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ANALYSIS OF APPROXIMATE SOLUTIONS OF COUPLED DYNAMICAL THERMOELASTICITY AND RELATED PROBLEMS

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Summary. The authors study problems of existence and uniqueness of solutions of various variational formulations of the coupled problem of dynamical thermoelasticity and of the convergence of approximate solutions of these problems.

First, the semidiscrete approximate solution is defined, which is obtained by time discretization of the original variational problem by Euler's backward formula. Under certain smoothness assumptions on the data authors prove existence and uniqueness of the solution and establish the rate of convergence $O(\Delta t^{1/2})$ of Rothe's functions in the spaces $C(I; W_2^1(\Omega))$ and $C(I; L_2(\Omega))$ for the displacement components and the temperature, respectively. Regularity of solutions is discussed.

In Part 2 the authors define the fully discretized solution of the original variational problem by Euler's backward formula and the simplest finite elements. Convergence of these approximate solutions is proved.

In Part 3, the weakest assumptions possible are imposed onto the data, which corresponds to a different definition of the variational solution. Existence and uniqueness of the variational solution, as well as convergence of the fully discretized solutions, are proved.

Keywords: Rothe's method, finite elements, coupled thermoelasticity, coupled consolidation of clay.

AMS classification: 65 M 20, 65 M 60, 65 N 30.

In the recent years several papers were devoted to the analysis of approximate solutions of coupled dynamical linear thermoelasticity: in [3] and [14] under the assumption that the exact solution is sufficiently smooth the rate of convergence of fully discrete schemes obtained by discretization by finite elements in space and by finite differences in time is established; schemes generated by various finite elements and various finite differences are studied. In [1] the rate of convergence for two semidiscrete schemes obtained by discretization in time is derived under some regularity assumptions. Besides these papers, in [5] an existence and uniqueness theorem is introduced.

Our paper completes the preceding investigations in the following directions: In Section 1 we analyze the simplest semidiscrete scheme (obtained by discretization in time by the Euler backward method). In Theorem 1 we establish the existence, uniqueness and some regularity properties of the solution and prove the strong convergence of the semidiscrete solution to the exact one in the space $C(I; L_2(\Omega))$. Our regularity results are stronger than in [5] under the same assumptions concerning the data. In Theorem 2 we prove without any additional regularity assumptions that the rate of convergence of the semidiscrete solution is $O(\Delta t^{1/2})$. In Theorem 3 we present stronger regularity properties in the space variables in the interior of the domain Ω.

In Section 2 we generalize the convergence results of Theorem 1 to the case of the simplest fully discrete scheme obtained by discretization in space by linear finite elements and in time by the Euler backward method.

In Section 3 a weaker variational formulation is presented. This formulation allows us to consider the data of the problem not so smooth as in Sections 1 and 2. As we cannot obtain results similar those in Theorems 2 and 3 we consider only a fully discrete scheme. We again prove the existence and uniqueness of the exact solution; however, the regularity results and convergence results are weaker than in Theorems 1 and 4.

In Section 4 we mention briefly two related problems: the quasistatical thermoelasticity and one of the models of consolidation of clay. The approach and results of Section 3 can be easily modified to these two cases.

1. ROTHE'S METHOD IN LINEAR THERMOELASTICITY

According to [2], the dynamical two-dimensional problem of coupled linear thermoelasticity can be formulated in the following way: Let Ω be a bounded domain in the x_1, x_2-plane with a sufficiently smooth boundary Γ. Find a vector $u(x_1, x_2, t)$ and a function $\vartheta(x_1, x_2, t)$ which satisfy the following initial-boundary value problem:

(1.1) $$\vartheta_{,ii} + Q = c_1 \dot{\vartheta} + c_2 \operatorname{div} \dot{u} \quad \text{in} \quad \Omega \times (0, T]$$

(1.2) $$\sigma_{ij,j} + f_i = c_4 \ddot{u}_i \quad (i = 1, 2) \quad \text{in} \quad \Omega \times (0, T]$$

(1.3) $$\vartheta(x_1, x_2, t) = \vartheta_B(x_1, x_2), \quad (x_1, x_2) \in \Gamma_{1\vartheta}, \quad t > 0$$

(1.4) $$\partial\vartheta/\partial\nu + \beta(\vartheta - g(x_1, x_2, t)) = 0, \quad (x_1, x_2) \in \Gamma_{2\vartheta}, \quad t > 0$$

(1.5) $$u(x_1, x_2, t) = u_B(x_1, x_2), \quad (x_1, x_2) \in \Gamma_{1u}, \quad t > 0$$

(1.6) $$\sigma_{ij}\nu_j = p_i(x_1, x_2, t), \quad i = 1, 2, \quad (x_1, x_2) \in \Gamma_{2u}, \quad t > 0$$

(1.7) $$\vartheta(x_1, x_2, 0) = \vartheta_0(x_1, x_2), \quad (x_1, x_2) \in \Omega$$

(1.8) $$u(x_1, x_2, 0) = u_0(x_1, x_2), \quad (x_1, x_2) \in \Omega$$

(1.9) $$\dot{u}(x_1, x_2, 0) = v_0(x_1, x_2), \quad (x_1, x_2) \in \Omega$$

where β, c_1, c_2, c_4 are positive constants, $Q(x_1, x_2, t)$, $\vartheta_B(x_1, x_2)$, $\vartheta_0(x_1, x_2)$,

$g(x_1, x_2, t)$ and $f(x_1, x_2, t)$, $u_B(x_1, x_2)$, $p(x_1, x_2, t)$, $u_0(x_1, x_2)$, $v_0(x_1, x_2)$ are given functions and vectors (their smoothness will be specified later), $\partial/\partial\nu$ is the normal derivative and where

(1.10) $$\sigma_{ij} \equiv \sigma_{ij}(u, \vartheta) = D_{ijkm}[\varepsilon_{km}(u) - \alpha\vartheta\delta_{km}] ,$$

(1.11) $$\varepsilon_{ij}(v) = (v_{i,j} + v_{j,i})/2$$

(1.12) $$D_{ijkm} = D_{jikm} = D_{kmij}$$

(1.13) $$D_{ijkm}\xi_{ij}\xi_{km} \geqq \mu_0\xi_{ij}\xi_{ij} \quad \forall\xi_{ij} = \xi_{ji} \in R$$

with $\mu_0 = \text{const.} > 0$. A summation convention over a repeated subscript is adopted. A comma is employed to denote partial differentiation with respect to spatial coordinates and a dot denotes the derivative with respect to time t. The symbols $\Gamma_{1\vartheta}, \Gamma_{2\vartheta}$ (and similarly Γ_{1u}, Γ_{2u}) denote two open subsets of the boundary such that mes $\Gamma_{1\vartheta}$ + mes $\Gamma_{2\vartheta}$ = mes Γ (mes Γ_{1u} + mes Γ_{2u} = mes Γ). The symbol $u(x_1, x_2, t)$ denotes the displacement vector and $\vartheta(x_1, x_2, t) = T(x_1, x_2, t) - T_r$, where $T(x_1, x_2, t)$ is the temperature and T_r the temperature for which the material is stress-free. Thus equation (1.1) is the coupled heat equation and equations (1.2) are coupled Cauchy's equations.

In relation (1.10) the symbol α is the coefficient of linear thermal expansion, δ_{ij} is the Kronecker delta and D_{ijkm} are constants depending on the material only. We shall consider isotropic materials only; in this case

(1.14) $$D_{ijkm}\delta_{km} = c_3\delta_{ij} , \quad c_3 = \text{const.} > 0 .$$

We restrict ourselves to the case

$$u_B \equiv 0 , \quad \vartheta_B \equiv 0 ;$$

in the opposite case we should use the transformations

$$\tilde{u} = u + \bar{u}_B , \quad \tilde{\vartheta} = \vartheta + \bar{\vartheta}_B$$

where $\bar{u}_B$ and $\bar{\vartheta}_B$ are appropriate extensions of u_B and ϑ_B from Γ onto Ω.

Let us introduce the spaces

(1.15) $$V = \{v \in W_2^1(\Omega) : v = 0 \text{ on } \Gamma_{1u}\} ,$$

(1.16) $$W = \{w \in W_2^1(\Omega) : w = 0 \text{ on } \Gamma_{1\vartheta}\} .$$

The norm in $L_2(\Omega)$ will be denoted by $\|\cdot\|_0$, the norm in $W_2^k(\Omega)$ by $\|\cdot\|_k$ and the seminorm by $|\cdot|_k$. In what follows we shall work in the function spaces of the types $C(I; V)$, $L_\infty(I; V)$, $L_2(I; V)$, where $I = [0, T]$, $T < \infty$, the basic properties of which can be found in [9]. For the sake of brevity we shall use the notation

$$H = L_2(\Omega) , \quad H^2 = H \times H , \quad V^2 = V \times V .$$

For the vector functions $v = (v_1, v_2) \in [W_2^k(\Omega)]^2$ we shall use the notation $\|v\|_k = (\|v_1\|_k^2 + \|v_2\|_k^2)^{1/2}$ and similar notation for the seminorm $|v|_k$.

We denote

$$(v, w)_u = \int_{\Gamma_{2u}} v_i w_i \, ds \,, \quad (v, w)_\vartheta = \int_{\Gamma_{2\vartheta}} vw \, ds \,,$$

$$a(v, w) = \int_\Omega D_{ijkm} \, \varepsilon_{ij}(v) \, \varepsilon_{km}(w) \, dx = \int_\Omega D_{ijkm} v_{i,j} w_{k,m} \, dx \,,$$

$$(u, v) = \int_\Omega uv \, dx \,, \quad (v, w) = (v_i, w_i) \,, \quad D(v, w) = (\text{grad } v, \text{ grad } w) \,.$$

According to (1.11)–(1.13), we have

$$a(v, v) \geqq C|v|_1^2 \quad \forall v \in [W_2^1(\Omega)]^2 \tag{1.17}$$

where C is a positive constant independent of v. Further

$$D(w, w) + \beta(w, w)_\vartheta \geqq C\|w\|_1^2 \quad \forall w \in W \tag{1.18}$$

where C is a positive constant independent of w. Because of a greater simplicity we do not consider the case $\beta = 0$. If $\beta = 0$, mes $\Gamma_{1\vartheta} > 0$ then inequality (1.18) remains valid and the case $\beta = 0$, $\Gamma_{2\vartheta} = \Gamma$ (which does not occur in applications) can be treated in the same way as the case $\Gamma_{2u} = \Gamma$, which is in this paper taken into account.

Definition 1. *The pair u, ϑ is a variational solution of problem (1.1)–(1.14) if the following conditions are satisfied:*

a) $u \in C(I; V^2)$, $\dot{u} \in L_\infty(I; V^2) \cap C(I; H^2)$, $\ddot{u} \in L_\infty(I; H^2)$; $u(0) = u_0 \in V^2$, $\dot{u}(0) = v_0 \in V^2$;

b) $\vartheta \in L_\infty(I; W)$, $\dot{\vartheta} \in L_\infty(I; H)$, $\vartheta(0) = \vartheta_0 \in W$;

c) *the following relations are satisfied:*

$$c_1(\dot{\vartheta}(t), w) + D(\vartheta(t), w) + c_2(\text{div } \dot{u}(t), w) + \beta(\vartheta(t), w)_\vartheta = (Q(t), w) + \beta(g(t), w)_\vartheta \quad \forall w \in W \quad \forall t \in I \setminus E_w \tag{1.19}$$

$$c_4(\ddot{u}(t), v) + a(u(t), v) - c_3(\vartheta(t), \text{div } v) = (f(t), v) + (p(t), v)_u \quad \forall v \in V^2 \quad \forall t \in I \setminus E_v \tag{1.20}$$

where $E_w \subset I$ and $E_v \subset I$ are subsets of measure zero depending on w and v, respectively. The problem defined by a)–c) will be briefly called problem PC - 1 (problem continuous 1).

In Section 3 a weaker variational solution will be defined (see Definition 2).

