Q) - u(P) + \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} \dots h_n^{2j_n} \times$$
$$\times \left(J_d w_{[j]}(P) + \Lambda_d w_{[j]}(Q) - w_{[j]}(P)\right).$$

Then, taking into account the smoothness of $w_{[j]}$ and Lemma 7 we have at $P \in \Omega_{h,ir}$

(4.4) $$z = r, \quad |r| \leqq \text{const} . |h|^{2m+1} .$$

So z satisfies

$$L_h z = \alpha, \quad P \in \Omega_{h,r},$$
$$z = r, \quad P \in \Omega_{h,ir},$$
$$z = 0 \quad P \in \Gamma,$$

(4.5) $$|\alpha| \leqq c . |h|^{2m+\lambda}$$

where

$$c = \text{const (independent of } h) .$$

Let us put

(4.6) $$z = z_1 + z_2$$

with

(4.7) $$L_h z_1 = 0, \quad P \in \Omega_{h,r},$$

(4.8) $$z_1 = r, \quad P \in \Omega_{h,ir}; \quad z_1 = 0, \quad P \in \Gamma,$$

(4.9) $$L_h z_2 = \alpha, \quad P \in \Omega_{h,r},$$

(4.10) $$z_2 = 0, \quad P \in \Omega_{h,ir} \cup \Gamma .$$

By the maximum principle (Lemma 2) we get from (4.7), (4.8)

(4.11) $$\|z_1\|_h \leqq \|r\|_{h,ir} .$$

To evaluate z_2 we consider the differential problem

$$Lw = -2, \quad P \in \Omega,$$
$$w = 2, \quad P \in \Gamma.$$

Thus w exists by Lemma 1 and

(4.12) $$0 < w \leqq K = \text{const (independent of } h).$$

At the same time

(4.13) for h small enough

we have

$$L_h w \leqq -1, \quad P \in \Omega_{h,r},$$
$$w \geqq 1, \quad P \in \Omega_{h,ir} \cup \Gamma.$$

Now we consider another differential problem

$$LW = -2K', \quad P \in \Omega,$$
$$W = 2K', \quad P \in \Gamma$$

where

(4.14) $$K' = \max |\alpha(P)|, \quad P \in \Omega_{h,r}.$$

So W exists and, in view of (4.12),

(4.15) $$0 < W \leqq KK'.$$

At the same time under the condition (4.13) we have

$$L_h W \leqq -K', \quad P \in \Omega_{h,r},$$
$$W \geqq K', \quad P \in \Omega_{h,ir} \cup \Gamma.$$

Therefore (4.9), (4.10) give

$$L_h(W \pm z_2) \leqq 0, \quad P \in \Omega_{h,r},$$
$$W \pm z_2 \geqq 0, \quad P \in \Omega_{h,ir} \cup \Gamma.$$

By the maximum principle we have

$$W \pm z_2 \geqq 0, \quad P \in \Omega_h,$$

that is, in view of (4.15),

$$|z_2| \leqq W \leqq KK', \quad P \in \Omega_h.$$

Taking into account the relations (4.14) and (4.5) we get

(4.16) $$\|z_2\|_h \leqq \text{const}\,.\,|h|^{2m+\lambda}.$$

Finally, the relations (4.6), (4.11), (4.4), (4.16) give

$$\|z\|_h \leqq \text{const (independent of } h) \, . \, |h|^{2m+\lambda}$$

and the theorem is proved.

Note 1. If $p_i = \text{const}$ then the restriction (4.13) is not necessary.

Note 2. The previous results still hold in the case when the term qu in Lu is replaced by $q(x, u)$ where q is smooth enough and $\partial q / \partial u \geqq 0$.

References

[1] *Г. И. Марчук, В. В. Шайдуров:* Повышение точности решений разностных схем. Москва, Наука, 1979.

[2] *О. А. Ладыженская, Н. Н. Уральцева:* Линейные и квазилинейные уравнения эллиптического типа. Москва, Наука, 1973.

Souhrn

O VÍCEPARAMETRICKÝCH ROZVOJÍCH CHYBY U SÍŤOVÝCH METOD PRO LINEÁRNÍ DIRICHLETOVU ÚLOHU

TA VAN DINH

Práce je věnována studiu diferenční aproximace Dirichletovy okrajové úlohy pro eliptickou parciální diferenciální rovnici druhého řádu v n-rozměrné oblasti.

K nejjednoduššímu diferenčnímu schématu odvozuje autor obecný rozvoj chyby, v němž jako parametry vystupují kroky (n-rozměrné) obdélníkové sítě ve směrech jednotlivých souřadnicových os. Předpokládá se přitom dostatečná hladkost oblasti, koeficientů rovnice, pravé strany a okrajové podmínky.

Резюме

О МНОГОПАРАМЕТРИЧЕСКИХ ФОРМУЛАХ ДЛЯ ПОГРЕШНОСТИ МЕТОДА СЕТОК ПРИ РЕШЕНИИ ЛИНЕЙНОЙ ЗАДАЧИ ДИРИХЛЕ

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Статья посвящена конечно-разностной аппроксимации краевой задачи Дирихле для эллиптического дифференциального уравнения второго порядка на n-мерной области.

Используя простейшую разностную схему и предполагая достаточную гладкость области, коэффициентов уравнения, правой части и краевого условия, автор выводит общую формулу для погрешности, в которой в качестве параметров выступают шаги (n-мерной) прямоугольной сетки по направлениям отдельных осей координат.

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