The existence and uniqueness of the solution of problem PC - 1 will be proved by Rothe's method. Let us choose an integer n, let us define $\Delta t = T/n$ and let us set

$$t_i = i\Delta t \quad (i = 0, 1, \ldots, n) \,.$$

For a function $F(t)$ let us write

$$F^i = F(t_i)\,, \quad \Delta F^i = F^i - F^{i-1}\,, \quad \Delta^2 F^i = \Delta F^i - \Delta F^{i-1}\,.$$

In Sections 1 and 2 let us assume

$$Q \in AC(I; H)\,, \quad \dot{Q} \in L_2(I; H) \tag{1.21}$$

$$g \in AC(I; L_2(\Gamma_{2\vartheta}))\,, \quad \dot{g} \in L_2(I; L_2(\Gamma_{2\vartheta})) \tag{1.22}$$

$$f \in AC(I; H^2)\,, \quad \dot{f} \in L_2(I; H^2) \tag{1.23}$$

$$p \in AC(I; [L_2(\Gamma_{2u})]^2)\,, \quad \dot{p} \in L_2(I; [L_2(\Gamma_{2u})]^2)\,, \quad \ddot{p} \in L_2(I; [L_2(\Gamma_{2u})]^2) \tag{1.24}$$

and in this section let us define the following semidiscrete problem:

Problem PD - 1: *Let* $U^0 = u_0$, $U^{-1} = u_0 - \Delta t v_0$, $\Theta^0 = \vartheta_0$. *Find* $U^i \in V^2$, $\Theta^i \in W$ $(i = 1, \ldots, n)$ *such that*

$$c_1 \Delta t^{-1}(\Delta\Theta^i, w) + D(\Theta^i, w) + c_2 \Delta t^{-1}(\operatorname{div} \Delta U^i, w) + \beta(\Theta^i, w)_\vartheta = = (Q(t_i), w) + \beta(g(t_i), w)_\vartheta \equiv G_i(w) \quad \forall w \in W\,, \tag{1.25}$$

$$c_4 \Delta t^{-2}(\Delta^2 U^i, v) + a(U^i, v) - c_3(\Theta^i, \operatorname{div} v) = = (f(t_i), v) + (p(t_i), v)_u \equiv F_i(v) \quad \forall v \in V^2\,. \tag{1.26}$$

Remark 1. In order to avoid the multiplying of equation (1.25) by c_3 and equation (1.26) by c_2 (when summing them up) we shall assume in the proofs of Lemmas 1–3 that $c_2 = c_3 = 1$.

Lemma 1. *Let* $\vartheta_0 \in W$, $u_0, v_0 \in V^2$. *Then there exists a unique solution* $U^i \in V^2$, $\Theta^i \in W$ $(i = 1, \ldots, n)$ *of problem PD - 1 where n is an arbitrary integer.*

Proof. Let us consider the space $Z = V^2 \times W$ with the norm $\|y\|_Z^2 = \|v\|_1^2 + \|w\|_1^2$ for $y = \{v, w\} \in Z$. Let us denote $z = \{U, \Theta\} \in Z$ and let us define a linear operator A: $Z \to Z^*$ by means of the bilinear form

$$[Az, y] = c_4 \Delta t^{-2}(U, v) + a(U, v) - (\Theta, \operatorname{div} v) + + c_1(\Theta, w) + \Delta t D(\Theta, w) + (\operatorname{div} U, w) + \beta \Delta t(\Theta, w)_\vartheta\,.$$

Setting $y = z$ and using (1.17), (1.18) we obtain (because Δt is fixed)

$$[Az, z] \geqq C\|z\|_Z^2 \quad \forall z \in Z$$

where C is a positive constant not depending on $z \in Z$. From here, from the boundedness of the bilinear form on $Z \times Z$ and from the estimates

$$|F_i(v)| \leqq C\|v\|_1\,, \quad |G_i(w)| \leqq C\|w\|_1\,, \quad |(\Theta^{i-1}, w)| \leqq C\|w\|_0\,,$$

$$|\Delta t^{-2}(2U^{i-1} - U^{i-2}, v)| \leqq C\|v\|_0\,, \quad |(\operatorname{div} U^{i-1}, w)| \leqq C\|w\|_0\,,$$

where C is a positive constant not depending on $y = \{v, w\}$, we obtain the existence of a solution by means of the Lax-Milgram lemma.

To prove the uniqueness let us assume that there exist two solutions $z_1^i = \{U_1^i, \Theta_1^i\}$, $z_2^i = \{U_2^i, \Theta_2^i\}$ $(i = 1, \ldots, n)$. Setting $i = 1$ we obtain from (1.25) and (1.26) that $[A(z_1^i - z_2^i), y] = 0$. Setting $y = z_1^i - z_2^i$ we find $z_1^i = z_2^i$. Repeating the consideration successively for $i = 2, \ldots, n$ we obtain $z_1^i = z_2^i$ $(i = 1, \ldots, n)$. Lemma 1 is proved.

Let us denote for the sake of brevity

$$Z^i = \Delta U^i / \Delta t\,, \quad S^i = \Delta Z^i / \Delta t \equiv \Delta^2 U^i / \Delta t^2\,, \quad R^i = \Delta \Theta^i / \Delta t\,,$$

where $i = 1, \ldots, n$. In order to obtain convenient a priori estimates we shall need a smoothness of data and validity of (1.25), (1.26) in the case of $i = 0$ in the following sense: There exist $S^0 \in H^2$ and $R^0 \in H$ such that

$$c_1(R^0, w) + D(\vartheta_0, w) + c_2(\operatorname{div} v_0, w) + \beta(\vartheta_0, w)_\vartheta = G_0(w) \quad \forall w \in W\,, \tag{1.27}$$

$$c_4(S^0, v) + a(u_0, v) - c_3(\vartheta_0, \operatorname{div} v) = F_0(v) \quad \forall v \in V^2\,. \tag{1.28}$$

Remark 2. Relations (2.3), (2.4) (together with (1.21)–(1.24)) are an example of sufficient conditions for (1.27), (1.28): Using Green's theorem and relations (1.10), (1.12), (1.14) we obtain for R^0, $S^0 = (S_1^0, S_2^0)$ from (1.27), (1.28):

$$R^0 = c_1^{-1}(Q(0) + \nabla^2 \vartheta_0 - c_2 \operatorname{div} v_0)\,, \quad S_i^0 = c_4^{-1}(\sigma_{ij,j}^0 + f_i(0))$$

where $\sigma_{ij}^0 = \sigma_{ij}(u_0, \vartheta_0)$ (see (1.10)).

Lemma 2. *Let relations* (1.27), (1.28) *hold, let the functions and vectors* Q, g, f, p *satisfy conditions* (1.21)–(1.24) *and let* $u_0, v_0 \in V^2$ *and* $\vartheta_0 \in W$. *Then there exists a positive constant* C *independent of* $j, n, \Delta t$ *such that*

$$\|Z^j\|_1 \leqq C\,, \quad \|S^j\|_0 \leqq C\,, \quad \|R^j\|_0 \leqq C \quad (1 \leqq j \leqq n)$$

$$\sum_{i=1}^n \|\Delta Z^i\|_1^2 \leqq C\,, \quad \Delta t^{-1} \sum_{i=1}^n \|\Delta \Theta^i\|_1^2 \leqq C\,.$$

Proof. According to Remark 1, we assume that $c_2 = c_3 = 1$. Let us set $i = j$ in (1.25) and then $i = j - 1$. Setting $w = R^j$ and subtracting the second result from the first we obtain (in the case $j = 1$ we use also (1.27))

$$c_1(R^j - R^{j-1}, R^j) + \Delta t^{-1} D(\Delta\Theta^j, \Delta\Theta^j) + (\operatorname{div} \Delta Z^j, R^j) + \\ + \Delta t^{-1} \beta \|\Delta\Theta^j\|_{L_2(\Gamma_{2\vartheta})} = (G_j - G_{j-1})(R^j)\,. \tag{1.29}$$

Repeating this procedure in the case of (1.26) with $v = S^j$ we obtain

$$c_4(S^j - S^{j-1}, S^j) + a(Z^j, Z^j - Z^{j-1}) - \\ - (R^j, \operatorname{div} \Delta Z^j) = (F_j - F_{j-1})(S^j)\,. \tag{1.30}$$

Summing (1.29) and (1.30) up and using (1.17), (1.18) we easily find

$$c_1(\|R^j\|_0^2 - \|R^{j-1}\|_0^2) + c_4(\|S^j\|_0^2 - \|S^{j-1}\|_0^2) +$$
$$+ C^*\Delta t^{-1}\|\Delta\Theta^j\|_1^2 + K_1\|Z^j\|_1^2 - K_2\|Z^{j-1}\|_1^2 + K_1\|\Delta Z^j\|_1^2 \leqq$$
$$\leqq 2[(Z^j, \Delta Z^j) + (F_j - F_{j-1})(S^j) + (G_j - G_{j-1})(R^j)],$$

where $K_1 > 0$, $K_2 > 0$. As $2|(Z^j, \Delta Z^j)| \leqq \|Z^j\|_0^2 \Delta t + \|S^j\|_0^2 \Delta t$ we obtain after summing from $j = 1$ to $j = k$ $(k \leqq n)$:

(1.31)
$$c_4\|S^k\|_0^2 + c_1\|R^k\|_0^2 + K_1\|Z^k\|_1^2 + K_1\sum_{j=1}^k \|\Delta Z^j\|_1^2 +$$
$$+ C^*\Delta t^{-1}\sum_{j=1}^k \|\Delta\Theta^j\|_1^2 \leqq c_4\|S^0\|_0^2 + c_1\|R^0\|_0^2 +$$
$$+ K_2\|v_0\|_1^2 + \Delta t\sum_{j=1}^k (\|Z^j\|_0^2 + \|S^j\|_0^2) +$$
$$+ 2\sum_{j=1}^k (F_j - F_{j-1})(S^j) + 2\sum_{j=1}^k (G_j - G_{j-1})(R^j),$$

Using the assumptions (1.21)–(1.24) we can estimate the last two terms on the right-hand side of (1.31):

$$\sum_{j=1}^k (p(t_j) - p(t_{j-1}), S^j)_u = \sum_{j=1}^k (\Delta p(t_j)/\Delta t, \Delta Z^j)_u =$$
$$= (\Delta p(t_k)/\Delta t, Z^k)_u - \sum_{j=2}^k (\Delta^2 p(t_j)/\Delta t, Z^{j-1})_u - (\Delta p(t_1)/\Delta t, Z^0) \leqq$$
$$\leqq K_3\|\dot{p}(t_k^*)\|_{[L_2(\Gamma_{2u})]^2}^2 + (K_1/2)\|Z^k\|_1^2 + C\|\dot{p}(t_1^*)\|_{[L_2(\Gamma_{2u})]^2}\|v_0\|_1 +$$
$$+ C\Delta t\sum_{j=2}^k \|Z^{j-1}\|_1^2 + \Delta t^{-1}\sum_{j=2}^k \|\Delta^2 p(t_j)/\Delta t\|_{[L_2(\Gamma_{2u})]^2}^2$$

where $t_{j-1} \leqq t_j^* \leqq t_j$ and $K_3 > 0$ depends on K_1 and on the constant appearing in the trace theorem. The last term on the right-hand side is bounded by $C\|\ddot{p}\|_{L_2(I;[L_2(\Gamma_{2u})]^2)}^2$ because

$$\Delta^2 p(t_j) = \int_{t_{j-2}}^{t_j} \ddot{p}(t)(t_j - t)\,dt - 2\int_{t_{j-2}}^{t_{j-1}} \ddot{p}(t)(t_{j-1} - t)\,dt.$$

Further, we have

$$2\sum_{j=1}^k (f(t_j) - f(t_{j-1}), S^j) \leqq 2\sum_{j=1}^k \left\{\Delta t\int_{t_{j-1}}^{t_j} \|\dot{f}\|_0^2\,dt\right\}^{1/2} \|S^j\|_0 \leqq$$
$$\leqq \|\dot{f}\|_{L_2(I;H^2)}^2 + \Delta t\sum_{j=1}^k \|S^j\|_0^2.$$

Thus

$$2\sum_{j=1}^k (F_j - F_{j-1})(S^j) \leqq C + C\Delta t\sum_{j=1}^k (\|Z^j\|_1^2 + \|S^j\|_0^2).$$

Similarly we obtain

$$2\sum_{j=1}^{k}(G_j - G_{j-1})(R^j) \leqq C + \Delta t\sum_{j=1}^{k}\|R^j\|_0^2 + K_4\Delta t^{-1}\sum_{j=1}^{k}\|\Delta\Theta^j\|_1^2$$

where $0 < K_4 < C^*$. Thus using Gronwall's lemma we obtain from (1.31) all assertions of Lemma 2. Lemma 2 is proved.

Let us define Rothe's functions

$$U_n(t) = U^{i-1} + (\Delta U^i/\Delta t)(t - t_{i-1}), \quad t_{i-1} \leqq t \leqq t_i \quad (i = 1, \ldots, n)$$

$$Z_n(t) = Z^{i-1} + (\Delta Z^i/\Delta t)(t - t_{i-1}), \quad t_{i-1} \leqq t \leqq t_i \quad (i = 1, \ldots, n)$$

$$\Theta_n(t) = \Theta^{i-1} + (\Delta\Theta^i/\Delta t)(t - t_{i-1}), \quad t_{i-1} \leqq t \leqq t_i \quad (i = 1, \ldots, n)$$

and corresponding step-functions

$$\bar{U}_n(t) = U^i, \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, n), \quad \bar{U}_n(0) = u_0,$$

$$\bar{Z}_n(t) = Z^i, \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, n),$$

$$\bar{\Theta}_n(t) = \Theta^i, \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, n), \quad \bar{\Theta}_n(0) = \vartheta_0$$

and let us prove the following lemma:

Lemma 3. *There exist functions $u(t), \vartheta(t)$ with the properties $u \in AC(I; V^2)$, $\dot{u} \in AC(I; H^2) \cap L_\infty(I; V^2)$, $\ddot{u} \in L_\infty(I; H^2)$, $u(0) = u_0$, $\dot{u}(0) = v_0$, $\vartheta \in AC(I; W)$, $\dot{\vartheta} \in L_2(I; W)$, $\vartheta(0) = \vartheta_0$ and such that*

$$U_m \to u \text{ in } C(I; H^2), \quad \bar{U}_m \rightharpoonup u \text{ in } L_2(I; V^2),$$

$$Z_m \to \dot{u} \text{ in } C(I; H^2), \quad \bar{Z}_m \rightharpoonup \dot{u} \text{ in } L_2(I; V^2),$$

$$\dot{Z}_m \rightharpoonup \ddot{u} \text{ in } L_2(I; H^2), \quad \Theta_m \to \vartheta \text{ in } C(I; H),$$

$$\bar{\Theta}_m \rightharpoonup \vartheta \text{ in } L_2(I; W), \quad \dot{\Theta}_m \rightharpoonup \dot{\vartheta} \text{ in } L_2(I; W),$$

where $\{U_m\}, \{Z_m\}$ and $\{\Theta_m\}$ are subsequences of $\{U_n\}, \{Z_n\}$ and $\{\Theta_n\}$, respectively.

Proof. The estimates introduced in Lemma 2 imply

(1.32) $$\|Z_n(t)\|_1 \leqq C \quad \forall t \in I, \quad \|\dot{Z}_n(t)\|_0 \leqq C \quad \forall t \in I \setminus E,$$

(1.33) $$\|\Theta_n(t)\|_1 \leqq C \quad \forall t \in I, \quad \|\dot{\Theta}_n\|_{L_2(I;W)} \leqq C,$$

(1.34) $$\int_0^T \|Z_n(t) - \bar{Z}_n(t)\|_1^2 \, dt \leqq \frac{C}{n}, \quad \int_0^T \|\Theta_n(t) - \bar{\Theta}_n(t)\|_1^2 \, dt \leqq \frac{C}{n^2}$$

where mes $E = 0$. Only $(1.33)_1$ needs an explanation. The relation

$$\Theta_n(t) = \Theta^0 + \int_0^t \dot{\Theta}_n(\tau) \, d\tau$$

together with the last estimate of Lemma 2 implies that

$$\|\Theta_n(t)\|_1 \leqq \|\Theta^0\|_1 + \left\{t \int_0^t \|\dot{\Theta}_n(\tau)\|_1^2 \, \mathrm{d}t\right\}^{1/2} \leqq$$
$$\leqq \|\Theta^0\|_1 + \{t \sum_{j=1}^n \|\Delta\Theta_n/\Delta t\|_1^2 \, \Delta t\}^{1/2} \leqq C .$$

The assertions of Lemma 3 are consequences of (1.32)–(1.34) and can be proved by standard devices used, e.g., in [6], [7], [8], [16]. Thus some parts of the proof are only sketched.

A) The relation

$$U_n(t) = u_0 + \int_0^t \bar{Z}_n(\tau) \, \mathrm{d}\tau \tag{1.35}$$

implies $\|U_n(t)\|_1 \leqq \|u_0\|_1 + t^{1/2}\|\bar{Z}_n\|_{L_2(I;V^2)}$. This result together with $(1.32)_1$ and $(1.34)_1$ give

$$\|U_n(t)\|_1 \leqq C , \quad \|\bar{U}_n(t)\|_1 \leqq C \quad \forall t \in I . \tag{1.36}$$

Further, we have

$$\|U_n(t'') - U_n(t')\|_0 \leqq \left\|\int_{t'}^{t''} \bar{Z}_n(t) \, \mathrm{d}t \right\|_0 \leqq C|t'' - t'|^{1/2} \quad \forall t', \, t'' \in I .$$

Thus, owing to the compactness of the imbedding $V \hookrightarrow H$, we can use the generalized Arzela-Ascoli theorem [9, p. 42] and find that $U_m \to u$ in $C(I; H^2)$. This result, relation $(1.36)_1$ and the compactness theorem imply

$$U_m(t) \rightharpoonup u(t) \quad \text{in} \quad V^2 \quad \forall t \in I . \tag{1.37}$$

As $\|\cdot\|_1$ is weakly lower semicontinuous relations $(1.36)_1$ and (1.37) imply $\|u(t)\|_1 \leqq \leqq \liminf_{m \to \infty} \|U_m(t)\|_1 \leqq C \ \forall t \in I$. Thus $u \in L_\infty(I; V^2)$.

B) Similarly as in the part A) estimates (1.32) and the generalized Arzela-Ascoli theorem give $Z_m \to z$ in $C(I; H^2)$. This result, relation $(1.32)_1$ and the compactness theorem imply both $Z_m(t) \rightharpoonup z(t)$ in V^2 $\forall t \in I$ and $Z_m \rightharpoonup z$ in $L_2(I; V^2)$. Then also, according to $(1.34)_1$, $\bar{Z}_m \rightharpoonup z$ in $L_2(I; V^2)$. It remains to prove that $z = \dot{u}$: We obtain from (1.35)

$$(U_m(t), v) = (u_0, v) + \int_0^t (\bar{Z}_m(\tau), v) \, \mathrm{d}\tau \quad \forall v \in V^2 .$$

Passing to the limit for $m \to \infty$ and using the preceding results we find (due to the density of V in H)

$$u(t) = u_0 + \int_0^t z(\tau) \, \mathrm{d}\tau .$$

Thus $u \in AC(I; V^2)$ and $u(0) = u_0$. Further $\dot{u}(t) = z(t)$ almost everywhere in I. The result $\dot{u} \in L_\infty(I; V^2)$ can be proved by the argument introduced at the end of part A).

C) Estimate $(1.32)_2$ and the compactness theorem imply that there exists $w \in L_2(I; H^2)$ such that $\dot{Z}_m \rightharpoonup w$ in $L_2(I; H^2)$. We have

$$\left(Z_m(t) - v_0 - \int_0^t \dot{Z}_m(\tau)\, d\tau, v\right) = 0 \quad \forall v \in V^2 .$$

Passing to the limit for $m \to \infty$ we obtain

$$\dot{u}(t) = v_0 + \int_0^t w(\tau)\, d\tau \quad \forall t \in I .$$

Thus $\dot{u}(0) = v_0$, $\dot{u} \in AC(I; H^2)$ and $\ddot{u}(t) = w(t)$ a.e. in I.

Let $v \in L_2(I; H^2)$. Then, according to $(1.32)_2$, we have

$$\left| \int_0^T (\ddot{u}(t), v(t))\, dt \right| = \left| \lim_{m \to \infty} \int_0^T (\dot{Z}_m(t), v(t))\, dt \right| \leqslant$$

$$\leqslant \limsup_{m \to \infty} \int_0^T \|\dot{Z}_m\|_0 \, \|v\|_0 \, dt \leqq C\|v\|_{L_1(I; H^2)} .$$

As $L_2(I; H^2)$ is dense in $L_1(I; H^2)$ we see that $\ddot{u} \in L_\infty(I; H^2)$.

D) Relation $(1.36)_1$ and the compactness theorem imply that $U_m \rightharpoonup w$ in $L_2(I; V^2)$. On the other hand $U_m \to u$ in $C(I; H^2)$; thus $w = u$ and

$$(1.38) \qquad U_m \rightharpoonup u \quad \text{in} \quad L_2(I; V^2) .$$

Further, we have $\|\Delta U^i\|_1^2 = \Delta t^2 \|Z^i\|_1^2 \leqq C \Delta t^2$; hence

$$(1.39) \qquad \|\bar{U}_n - U_n\|^2_{L_2(I; V^2)} \leqq \sum_{i=1}^n \|\Delta U^i\|_1^2 \, \Delta t \leqq C/n^2 .$$

Relations (1.38) and (1.39) imply $\bar{U}_m \rightharpoonup u$ in $L_2(I; V^2)$.

E) The remaining assertions of Lemma 3, which concern the function ϑ and its derivative $\dot{\vartheta}$, can be proved in the same way.

Theorem 1. *Let the assumptions of Lemma 2 be satisfied. Then problem PC - 1 has a unique solution u, ϑ and we have*

$$U_n \to u \text{ in } C(I; H^2) , \quad Z_n \to \dot{u} \text{ in } C(I; H^2) , \quad \Theta_n \to \vartheta \text{ in } C(I; H) ,$$

where $\{U_n\}$, $\{Z_n\}$ and $\{\Theta_n\}$ are arbitrary sequences of Rothe's functions.

Proof. A) Let us write relations (1.25), (1.26) in the form

$$(1.40) \qquad c_1(\dot{\Theta}_m(t), w) + D(\bar{\Theta}_m(t), w) + c_2(\operatorname{div} \bar{Z}_m(t), w) + \\ + \beta(\bar{\Theta}_m(t), w)_\vartheta = \bar{G}_m(w) \quad \forall w \in W \quad \text{a.e. in} \quad I ,$$

$$(1.41) \qquad c_4(\dot{Z}_m(t), v) + a(\bar{U}_m(t), v) - c_3(\bar{\Theta}_m(t), \operatorname{div} v) = \bar{F}_m(v) \\ \forall v \in V^2 \quad \text{a.e. in} \quad I ,$$

where $\bar{F}_m(v) = F_i(v)$ for $t_{i-1} < t \leqq t_i$ $(i = 1, \ldots, m)$; the functional $\bar{G}_m(w)$ is defined analogously.

It is not difficult to find that

$$(1.42) \qquad \lim_{m\to\infty} \int_{t'}^{t''} \bar{F}_m(v)\,\mathrm{d}t = \int_{t'}^{t''} [(f(t), v) + (p(t), v)_u]\,\mathrm{d}t\,,$$

$$(1.43) \qquad \lim_{m\to\infty} \int_{t'}^{t''} \bar{G}_m(w)\,\mathrm{d}t = \int_{t'}^{t''} [(Q(t), w) + \beta(g(t), w)_\vartheta]\,\mathrm{d}t\,,$$

where $t' < t''$ are arbitrary numbers in I. Integrating (1.40) and (1.41) in $[t', t''] \subset I$ and passing to the limit for $m \to \infty$ we obtain by means of Lemma 3 and relations (1.42), (1.43):

$$(1.44) \qquad \int_{t'}^{t''} \{c_1(\dot{\vartheta}(t), w) + D(\vartheta(t), w) + c_2(\operatorname{div} \dot{u}(t), w) + \\ + \beta(\vartheta(t), w)_\vartheta - (Q(t), w) - \beta(g(t), w)_\vartheta\}\,\mathrm{d}t = 0 \quad \forall w \in W\,,$$

$$(1.45) \qquad \int_{t'}^{t''} \{c_4(\ddot{u}(t), v) + a(u(t), v) - c_3(\vartheta(t), \operatorname{div} v) - \\ - (f(t), v) - (p(t), v)_u\}\,\mathrm{d}t = 0 \quad \forall v \in V^2\,.$$

As $t' < t''$ are arbitrary we see that $u(t)$, $\vartheta(t)$ satisfy (1.19) and (1.20), i.e. property c) of Definition 1 is proved.

B) Functions $u(t)$, $\vartheta(t)$ and their derivatives satisfy properties a), b) of Definition 1, as follows from Lemma 3.

C) Now we prove the uniqueness of the solution of problem PC - 1. Let $u_0 = v_0 = f(t) = p(t) = 0$, $\vartheta_0 = Q(t) = g(t) = 0$. Let us choose $w(t) \in L_2(I; W)$ and $v(t) \in L_2(I; V^2)$ arbitrarily. As the set of all step-functions belonging to $L_2(I; W)$ is dense in $L_2(I; W)$ we can find a sequence $\{\bar{w}_n(t)\} \subset L_2(I; W)$ of step-functions such that

$$\bar{w}_n \to w \quad \text{in} \quad L_2(I; W)\,.$$

On the other hand, using (1.44) we can write

$$\int_0^t \{c_1(\dot{\vartheta}(\tau), \bar{w}_n(\tau)) + D(\vartheta(\tau), \bar{w}_n(\tau)) + \\ + c_2(\operatorname{div} \dot{u}(\tau), \bar{w}_n(\tau)) + \beta(\vartheta(\tau), \bar{w}_n(\tau))_\vartheta\}\,\mathrm{d}\tau = 0\,.$$

Passing to the limit for $n \to \infty$ we obtain

$$(1.46) \qquad \int_0^t \{c_1(\dot{\vartheta}(\tau), w(\tau)) + D(\vartheta(\tau), w(\tau)) + c_2(\operatorname{div} \dot{u}(\tau), w(\tau)) + \\ + \beta(\vartheta(\tau), w(\tau))_\vartheta\}\,\mathrm{d}\tau = 0 \quad \forall w \in L_2(I; W)\,.$$

Similarly we find

$$(1.47) \qquad \int_0^T \{c_4(\ddot{u}(\tau), v(\tau)) + a(u(\tau), v(\tau)) - c_3(\vartheta(\tau), \operatorname{div} v(\tau))\} \, d\tau = 0 \quad \forall v \in L_2(I; V^2).$$

Multiplying (1.46) by c_3 and (1.47) by c_2, setting $w(\tau) = \vartheta(\tau)$ and $v(\tau) = \dot{u}(\tau)$ we obtain after summing up

$$\tfrac{1}{2}c_2c_4\|\dot{u}(t)\|_0^2 + c_2a(u(t), u(t))/2 + \tfrac{1}{2}c_1c_3\|\vartheta(t)\|_0^2 + \\ + c_3\int_0^t \{D(\vartheta(\tau), \vartheta(\tau)) + \beta(\vartheta(\tau), \vartheta(\tau))_\vartheta\} \, d\tau = 0 .$$

Adding to the both sides the expression $\|u(t)\|_0^2$ we find by virtue of (1.17) and (1.18):

$$\|\dot{u}(t)\|_0^2 + \|u(t)\|_1^2 + \|\vartheta(t)\|_0^2 \leqq C\|u(t)\|_0^2 .$$

We have

$$\|u(t)\|_0^2 = \|u(t)\|_0^2 - \|u(0)\|_0^2 = \int_0^t \frac{d}{d\tau} \|u(\tau)\|_0^2 \, d\tau = \\ = 2\int_0^t (u(\tau), \dot{u}(\tau)) \, d\tau \leqq \int_0^t (\|u(\tau)\|_0^2 + \|\dot{u}(\tau)\|_0^2) \, d\tau .$$

The last two inequalities and the Gronwall's lemma imply

$$\|u(t)\|_0 = 0 , \quad \|\vartheta(t)\|_0 = 0 \quad \forall t \in I ,$$

thus $u = 0$, $\vartheta = 0$ a.e. in I.

D) The assertion concerning sequences of Rothe's functions, i.e. that $\{U_m\} = \{U_n\}$, $\{\Theta_m\} = \{\Theta_n\}$, follows from the uniqueness of the solution. Theorem 1 is proved.

Now we prove stronger convergence results without additional regularity assumptions and without changing the assumptions of Theorem 1.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$\|U_n - u\|_{C(I;V^2)} \leqq K\Delta t^{1/2} , \quad \|Z_n - \dot{u}\|_{C(I;H^2)} \leqq K\Delta t^{1/2} ,$$
$$\|\Theta_n - \vartheta\|_{C(I;H)} \leqq K\Delta t^{1/2} , \quad \|\Theta_n - \vartheta\|_{L_2(I;W)} \leqq K\Delta t^{1/2} ,$$

where K is a constant independent of n and Δt.

Proof. Let us subtract (1.40) and (1.41) with $m = s$ from (1.40) and (1.41) with $m = r$, respectively. Let us set $v = \bar{Z}_r(t) - \bar{Z}_s(t)$, $w = \bar{\Theta}_r(t) - \bar{\Theta}_s(t)$, let us sum up the resulting relations and let us integrate over $[0, t] \subset I$. As $c_2 = c_3 = 1$, according to Remark 1, we obtain

$$\int_0^t \{c_4(\dot{Z}_r - \dot{Z}_s, \bar{Z}_r - \bar{Z}_s) + a(\bar{U}_r - \bar{U}_s, \dot{U}_r - \dot{U}_s) +$$

$$+ c_1(\dot{\Theta}_r - \dot{\Theta}_s, \bar{\Theta}_r - \bar{\Theta}_s) + D(\bar{\Theta}_r - \bar{\Theta}_s, \bar{\Theta}_r - \bar{\Theta}_s) +$$
$$+ \beta(\bar{\Theta}_r - \bar{\Theta}_s, \bar{\Theta}_r - \bar{\Theta}_s)_\vartheta\} \, d\tau =$$
$$= \int_0^t \{(\bar{F}_r - \bar{F}_s)(\bar{Z}_r - \bar{Z}_s) + (\bar{G}_r - \bar{G}_s)(\bar{\Theta}_r - \bar{\Theta}_s)\} \, d\tau \, .$$

Adding to the both sides the expression

$$(1.48) \int_0^t \{c_4(\dot{Z}_r - \dot{Z}_s, Z_r - \bar{Z}_r - (Z_s - \bar{Z}_s)) + a(U_r - \bar{U}_r - (U_s - \bar{U}_s), \dot{U}_r - \dot{U}_s) +$$
$$+ c_1(\dot{\Theta}_r - \dot{\Theta}_s, \Theta_r - \bar{\Theta}_r - (\Theta_s - \bar{\Theta}_s))\} \, d\tau$$

we find

$$(1.49) \qquad \|Z_r(t) - Z_s(t)\|_0^2 + \|U_r(t) - U_s(t)\|_1^2 + \|\Theta_r(t) - \Theta_s(t)\|_0^2 +$$
$$+ \int_0^t \|\bar{\Theta}_r(\tau) - \bar{\Theta}_s(\tau)\|_1^2 \, d\tau \leqq K \left\{\frac{1}{r} + \frac{1}{s}\right\} + C\|U_r(t) - U_s(t)\|_0^2 +$$
$$+ C \left| \int_0^t (\bar{F}_r - \bar{F}_s)(\bar{Z}_r - \bar{Z}_s) \, d\tau \right| + C \left| \int_0^t (\bar{G}_r - \bar{G}_s)(\bar{\Theta}_r - \bar{\Theta}_s) \, d\tau \right|$$

where C is a positive constant independent of r, s. We have obtained the left-hand side of (1.49) by integrating and using relations (1.17), (1.18). (In the case mes $\Gamma_{1u} = 0$ we added to the both sides the term $\|U_r(t) - U_s(t)\|_0^2$ before using (1.17).) The first term on the right-hand side of (1.49) was found by estimating (1.48) by means of (1.32)–(1.34), (1.39) and the fact that $\{\dot{U}_n\} \equiv \{\bar{Z}_n\}$ is a bounded sequence.

The second term on the right-hand side of (1.49) can be estimated as follows:

$$\|U_r(t) - U_s(t)\|_0^2 = \frac{1}{2}\int_0^t (U_r(\tau) - U_s(\tau), \bar{Z}_r(\tau) - \bar{Z}_s(\tau)) \, d\tau \leqq$$
$$\leqq \frac{1}{4}\int_0^t \{\|U_r(\tau) - U_s(\tau)\|_0^2 + \|\bar{Z}_r(\tau) - \bar{Z}_s(\tau)\|_0^2\} \, d\tau \leqq$$
$$\leqq C\left(\frac{1}{r} + \frac{1}{s}\right) + \int_0^t \{\|U_r(\tau) - U_s(\tau)\|_1^2 + \|Z_r(\tau) - Z_s(\tau)\|_0^2\} \, d\tau \, ,$$

according to the Cauchy inequality and estimate $(1.34)_1$.

Now we estimate the remaining two terms on the right-hand side of (1.49). Let us write $\bar{f}_r(t) = f(t_i)$ and $\bar{p}_r(t) = p(t_i)$, $t_{i-1} < t \leqq t_i$ $(i = 1, \ldots, r)$. Denoting $A = L_2(I; H^2)$, $B = L_2(I; [L_2(\Gamma_{2u})]^2)$ we have

$$\int_0^t (\bar{F}_r - \bar{F}_s)(\bar{Z}_r - \bar{Z}_s) \, d\tau \leqq \int_0^t |(\bar{f}_r(\tau) - \bar{f}_s(\tau), \bar{Z}_r - \bar{Z}_s)| \, d\tau +$$
$$+ \int_0^t |(\bar{p}_r(\tau) - \bar{p}_s(\tau), \bar{Z}_r - \bar{Z}_s)_u| \, d\tau \leqq \|\bar{f}_r - \bar{f}_s\|_A \, \|\bar{Z}_r - \bar{Z}_s\|_A +$$

$$+ C\|\bar{p}_r - \bar{p}_s\|_B \|\bar{Z}_r - \bar{Z}_s\|_{L_2(I;V^2)} \leqq C\{\|\bar{f}_r - f\|_A +$$

$$+ \|f - \bar{f}_s\|_A + \|\bar{p}_r - p\|_B + \|p - \bar{p}_s\|_B\} \leqq K\left(\frac{1}{r} + \frac{1}{s}\right)$$

because

$$\|\bar{f}_r - f\|_A^2 = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|f(t_i) - f(t)\|_0^2 \, dt =$$

$$= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left\|\int_t^{t_i} \dot{f}(s) \, ds\right\|_0^2 dt \leqq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left\{\int_t^{t_i} \|\dot{f}(s)\|_0 \, ds\right\}^2 dt \leqq$$

$$\leqq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left\{\int_t^{t_i} ds \int_t^{t_i} \|\dot{f}(s)\|_0^2 \, ds\right\} dt \leqq \Delta t^2 \|\dot{f}\|_A^2$$

and similarly

$$\|\bar{p}_r - p\|_B^2 \leqq \Delta t^2 \|\dot{p}\|_B^2 .$$

In the same way we obtain

$$\left|\int_0^t (\bar{G}_r - \bar{G}_s)(\bar{\Theta}_r - \bar{\Theta}_s) \, d\tau\right| \leqq K\left(\frac{1}{r} + \frac{1}{s}\right).$$

Thus relation (1.49) and Gronwall's lemma imply the first three assertions of Theorem 2. Moreover, we see that $\|\bar{\Theta}_r - \bar{\Theta}_s\|_M^2 \leqq K(1/r + 1/s)$, where $M = L_2(I; W)$. This relation and $(1.34)_2$ imply

$$\|\Theta_r - \Theta_s\|_M \leqq \|\Theta_r - \bar{\Theta}_r\|_M + \|\bar{\Theta}_r - \bar{\Theta}_s\|_M +$$

$$+ \|\bar{\Theta}_s - \Theta_s\|_M \leqq K(1/r + 1/s)^{1/2} .$$

Theorem 2 is proved.

A priori estimates of Lemma 2 enable us to prove stronger regularity properties of $u(t)$, $\vartheta(t)$ in space variables.

Theorem 3. *Let the assumptions of Theorem* 1 *be satisfied. Then* $u \in L_\infty(I; V^2 \cap \cap [W^2_{2,\mathrm{loc}}(\Omega)]^2)$ *and* $\vartheta \in L_\infty(I; W \cap W^2_{2,\mathrm{loc}}(\Omega))$.

Proof. If $w \in C_0^\infty(\Omega)$, $v \in [C_0^\infty(\Omega)]^2$ then relations (1.40) and (1.41) can be written in the form

$$D(\dot{\bar{\Theta}}_n(t), w) = (A_n(t), w) \quad \forall t \in I \quad \forall w \in C_0^\infty(\Omega) , \tag{1.50}$$

$$a(\bar{U}_n(t), v) = (B_n(t), v) \quad \forall t \in I \quad \forall v \in [C_0^\infty(\Omega)]^2 \tag{1.51}$$

where

$$A_n(t) = Q_n(t) - c_1 \bar{R}_n(t) - c_2 \operatorname{div} \bar{Z}_n(t) ,$$

$$\bar{R}_n(t) = R^i , \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, n) ,$$

$$B_n(t) = \bar{f}_n(t) - c_3 \operatorname{grad} \bar{\Theta}_n(t) - c_4 \bar{Z}_n(t) .$$

We have, according to (1.21), (1.23), Lemma 2 and $(1.33)_1$,

(1.52) $$\|A_n(t)\|_0 \leqq C\,, \quad \|B_n(t)\|_0 \leqq C \quad \forall t \in I$$

and, according to $(1.33)_1$ and $(1.36)_2$,

(1.53) $$\|\bar{\Theta}_n(t)\|_1 \leqq C\,, \quad \|\bar{U}_n(t)\|_1 \leqq C \quad \forall t \in I\,.$$

Further, using the methods of the proof of Lemma 3 we can find

(1.54) $$\bar{U}_n(t) \rightharpoonup u(t) \quad \text{in} \quad [W_2^1(\Omega^*)]^2 \quad \forall t \in I\,,$$

(1.55) $$\bar{\Theta}_n(t) \rightharpoonup \vartheta(t) \quad \text{in} \quad W_2^1(\Omega^*) \qquad \forall t \in I$$

where Ω^* is an arbitrary subdomain of Ω.

According to (1.50) and (1.51), $\bar{\Theta}_n(t)$ and $\bar{U}_n(t)$ (t is fixed) are weak solutions of certain linear boundary value elliptic problems (in the interior of Ω). The regularity results (see [11, Chapter 4, § 1]) imply $\bar{\Theta}_n(t) \in W^2_{2,\text{loc}}(\Omega)$, $\bar{U}_n(t) \in [W^2_{2,\text{loc}}(\Omega)]^2$ and the estimates

(1.56) $$\|\bar{\Theta}_n(t)\|_{2,\Omega'} \leqq C(\Omega')\left(\|\bar{\Theta}_n(t)\|_1 + \|A_n(t)\|_0\right) \leqq C(\Omega')\,,$$

(1.57) $$\|\bar{U}_n(t)\|_{2,\Omega'} \leqq C(\Omega')\left(\|\bar{U}_n(t)\|_1 + \|B_n(t)\|_0\right) \leqq C(\Omega')$$

where Ω' is an arbitrary subdomain of Ω such that $\bar{\Omega}' \subset \Omega$ and $C(\Omega')$ is a constant depending on Ω'. The symbol $\|\cdot\|_{2,\Omega'}$ denotes the norm in the space $W_2^2(\Omega')$.

Relations (1.54)–(1.57) imply

$$\bar{U}_n(t) \rightharpoonup u(t) \quad \text{in} \quad [W_2^2(\Omega')]^2 \quad \forall t \in I\,,$$

$$\bar{\Theta}_n(t) \rightharpoonup \vartheta(t) \quad \text{in} \quad W_2^2(\Omega') \qquad \forall t \in I\,.$$

Thus passing to the limit for $n \to \infty$ in (1.56) and (1.57) we obtain

$$\|\vartheta(t)\|_{2,\Omega'} \leqq C(\Omega')\,, \quad \|u(t)\|_{2,\Omega'} \leqq C(\Omega') \quad \forall t \in I\,.$$

This result together with Lemma 3 proves Theorem 3.

2. CONVERGENCE OF FINITE ELEMENT ROTHE'S FUNCTIONS

In this section we consider in addition discretization in space by the finite element method. We restrict ourselves to triangulations which cover the domain Ω exactly. This means that in the case of a curved boundary Γ triangles along the boundary have one curved side which is part of Γ. These triangles are called ideal curved triangles.

With every triangulation $\mathscr{T}$ we associate three parameters

$$h = \max_{K \in \mathscr{T}} h_K\,, \quad \bar{h} = \min_{K \in \mathscr{T}} h_K\,, \quad \omega = \min_{K \in \mathscr{T}} \omega_K$$

where h_K is the length of the greatest side and ω_K the magnitude of the smallest angle of the triangle having the same vertices as K. We choose a sequence $\{\mathscr{T}_n\}_{n=1}^{\infty}$ of triangulations which satisfy the following conditions:

$$\lim_{n\to\infty} h_n = 0\,, \quad \bar{h}_n/h_n \geqq c_0 > 0\,, \quad \omega_n \geqq \omega_0 > 0 \quad \forall n$$

where c_0 and ω_0 are constants independent of n.

On every triagulation $\mathscr{T}_n$ we choose a finite dimensional space X_n with the following properties:

a) $X_n \subset C^0(\bar{\Omega})$;

b) every function $v \in X_n$ is uniquely determined by its values $v(P_k)$, P_k being the nodal points of $\mathscr{T}_n$ (i.e. the vertices of $K \in \mathscr{T}_n$);

c) the restriction of $v \in X_n$ to a triangle $K \in \mathscr{T}_n$ with straight sides is a linear function.

Let us note that the definition of the restriction of $v \in X_n$ to an ideal curved triangle K is given in [12].

For every n we define two subspaces of X_n:

$$V_n = X_n \cap V = \{v \in X_n\colon\ v = 0 \text{ on } \Gamma_{1u}\}\,,$$

$$W_n = X_n \cap W = \{w \in X_n\colon\ w = 0 \text{ on } \Gamma_{1\vartheta}\}\,.$$

Our starting point is the following completely discrete problem:

Problem PD - 2: *Let $U^0, U^{-1} \in V_n^2$ and $\Theta^0 \in W_n$ be given and let $\Delta t = T/r$, where r is an integer. Find $U^i \in V_n^2$ and $\Theta^i \in W_n$ $(i = 1, \ldots, r)$ such that*

$$\text{(2.1)} \qquad c_1 \Delta t^{-1}(\Delta\Theta^i, w) + D(\Theta^i, w) + c_2 \Delta t^{-1}(\operatorname{div} \Delta U^i, w) + \beta(\Theta^i, w)_\vartheta =$$
$$= (Q(t_i), w) + \beta(g(t_i), w)_\vartheta \quad \forall w \in W_n\,,$$

$$\text{(2.2)} \qquad c_4 \Delta t^{-2}(\Delta^2 U^i, v) + a(U^i, v) - c_3(\Theta^i, \operatorname{div} v) =$$
$$= (f(t_i), v) + (p(t_i), v)_u \quad \forall v \in V_n^2\,.$$

Lemma 4. *There exists a unique solution U^i, Θ^i $(i = 1, \ldots, r)$ of problem PD - 2.*

Proof. As (2.1), (2.2) represent a system of linear algebraic equations for the values $\Theta^i(P_k)$, $U^i(P_k)$ it is sufficient to prove the uniqueness. Let $U^0 = U^{-1} = 0$, $\Theta^0 = 0$ and $Q(t_i) = g(t_i) = 0$, $f(t_i) = p(t_i) = 0$ $(i = 1, \ldots, r)$. In the case $i = 1$ we set $w = \Theta^i$ in (2.1) and $v = U^i$ in (2.2), multiply (2.1) by Δt and sum the obtained equations up (we again assume $c_2 = c_3 = 1$). We get

$$c_1 \|\Theta^i\|_0^2 + \Delta t\{D(\Theta^i, \Theta^i) + \beta(\Theta^i, \Theta^i)_\vartheta\} + c_4 \Delta t^{-2} \|U^i\|_0^2 + a(U^i, U^i) = 0\,.$$

Thus $U^i = 0$, $\Theta^i = 0$ $(i = 1)$, according to (1.17) and (1.18). In the case $i > 1$ let us assume that we have proved $U^j = 0$, $\Theta^j = 0$, where $j < i$. Repeating the preceding consideration we obtain that $U^i = 0$, $\Theta^i = 0$. Lemma 4 is proved.

In this section we shall assume

$$\vartheta_0 \in W \cap W_2^2(\Omega)\,, \quad u_0 \in V^2 \cap [W_2^2(\Omega)]^2\,, \quad v_0 \in V^2 \cap [W_2^2(\Omega)]^2\,, \tag{2.3}$$

$$\left[\beta\vartheta_0 + \frac{\partial\vartheta_0}{\partial v}\right]_{\Gamma_{2\vartheta}} = \beta\, g(0)\,, \quad \sigma_{ij}(u_0, \vartheta_0)\, v_j\Big|_{\Gamma_{2u}} = p_i(0)\,. \tag{2.4}$$

Assumptions (2.3) enable us to define Θ^0, U^0, U^{-1} in the simplest and most natural way, assumptions (2.3) and (2.4) enable us to give a discrete analogy of (1.27), (1.18) and to define $\Delta\Theta^0/\Delta t$ and $\Delta^2 U^0/\Delta t^2$ which are bounded in the L_2-norm by a constant independent of W_n and V_n, respectively.

Let us set

$$\Theta^0 = I_n\vartheta_0\,, \quad U^0 = I_n u_0\,, \quad U^{-1} = U^0 - \Delta t I_n v_0 \tag{2.5}$$

where $I_n w \in W_n$ and $I_n v \in V_n^2$ are W_n-interpolate and V_n^2-interpolate of a function $w \in W$ and of a vector $v \in V^2$, respectively. This means that

$$(I_n w)(P_k) = w(P_k)\,, \quad (I_n v)(P_k) = v(P_k)$$

for all nodal points P_k in $\mathscr{T}_n$. Using standard interpolation theorems (see, e.g., [12], [13]) we can see that

$$\|\vartheta_0 - I_n\vartheta_0\|_1 \leqq Ch_n\|\vartheta_0\|_2\,, \quad \|u_0 - I_n u_0\|_1 \leqq Ch_n\|u_0\|_2\,, \tag{2.6}$$
$$\|v_0 - I_n v_0\|_1 \leqq Ch_n\|v_0\|_2\,,$$

where C is a constant independent of n, u_0, v_0 and ϑ_0.

We define $\Delta\Theta^0/\Delta t \in W_n$ and $\Delta^2 U^0/\Delta t^2 \in V_n^2$ by the relations

$$c_1(\Delta\Theta^0/\Delta t, w) + D(I_n\vartheta_0, w) + c_2(\operatorname{div} I_n v_0, w) + \beta(I_n\vartheta_0, w)_\vartheta = \tag{2.7}$$
$$= (Q(0), w) + \beta(g(0), w)_\vartheta \quad \forall w \in W_n\,,$$

$$c_4(\Delta^2 U^0/\Delta t^2, v) + a(I_n u_0, v) - c_3(I_n\vartheta_0, \operatorname{div} v) = \tag{2.8}$$
$$= (f(0), v) + (p(0), v)_u \quad \forall v \in V_n^2\,.$$

As $\Delta U^0/\Delta t = I_n v_0$ relations (2.7), (2.8) are relations (2.1), (2.2) written for $i = 0$.

Lemma 5. *The solutions of both relations* (2.7) *and* (2.8) *exist and are unique and satisfy*

$$\|\Delta\Theta^0/\Delta t\|_0 \leqq C\,, \quad \|\Delta^2 U^0/\Delta t^2\|_0 \leqq C \tag{2.9}$$

where C is a constant independent of n.

Proof. A) Existence and uniqueness: Similarly as in the proof of Lemma 4 it is sufficient to prove the uniqueness. Let the data $I_n\vartheta_0$, $I_n u_0$, $I_n v_0$, $Q(0)$, $g(0)$, $f(0)$, $p(0)$ be equal to zero. Then relations (2.7) and (2.8) reduce to

$$(\Delta\Theta^0/\Delta t, w) = 0 \quad \forall w \in W_n\,, \qquad (\Delta^2 U^0/\Delta t^2, v) = 0 \quad \forall v \in V_n^2\,.$$

Setting $w = \Delta\Theta^0/\Delta t$, $v = \Delta^2 U^0/\Delta t^2$ we see that $\Delta\Theta^0/\Delta t = 0$, $\Delta^2 U^0/\Delta t^2 = 0$.

B) Estimates (2.9): According to (2.7), we have

$$\text{(2.10)} \qquad c_1|(\Delta\Theta^0/\Delta t, w)| \leqq |D(\vartheta_0 - I_n\vartheta_0, w)| + \beta|(\vartheta_0 - I_n\vartheta_0, w)_\vartheta| + \\ + |\beta(g(0) - \vartheta_0, w)_\vartheta - D(\vartheta_0, w)| + |(Q(0), w)| + \\ + c_2|(\operatorname{div}(v_0 - I_n v_0), w)| + c_2|(\operatorname{div} v_0, w)| .$$

The inequality (a consequence of [13, (106) and (153)])

$$\text{(2.11)} \qquad h_n\|w\|_1 \leqq C\|w\|_0 \quad \forall w \in X_n ,$$

where the constant C does not depend on w and n, estimates (2.6) and the trace theorem imply

$$|D(\vartheta_0 - I_n\vartheta_0, w)| \leqq \|\vartheta_0 - I_n\vartheta_0\|_1 \, \|w\|_1 \leqq C\|\vartheta_0\|_2 \, \|w\|_0 ,$$
$$|(\vartheta_0 - I_n\vartheta_0, w)_\vartheta| \leqq C\|\vartheta_0 - I_n\vartheta_0\|_1 \, \|w\|_1 \leqq C\|\vartheta_0\|_2 \, \|w\|_0 .$$

Using Green's theorem and assumption $(2.4)_1$ we find

$$|\beta(g(0) - \vartheta_0, w)_\vartheta - D(\vartheta_0, w)| = |(\nabla^2\vartheta_0, w)| \leqq \|\vartheta_0\|_2 \, \|w\|_0 .$$

The last three terms on the right-hand side of (2.10) are bounded by $C(\|Q(0)\|_0 + \|v_0\|_2) \, \|w\|_0$. Inserting all these results into (2.10) and setting $w = \Delta\Theta^0/\Delta t$ we obtain the first estimate (2.9). The second estimate (2.9) can be obtained similarly. Lemma 5 is proved.

Let $\{\Delta t_n\}_{n=1}^{\infty}$ be an arbitrary sequence with the properties

$$\lim_{n \to \infty} \Delta t_n = 0 , \quad r_n = T/\Delta t_n = \text{integer} .$$

Let us set $r = r_n$, $\Delta t = \Delta t_n$ in problem PD - 2 and let Θ^i, U^i $(i = 1, \ldots, r_n)$ be the corresponding solution of PD - 2. We define the finite element Rothe's functions

$$\Theta_n(t) = \Theta^{i-1} + (\Delta\Theta^i/\Delta t_n)(t - t_{i-1}) , \quad t_{i-1} \leqq t \leqq t_i ,$$
$$U_n(t) = U^{i-1} + (\Delta U^i/\Delta t_n)(t - t_{i-1}) , \quad t_{i-1} \leqq t \leqq t_i ,$$
$$Z_n(t) = Z^{i-1} + (\Delta Z^i/\Delta t_n)(t - t_{i-1}) , \quad t_{i-1} \leqq t \leqq t_i ,$$

where $i = 1, \ldots, r_n$ and $Z^j = \Delta U^j/\Delta t_n$, and the corresponding step-functions

$$\bar{\Theta}_n(t) = \Theta^i , \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, r_n) , \quad \bar{\Theta}_n(0) = \Theta^0 ,$$
$$\bar{U}_n(t) = U^i , \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, r_n) , \quad \bar{U}_n(0) = U^0 ,$$
$$\bar{Z}_n(t) = Z^i , \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, r_n) .$$

Similarly as in [16] the functions $\Theta_n(t)$, $U_n(t)$, $Z_n(t)$ are called the finite element Rothe's functions in order to stress the discretization in space.

Owing to Lemma 5 we can prove the following lemma (the proof follows the same lines as those of Lemmas 1 and 3 and thus it is omitted):

Lemma 6. *Let the initial data ϑ_0, u_0, v_0 satisfy (2.3), (2.4) and let the functions Q, g and the vectors f, p satisfy conditions (1.21)–(1.24). Then there exist a vector $u(t)$ and a function $\vartheta(t)$ with the properties*

$$u \in AC(I; V^2), \quad \dot{u} \in AC(I; H^2) \cap L_\infty(I; V^2), \quad \ddot{u} \in L_\infty(I; H^2),$$

$$\vartheta \in AC(I; W), \quad \dot{\vartheta} \in L_2(I; W),$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \vartheta(0) = \vartheta_0$$

and such that

$$U_m \to u \text{ in } C(I; H^2), \quad \bar{U}_m \rightharpoonup u \text{ in } L_2(I; V^2),$$

$$Z_m \to \dot{u} \text{ in } C(I; H^2), \quad \bar{Z}_m \rightharpoonup \dot{u} \text{ in } L_2(I; V^2),$$

$$\dot{Z}_m \rightharpoonup \ddot{u} \text{ in } L_2(I; H^2), \quad \Theta_m \to \vartheta \text{ in } C(I; H),$$

$$\bar{\Theta}_m \rightharpoonup \vartheta \text{ in } L_2(I; W), \quad \dot{\Theta}_m \rightharpoonup \dot{\vartheta} \text{ in } L_2(I; W),$$

where $\{m\}$ is a subsequence of $\{n\}$ and $\{U_n\}$, $\{Z_n\}$ and $\{\Theta_n\}$ are arbitrary sequences of finite element Rothe's functions.

Theorem 4. *Let the assumptions of Lemma 6 be satisfied. Then we have*

$$U_n \to u \text{ in } C(I; H^2), \quad Z_n \to \dot{u} \text{ in } C(I; H^2), \quad \Theta_n \to \vartheta \text{ in } C(I; H),$$

where $\{U_n\}$, $\{Z_n\}$ and $\{\Theta_n\}$ are arbitrary sequences of finite element Rothe's functions and u, ϑ is the solution of problem PC - 1.

Proof. It suffices to prove that the limit functions from Lemma 6 and their derivatives satisfy equations (1.19), (1.20).

Let us choose $w \in W$, $v \in V^2$ arbitrarily and let $\{w_n\}$, $w_n \in W_n$ and $\{v_n\}$, $v_n \in V_n^2$ be such sequences that

$$\lim_{n\to\infty} \|w_n - w\|_1 = 0, \quad \lim_{n\to\infty} \|v_n - v\|_1 = 0. \tag{2.12}$$

The existence of such sequences follows from the interpolation properties of functions belonging to X_n and can be proved by using the same considerations as in [17] or [4, pp. 134–135]. According to (2.1), (2.2), we can write

$$c_1(\dot{\Theta}_m(t), w_m) + D(\bar{\Theta}_m(t), w_m) + c_2(\operatorname{div} \bar{Z}_m(t), w_m) +$$
$$+ \beta(\bar{\Theta}_m(t), w_m)_\vartheta = (\bar{Q}_m(t), w_m) + \beta(\bar{g}_m(t), w_m)_\vartheta,$$
$$c_4(\dot{Z}_m(t), v_m) + a(\bar{U}_m(t), v_m) - c_3(\bar{\Theta}_m(t), \operatorname{div} v_m) =$$
$$= (\bar{f}_m(t), v_m) + (\bar{p}_m(t), v_m)_u \quad \text{a.e. in } I,$$

where $\bar{Q}_m(t) = Q(t_i)$, $t_{i-1} < t \leqq t_i$ $(i = 1, \ldots, r_m)$ and $\bar{g}_m(t)$, $\bar{f}_m(t)$ and $\bar{p}_m(t)$ are defined in the same way.

Integrating the last two relations over $[t', t''] \subset I$, where $t' < t''$ are arbitrary numbers, and passing to the limit for $n \to \infty$ we obtain (due to Lemma 6 and (2.12)) equations (1.19) and (1.20). Theorem 4 is proved.

3. WEAKER VARIATIONAL FORMULATION

Assumptions (1.27), (1.28) or (2.3), (2.4) are not very often satisfied in applications. In this section we consider a more realistic situation:

$$\vartheta_0 \in H\,, \quad u_0 \in V^2\,, \quad v_0 \in H^2\,. \tag{3.1}$$

Thus the second requirement in (3.1) is the only restrictive assumption. As the initial data are not so smooth as in the two preceding sections we obtain a less regular solution. Here our weak solution will be in the sense of the following definition:

Definition 2. *A pair $u(t)$, $\vartheta(t)$ is a weak (variational) solution of problem* (1.1) *to* (1.14) *iff*

a) $u \in AC(I; H^2) \cap L_\infty(I; V^2)$, $\dot{u} \in L_2(I; H^2)$, $u(0) = u_0$, $\vartheta \in L_2(I; W)$;

b) *the following relations are satisfied*:

$$\begin{aligned}\int_0^T D(\vartheta(t), w(t))\,\mathrm{d}t - c_1 \int_0^T (\vartheta(t), \dot{w}(t))\,\mathrm{d}t - c_1(\vartheta_0, w(0)) - \\ - c_2 \int_0^T (\operatorname{div} u(t), \dot{w}(t))\,\mathrm{d}t - c_2(\operatorname{div} u_0, w(0)) + \\ + \beta \int_0^T (\vartheta(t), w(t))_\vartheta\,\mathrm{d}t = \int_0^T (Q(t), w(t))\,\mathrm{d}t + \\ + \beta \int_0^T (g(t), w(t))_\vartheta\,\mathrm{d}t \quad \forall w \in \mathscr{F}_1\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\int_0^T a(u(t), v(t))\,\mathrm{d}t - c_4 \int_0^T (\dot{u}(t), \dot{v}(t))\,\mathrm{d}t - \\ - c_4(v_0, v(0)) - c_3 \int_0^T (\vartheta(t), \operatorname{div} v(t))\,\mathrm{d}t = \\ = \int_0^T [(f(t), v(t)) + (p(t), v(t))_u]\,\mathrm{d}t \quad \forall v \in \mathscr{F}_2\end{aligned} \tag{3.3}$$

where

$$\mathscr{F}_1 = \{v \in L_2(I; W) \colon \dot{v} \in L_2(I; H),\ v(T) = 0\}\,,$$

$$\mathscr{F}_2 = \{v \in L_2(I; V^2) \colon \dot{v} \in L_2(I; H^2),\ v(T) = 0\}.$$

The problem defined by a), b) will be briefly called problem PC - 2.

We shall use the same finite element spaces V_n and W_n as in Section 2. We make only one restriction concerning the sequence $\{\mathscr{T}_n\}$ of triangulations:

$$h_n \geqq h_{n+1} \quad \forall n. \tag{3.4}$$

Requirement (3.4) is easy to satisfy.

Lemma 7. *There exist sequences $\{u_{0n}\}$, $\{v_{0n}\}$, $\{\vartheta_{0n}\}$, where $u_{0n} \in V_n^2$, $v_{0n} \in V_n^2$, $\vartheta_{0n} \in W_n$, such that*

$$\lim_{n\to\infty} \|u_{0n} - u_0\|_1 = 0, \tag{3.5}$$

$$\lim_{n\to\infty} \|\vartheta_{0n} - \vartheta_0\|_0 = 0, \quad \lim_{n\to\infty} \|v_{0n} - v_0\|_0 = 0. \tag{3.6}$$

Proof. The proof of (3.5) is the same as the proof of $(2.12)_2$. We prove $(3.6)_1$: Let $\{\varepsilon_k\}$ be an arbitrary sequence with the properties $\varepsilon_k > 0$, $\varepsilon_k > \varepsilon_{k+1}$, $\lim \varepsilon_k = 0$. As $C_0^\infty(\Omega)$ is dense in $H = L_2(\Omega)$ we can find a sequence $\{w_{\varepsilon_k}\} \subset C_0^\infty(\Omega)$ such that $\|\vartheta_0 - w_{\varepsilon_k}\|_0 < \varepsilon_k/2$. In wiew of the well known interpolation properties of the finite element spaces W_n, we have

$$\|w_{\varepsilon_k} - I_n w_{\varepsilon_k}\|_0 \leqq Ch_n^2 \|w_{\varepsilon_k}\|_2,$$

where $I_n w_{\varepsilon_k} \in W_n$ is the interpolate of w_{ε_k} in W_n. Thus, according to (3.4) and the property $\lim h_n = 0$, there exists n_k such that

$$\|w_{\varepsilon_k} - I_n w_{\varepsilon_k}\|_0 \leqq \varepsilon_k/2 \quad n \geqq n_k.$$

Hence

$$\|\vartheta_0 - I_n w_{\varepsilon_k}\|_0 < \varepsilon_k \quad \forall n \geqq n_k.$$

Thus we can construct a sequence $\{\vartheta_{0n}\}$ satisfying $(3.6)_1$ in the following way: We set $\vartheta_{0n_k} = I_{n_k} w_{\varepsilon_k} \in W_{n_k}$ and in the case $n_k < j < n_{k+1}$ we set $\vartheta_{0j} = I_j w_{\varepsilon_k} \in W_j$. The construction of a sequence $\{v_{0n}\}$ satisfying $(3.6)_2$ is similar. Lemma 7 is proved.

As to the functions $f(t)$, $Q(t)$, $g(t)$ and $p(t)$ let us assume less than in Sections 1 and 2:

$$f \in L_2(I; H^2), \quad Q \in L_2(I; H), \quad g \in L_2(I; L_2(\Gamma_{2\vartheta})), \tag{3.7}$$

$$p \in AC(I; [L_2(\Gamma_{2u})]^2), \quad \dot{p} \in L_2(I; [L_2(\Gamma_{2u})]^2). \tag{3.8}$$

Let us set $\Delta t_n = T/r_n$, r_n being an integer ($r_{n+1} > r_n\ \forall n$) and let us define

$$f^i = \frac{1}{\Delta t_n} \int_{t_{i-1}}^{t_i} f \, dt, \quad Q^i = \frac{1}{\Delta t_n} \int_{t_{i-1}}^{t_i} Q \, dt, \quad g^i = \frac{1}{\Delta t_n} \int_{t_{i-1}}^{t_i} g \, dt \tag{3.9}$$

where $t_i \equiv t_i^n = i\Delta t_n$ and $i = 1, \ldots, r_n$. Now we can formulate the following discrete problem:

Problem PD - 3: *Let the functions f, Q, g and p satisfy assumptions* (3.7) *and* (3.8), *respectively. Let* $U^0 = u_{0n} \in V_n^2$, $U^{-1} = u_{0n} - \Delta t_n v_{0n} \in V_n^2$ *and* $\Theta^0 = \vartheta_{0n} \in W_n$, *where* u_{0n}, v_{0n} *and* ϑ_{0n} *were introduced in Lemma 7. Find* $U^i \in V_n^2$, $\Theta^i \in W_n$ $(i = 1, \ldots, r_n)$ *such that*

$$(3.10)\quad c_1 \Delta t_n^{-1}(\Delta\Theta^i, w) + D(\Theta^i, w) + c_2 \Delta t_n^{-1}(\operatorname{div} \Delta U^i, w) + \beta(\Theta^i, w)_\vartheta = (Q^i, w) + \beta(g^i, w)_\vartheta \quad \forall w \in W_n\,,$$

$$(3.11)\quad c_4 \Delta t_n^{-2}(\Delta^2 U^i, v) + a(U^i, v) - c_3(\Theta^i, \operatorname{div} v) = (f^i, v) + (p(t_i), v)_u \quad \forall v \in V_n^2\,.$$

The proof of existence and uniqueness of the solution of problem PD - 3 is the same as the proof of Lemma 4.

Using the solution $\Theta^i \in W_n$, $U^i \in V_n^2$ $(i = 1, \ldots, r_n)$ of PD - 3 we define the following finite element Rothe's function $U_n(t)$ and step-functions $\bar{U}_n(t)$, $\bar{\Theta}_n(t)$:

$$(3.12)\quad U_n(t) = U^{i-1} + \frac{\Delta U^i}{\Delta t_n}(t - t_{i-1}^n)\,, \quad t_{i-1}^n \leqq t \leqq t_i^n \quad (i = 1, \ldots, r_n)\,,$$

$$(3.13)\quad \bar{U}_n(t) = U^i\,, \quad t_{i-1}^n < t \leqq t_i^n \quad (i = 1, \ldots, r_n)\,,$$

$$(3.14)\quad \bar{\Theta}_n(t) = \Theta^i\,, \quad t_{i-1}^n < t \leqq t_i^n \quad (i = 1, \ldots, r_n)\,.$$

Lemma 8. *Let the initial data* ϑ_0, u_0, v_0 *satisfy* (3.1) *and let* f, Q, g *and* p *satisfy* (3.7) *and* (3.8), *respectively. Then there exist functions* $u(t)$, $\vartheta(t)$ *with the properties* $u \in AC(I; H^2) \cap L_\infty(I; V^2)$, $\dot{u} \in L_2(I; H^2)$, $u(0) = u_0$, $\vartheta \in L_2(I; W)$ *and such that*

$$(3.15)\quad U_m \to u \text{ in } C(I; H^2)\,, \quad \bar{U}_m \rightharpoonup u \text{ in } L_2(I; V^2)\,,$$

$$(3.16)\quad \dot{U}_m \rightharpoonup \dot{u} \text{ in } L_2(I; H^2)\,, \quad \bar{\Theta}_m \rightharpoonup \vartheta \text{ in } L_2(I; W)$$

where $\{m\}$ *is a subsequence of* $\{n\}$ *and* $\{U_n\}$, $\{\bar{U}_n\}$ *and* $\{\bar{\Theta}_n\}$ *are sequences of functions* (3.12)–(3.14).

Proof. Let us assume $c_2 = c_3 = 1$ (see Remark 1). We multiply (3.10) by Δt_n, set $w = \Theta^i$ in (3.10) and $v = \Delta U^i$ in (3.11), add (3.10) and (3.11) up and sum the result from $i = 1$ to $i = j$, where $j \leqq r_n$. We obtain

$$c_1 \sum_{i=1}^{j} (\Delta\Theta^i, \Theta^i) + \Delta t_n \sum_{i=1}^{j} \{D(\Theta^i, \Theta^i) + \beta(\Theta^i, \Theta^i)_\vartheta\} + c_4 \Delta t_n^{-2} \sum_{i=1}^{j} (\Delta^2 U^i, \Delta U^i) + \sum_{i=1}^{j} a(U^i, \Delta U^i) = \Delta t_n \sum_{i=1}^{j} (Q^i, \Theta^i) + \Delta t_n \beta \sum_{i=1}^{j} (g^i, \Theta^i)_\vartheta + \sum_{i=1}^{j} (f^i, \Delta U^i) + \sum_{i=1}^{j} (p(t_i), \Delta U^i)_u\,.$$

Adding to the both sides the expression $\sum_{i=1}^{j} (U^i, \Delta U^i)$, using the relation $(\Delta b^i, b^i) =$

$= (b^i, b^i)/2 - (b^{i-1}, b^{i-1})/2 + (\Delta b^i, \Delta b^i)/2$ and inequalities (1.17), (1.18) we get

$$(3.17) \qquad \tfrac{1}{2}c_1\|\Theta^j\|_0^2 + \tfrac{1}{2}c_1\sum_{i=1}^{j}\|\Delta\Theta^i\|_0^2 + C_1\Delta t_n\sum_{i=1}^{j}\|\Theta^i\|_1^2 +$$

$$+ \tfrac{1}{2}c_4\left\|\frac{\Delta U^j}{\Delta t_n}\right\|_0^2 + \tfrac{1}{2}c_4\sum_{i=1}^{j}\left\|\frac{\Delta^2 U^i}{\Delta t_n}\right\|_0^2 + C_2\|U^j\|_1^2 +$$

$$+ C_2\sum_{i=1}^{j}\|\Delta U^i\|_1^2 \leqq C\left(\|\Theta^0\|_0^2 + \left\|\frac{\Delta U^0}{\Delta t_n}\right\|_0^2 + \|U^0\|_1^2\right) +$$

$$+ \Delta t_n\sum_{i=1}^{j}(Q^i, \Theta^i) + \Delta t_n\beta\sum_{i=1}^{j}(g^i, \Theta^i)_\vartheta + \sum_{i=1}^{j}(f^i, \Delta U^i) +$$

$$+ \sum_{i=1}^{j}(p(t_i), \Delta U^i)_u + \sum_{i=1}^{j}(U^i, \Delta U^i)\,.$$

As $\|\Theta^0\|_0^2 + \|\Delta U^0/\Delta t_n\|_0^2 + \|U^0\|_1^2 = \|\vartheta_{0n}\|_0^2 + \|v_{0n}\|_0^2 + \|u_{0n}\|_1^2 \leqq C$ we can find

$$(3.18) \qquad R.H.S. \leqq C + C\left\{\int_0^T\|Q\|_0^2\,dt + \int_0^T\|g\|_A^2\,dt + \int_0^T\|f\|_0^2\,dt +\right.$$

$$\left.+ \max_{t\in I}\|p(t)\|_B^2 + \int_0^T\|\dot{p}\|_B^2\,dt\right\} + \tfrac{1}{2}C_1\Delta t_n\sum_{i=1}^{j}\|\Theta^i\|_1^2 +$$

$$+ \tfrac{1}{2}C_2\|U^j\|_1^2 + C\Delta t_n\sum_{i=1}^{j}\left\{\|U^i\|_1^2 + \left\|\frac{\Delta U^i}{\Delta t_n}\right\|_0^2\right\},$$

where $R.H.S.$ denotes the right-hand side of (3.17) and $A = L_2(\Gamma_{2\vartheta})$, $B = [L_2(\Gamma_{2u})]^2$. The proof of (3.18) follows from the estimates

$$\Delta t_n\sum_{i=1}^{j}(Q^i, \Theta^i) \leqq \sum_{i=1}^{j}\left(\Delta t_n\int_{t_{i-1}}^{t_i}\|Q\|_0^2\,dt\right)^{1/2}\|\Theta^i\|_0 \leqq$$

$$\leqq C_1^{-1}\int_0^T\|Q\|_0^2\,dt + \tfrac{1}{4}C_1\Delta t_n\sum_{i=1}^{j}\|\Theta^i\|_1^2\,,$$

$$\sum_{i=1}^{j}(f^i, \Delta U^i) \leqq \sum_{i=1}^{j}\|f^i\|_0^2\,\Delta t_n + \sum_{i=1}^{j}\|\Delta U^i/\Delta t_n\|_0^2\,\Delta t_n\,,$$

$$\sum_{i=1}^{j}\|f^i\|_0^2\,\Delta t_n \leqq \frac{1}{\Delta t_n}\sum_{i=1}^{j}\left(\int_{t_{i-1}}^{t_i}\|f\|_0\,dt\right)^2 \leqq$$

$$\leqq \frac{1}{\Delta t_n}\sum_{i=1}^{r_n}\int_{t_{i-1}}^{t_i}dt\int_{t_{i-1}}^{t_i}\|f\|_0^2\,dt = \int_0^T\|f\|_0^2\,dt$$

$$\sum_{i=1}^{j}(p(t_i^n), \Delta U^i)_u = (p(t_j^n), U^j)_u - (p(t_1^n), U^0) - \sum_{i=1}^{j-1}(\Delta p(t_{i+1}^n), U^i)_u \leqq$$

$$\leqq C \max_{t\in I} \|p(t)\|_B (\|U^j\|_1 + \|u_{0n}\|_1) + C\Delta t_n \sum_{i=1}^{j-1} \left\| \frac{\Delta p(t_{i+1}^n)}{\Delta t_n} \right\|_B \|U^i\|_1 ,$$

$$\Delta t_n \sum_{i=1}^{j-1} \left\| \frac{\Delta p(t_{i+1}^n)}{\Delta t_n} \right\|_B \|U^i\|_1 \leqq \int_0^T \|\dot{p}\|_0^2 \, dt + \Delta t_n \sum_{i=1}^{j-1} \|U^i\|_1^2 ,$$

$$\sum_{i=1}^{j} (U^i, \Delta U^i) \leqq \Delta t_n \sum_{i=1}^{j} \left\{ \|U^i\|_1^2 + \left\| \frac{\Delta U^i}{\Delta t_n} \right\|_0^2 \right\} .$$

The term containing $(g^i, \Theta^i)_\vartheta$ can be estimated similarly as the term depending on (Q^i, Θ^i).

Using the discrete form of Gronwall's lemma we obtain from (3.17) and (3.18):

$$\|U^j\|_1 \leqq C , \quad \left\| \frac{\Delta U^j}{\Delta t_n} \right\|_0 \leqq C , \quad \|\Theta^j\|_0 \leqq C \quad (1 \leqq j \leqq r_n) , \tag{3.19}$$

$$\sum_{i=1}^{j} \|\Delta U^i\|_1^2 \leqq C , \quad \Delta t_n \sum_{i=1}^{j} \|\Theta^i\|_1^2 \leqq C \quad (1 \leqq j \leqq r_n) . \tag{3.20}$$

Estimates (3.19), (3.20) and relations (3.12)–(3.14) imply:

$$\|U_n(t)\|_1 \leqq C \quad \forall t \in I , \quad \|\dot{U}_n\|_{L_2(I; H^2)} \leqq C ,$$

$$\int_0^T \|\bar{U}_n(t) - U_n(t)\|_1^2 \, dt \leqq C/n , \quad \|\bar{\Theta}_n\|_{L_2(I;W)} \leqq C .$$

The last estimate immediately implies assertion $(3.15)_2$ with $\vartheta \in L_2(I; W)$. The other three estimates together with (3.5) imply the remaining assertions of Lemma 8. The proof is almost the same as that of Lemma 3, parts A), B) and D). Details are omitted. Lemma 8 is proved.

The proof of the following lemma can be found in [10].

Lemma 9. *The space $C^1(I; W_2^2(\Omega))$ is dense in $C^1(I; W_2^1(\Omega))$ and the sets $\mathscr{S}_1 = \{w \in C^1(I; W_2^2(\Omega) \cap W): w(T) = 0\}$ and $\mathscr{S}_2 = \{v \in C^1(I; [W_2^2(\Omega)]^2 \cap V^2): v(T) = 0\}$ are dense in the sets $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively, where $\mathscr{F}_1$ and $\mathscr{F}_2$ are defined in Definition* 2.

Now we are ready to prove the main results of this section:

Theorem 5. *Let the assumptions of Lemma* 8 *be satisfied. Then the solution of problem PC - 2 exists and is unique and we have*

$$U_n \to u \ \text{in} \ C(I; H^2) , \quad \dot{U}_n \rightharpoonup \dot{u} \ \text{in} \ L_2(I; H^2) , \quad \bar{\Theta}_n \rightharpoonup \vartheta \ \text{in} \ L_2(I; W)$$

where $\{U_n\}$ and $\{\bar{\Theta}_n\}$ are arbitrary sequences of functions (3.12) *and* (3.14), *respectively, and u, ϑ is the solution of PC - 2.*

Proof. A) First we prove that the limit functions u, ϑ from Lemma 8 and their derivatives satisfy relations (3.2), (3.3). Let $w^* \in \mathscr{S}_1$, $v^* \in \mathscr{S}_2$, the sets $\mathscr{S}_1, \mathscr{S}_2$ being defined in Lemma 9, and let us set

$$(3.21) \qquad w^i = \frac{1}{\Delta t_n} \int_{t_{i-1}}^{t_i} w^*(s)\, ds\,, \quad v^i = \frac{1}{\Delta t_n} \int_{t_{i-1}}^{t_i} v^*(s)\, ds\,.$$

In (3.21) and in the sequel we omit the superscript n at the symbol t_j^n. Let $I_n w^i \in W_n$ and $I_n v^i \in V_n^2$ be the interpolates of w^i and v^i, respectively. Let us set $w = I_n w^i$ in (3.10) and $v = I_n v^i$ in (3.11) and let us sum from $i = 1$ to $i = r_n$. After summing by parts and multiplying by Δt_n we obtain (in (3.22), (3.23) and in what follows we write r instead of r_n):

$$(3.22) \qquad \sum_{i=1}^{r} D(\Theta^i, I_n w^i)\, \Delta t_n + c_1(\Theta^r, I_n w^r) - c_1(\vartheta_{0n}, I_n w^1) -$$
$$- c_1 \sum_{i=1}^{r-1} (\Theta^i, \Delta(I_n w^{i+1})/\Delta t_n)\, \Delta t_n + c_2(\operatorname{div} U^r, I_n w^r) -$$
$$- c_2(\operatorname{div} u_{0n}, I_n w^1) - c_2 \sum_{i=1}^{r-1} (\operatorname{div} U^i, \Delta(I_n w^{i+1})/\Delta t_n)\, \Delta t_n +$$
$$+ \beta \sum_{i=1}^{r} (\Theta^i, I_n w^i)_\vartheta\, \Delta t_n = \sum_{i=1}^{r} \{(Q^i, I_n w^i) + \beta(g^i, I_n w^i)_\vartheta\}\, \Delta t_n\,,$$

$$(3.23) \qquad \sum_{i=1}^{r} a(U^i, I_n v^i)\, \Delta t_n + c_4(\Delta U^r, I_n v^r/\Delta t_n) - c_4(v_{0n}, I_n v^1) -$$
$$- c_4 \sum_{i=1}^{r-1} (\Delta U^i/\Delta t_n, \Delta(I_n v^{i+1})/\Delta t_n)\, \Delta t_n -$$
$$- c_3 \sum_{i=1}^{r} (\Theta^i, \operatorname{div} I_n v^i)\, \Delta t_n = \sum_{i=1}^{r} (f^i, I_n v^i)\, \Delta t_n + \sum_{i=1}^{r} (p(t_i), I_n v^i)_u\, \Delta t_n\,.$$

Let us define the following step-functions:

$$\bar{w}_n(t) = w^i\,, \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r)\,;$$
$$\bar{v}_n(t) = v^i\,, \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r)\,;$$
$$\bar{q}_n(t) = \Delta w^{i+1}/\Delta t_n\,, \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r-1)\,, \quad \bar{q}_n(t) = 0\,, \quad t \in (t_{r-1}, T]\,;$$
$$\bar{z}_n(t) = \Delta v^{i+1}/\Delta t_n\,, \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r-1)\,, \quad \bar{z}_n(t) = 0\,, \quad t \in (t_{r-1}, T]\,.$$

As $w^* \in \mathscr{S}_1$ and $v^* \in \mathscr{S}_2$ it is not difficult to see that

$$(3.24) \qquad \bar{w}_n \to w^* \text{ in } L_2(I; W)\,, \quad \bar{v}_n \to v^* \text{ in } L_2(I; V^2)\,,$$

$$(3.25) \qquad \bar{q}_n \to \dot{w}^* \text{ in } L_2(I; W)\,, \quad \bar{z}_n \to \dot{v}^* \text{ in } L_2(I; V^2)\,.$$

As an example let us prove $(3.25)_1$. We have

$$\int_0^T \|\bar{q}_n(t) - \dot{w}^*(t)\|_1^2\, dt = \int_{t_{r-1}}^T \|\dot{w}^*(t)\|_1^2\, dt +$$

$$+\sum_{i=1}^{r-1}\int_{t_{i-1}}^{t_i}\left\|\Delta t_n^{-2}\int_{t_{i-1}}^{t_i}\{w^*(s+\Delta t_n)-w^*(s)\}\,\mathrm{d}s-\dot{w}^*(t)\right\|_1^2\mathrm{d}t\,.$$

The first term tends to zero if $\Delta t_n \to 0$. The second term can be written in the form, according to Taylor's theorem,

$$\sum_{i=1}^{r-1}\int_{t_{i-1}}^{t_i}\left\|\Delta t_n^{-1}\int_{t_{i-1}}^{t_i}(\dot{w}^*(\tilde{s})-\dot{w}^*(t))\,\mathrm{d}s\right\|_1^2\mathrm{d}t$$

where $\tilde{s}$ depends on $s\in[t_{i-1},t_i]$ and belongs to $[t_{i-1},t_{i+1}]$. As I is a closed interval we have

$$\|\dot{w}^*(\tilde{s})-\dot{w}^*(t)\|_1\leqq\eta_n\,,\quad \tilde{s}\in[t_{i-1},t_{i+1}]\,,\quad t\in[t_{i-1},t_i]\quad(i=1,\dots,r-1)$$

where $\eta_n \to 0$ if $\Delta t_n \to 0$. Thus the second term is bounded by $\eta_n^2 T$. This proves $(3.25)_1$.

Using the step-functions $\bar{w}_n(t)$, $\bar{v}_n(t)$, $\bar{q}_n(t)$, $\bar{z}_n(t)$ let us write relations (3.22) and (3.23) in the following form:

$$\begin{aligned}(3.26)\qquad &\int_0^T D(\bar{\Theta}_n(t),\bar{w}_n(t))\,\mathrm{d}t+c_1(\Theta^r,w^r)-c_1(\vartheta_{0n},w^1)-\\
&-c_1\int_0^T(\bar{\Theta}_n(t),\bar{q}_n(t))\,\mathrm{d}t+c_2(\operatorname{div}U^r,w^r)-\\
&-c_2(\operatorname{div}u_{0n},w^1)-c_2\int_0^T(\operatorname{div}\bar{U}_n(t),\bar{q}_n(t))\,\mathrm{d}t+\\
&+\beta\int_0^T(\bar{\Theta}_n(t),\bar{w}_n(t))_\vartheta\,\mathrm{d}t+\{\sum_{i=1}^{r}D(\Theta^i,I_nw^i-w^i)\,\Delta t_n+c_1(\Theta^r,I_nw^r-w^r)-\\
&-c_1(\vartheta_{0n},I_nw^1-w^1)-c_1\sum_{i=1}^{r-1}(\Theta^i,I_n(\Delta w^{i+1}/\Delta t_n)-\Delta w^{i+1}/\Delta t_n)\,\Delta t_n+\\
&+c_2(\operatorname{div}U^r,I_nw^r-w^r)-c_2(\operatorname{div}u_{0n},I_nw^1-w^1)-\\
&-c_2\sum_{i=1}^{r-1}(\operatorname{div}U^i,I_n(\Delta w^{i+1}/\Delta t_n)-\Delta w^{i+1}/\Delta t_n)\,\Delta t_n+\\
&+\beta\sum_{i=1}^{r}(\Theta^i,I_nw^i-w^i)_\vartheta\,\Delta t_n\}=\int_0^T(\bar{Q}_n(t),\bar{w}_n(t))\,\mathrm{d}t+\beta\int_0^T(\bar{g}_n(t),\bar{w}_n(t))_\vartheta\,\mathrm{d}t+\\
&+\{\sum_{i=1}^{r}(Q^i,I_nw^i-w^i)\,\Delta t_n+\beta\sum_{i=1}^{r}(g^i,I_nw^i-w^i)_\vartheta\,\Delta t_n\}\,,\end{aligned}$$

$$\begin{aligned}(3.27)\qquad &\int_0^T a(\bar{U}_n(t),\bar{v}_n(t))\,\mathrm{d}t+c_4(\Delta U^r,v^r/\Delta t_n)-c_4(v_{0n},v^1)-\\
&-c_4\int_0^T(\dot{U}_n(t),\bar{z}_n(t))\,\mathrm{d}t-c_3\int_0^T(\bar{\Theta}_n(t),\operatorname{div}\bar{v}_n(t))\,\mathrm{d}t+\end{aligned}$$

$$+ \{ \sum_{i=1}^{r} a(U^i, I_n v^i - v^i)\, dt + c_4(\Delta U^r, I_n(v^r/\Delta t_n) - v^r/\Delta t_n) -$$

$$- c_4 \sum_{i=1}^{r-1} (\Delta U^i/\Delta t_n, I_n(\Delta v^{i+1}/\Delta t_n) - \Delta v^{i+1}/\Delta t_n)\, \Delta t_n -$$

$$- c_4(v_{0n}, I_n v^1 - v^1) - c_3 \sum_{i=1}^{r} (\Theta^i, \operatorname{div}(I_n v^i - v^i))\, \Delta t_n \} =$$

$$= \int_0^T (\bar{f}_n(t), \bar{v}_n(t))\, dt + \int_0^T (\bar{p}_n(t), \bar{v}_n(t))_u\, dt +$$

$$+ \sum_{i=1}^{r} \{ (f^i, I_n v^i - v^i) + (p(t_i), I_n v^i - v^i)_u \}\, \Delta t_n ,$$

where

$$\bar{Q}_n(t) = Q^i , \quad t \in (t_{i-1}, t_i] , \quad \bar{g}_n(t) = g^i , \quad t \in (t_{i-1}, t_i] ,$$

$$\bar{f}_n(t) = f^i , \quad t \in (t_{i-1}, t_i] , \quad \bar{p}_n(t) = p(t_i) , \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r) .$$

Assumptions (3.7) and (3.8) imply

(3.28) $$\bar{Q}_n \to Q \text{ in } L_2(I; H) , \quad \bar{g}_n \to g \text{ in } L_2(I; L_2(\Gamma_{2\vartheta})) ,$$

(3.29) $$\bar{f}_n \to f \text{ in } L_2(I; H^2) , \quad \bar{p}_n \to p \text{ in } L_2(I; [L_2(\Gamma_{2u})]^2) .$$

We prove, for example, $(3.29)_1$: As $C(I; H^2)$ is dense in $L_2(I; H^2)$ we can find a sequence $\{f^{(k)}\} \subset C(I; H^2)$ such that $\|f^{(k)} - f\|_A \to 0$, where we denote $A = L_2(I; H^2)$. For every k let us define the step-functions

$$\bar{f}_n^{(k)}(t) = \Delta t_n^{-1} \int_{t_{i-1}}^{t_i} f^{(k)}(s)\, ds , \quad t \in (t_{i-1}, t_i] , \quad i = 1, \ldots, r .$$

We have

$$\|\bar{f}_n - f\|_A \leqq \|\bar{f}_n - \bar{f}_n^{(k)}\|_A + \|\bar{f}_n^{(k)} - f^{(k)}\|_A + \|f^{(k)} - f\|_A .$$

It is easy to see that

$$\|\bar{f}_n - \bar{f}_n^{(k)}\|_A \leqq \|f - f^{(k)}\|_A , \quad \|\bar{f}_n^{(k)} - f^{(k)}\|_A \leqq \eta_n^{(k)} T^{1/2}$$

where for every k, $\eta_n^{(k)} \to 0$ if $\Delta t_n \to 0$. The first inequality follows immediately from the definition of the functions $\bar{f}_n(t)$, $\bar{f}_n^{(k)}(t)$. The proof of the second inequality is the same as that of (3.24). Let us choose $\varepsilon > 0$ and let k be such an integer that $\|f^{(k)} - f\|_A < \varepsilon/3$. Then there exists such N that $\eta_n^{(k)} T^{1/2} < \varepsilon/3\ \forall n \geqq N$. This means that for every $\varepsilon > 0$ we can find N such that $\|\bar{f}_n - f\|_A < \varepsilon\ \forall n \geqq N$ and $(3.29)_1$ is proved.

To be able to prove (3.38) and (3.39) we shall need also the following relations:

(3.30) $$\|w^i - I_n w^i\|_1 \leqq C h_n \max_{t \in I} \|w^*(x, t)\|_2 ,$$

$$(3.31) \qquad \|\Delta w^{i+1}/\Delta t_n - I_n(\Delta w^{i+1}/\Delta t_n)\|_1 \leqq Ch_n \max_{t\in I} \|\dot{w}^*(x, t)\|_2 ,$$

$$(3.32) \qquad \|v^i - I_n v^i\|_1 \leqq Ch_n \max_{t\in I} \|v^*(x, t)\|_2 ,$$

$$(3.33) \qquad \|\Delta v^{i+1}/\Delta t_n - I_n(\Delta v^{i+1}/\Delta t_n)\|_1 \leqq Ch_n \max_{t\in I} \|\dot{v}^*(x, t)\|_2 ,$$

which follow from the finite element interpolation theorems. For example,

$$\|w^i - I_n w^i\|_1 \leqq Ch_n \|w^i\|_2 = Ch_n \left\| \Delta t_n^{-1} \int_{t_{i-1}}^{t_i} w^*(x, s)\, ds \right\|_2 \leqq$$

$$\leqq Ch_n\, \Delta t_n^{-1} \int_{t_{i-1}}^{t_i} \|w^*(x, s)\|_2\, ds \leqq Ch_n \max_{t\in I} \|w^*(x, t)\|_2 ,$$

$$\|\Delta w^{i+1}/\Delta t_n - I_n(\Delta w^{i+1}/\Delta t_n)\|_1 \leqq Ch_n \|\Delta w^{i+1}/\Delta t_n\|_2 =$$

$$= Ch_n\, \Delta t_n^{-2} \left\| \int_{t_i}^{t_{i+1}} \{w^*(x, s) - w^*(x, s - \Delta t_n)\}\, ds \right\|_2 \leqq$$

$$\leqq Ch_n \max_{t\in I} \|\dot{w}^*(x, t)\|_2 .$$

Finally, we shall need:

$$(3.34) \qquad \lim_{n\to\infty} (\Delta U^r, v^r/\Delta t_n) = \lim_{n\to\infty} \|w^r\|_1 = \lim_{n\to\infty} \|v^r\|_1 = 0 ,$$

$$(3.35) \qquad \lim_{n\to\infty} \|w^1 - w^*(0)\|_1 = \lim_{n\to\infty} \|v^1 - v^*(0)\|_1 = 0 ,$$

$$(3.36) \qquad \|v^r/\Delta t_n - I_n(v^r/\Delta t_n)\|_1 \leqq Ch_n \max_{t\in I} \|\dot{v}^*(x, t)\|_2 .$$

Let us prove these relations. As $v^*(x, T) = 0$ we have

$$v^r/\Delta t_n = \Delta t_n^{-2} \int_{t_{r-1}}^{T} [v^*(x, s) - v^*(x, T)]\, ds = \Delta t_n^{-2} \int_{t_{r-1}}^{T} (s - T)\, \dot{v}^*(x, \tilde{s})\, ds$$

where $\tilde{s} = T - \delta(T - s)$, $\delta \in (0, 1)$. Thus

$$(3.37) \qquad \|v^r/\Delta t_n\|_j \leqq \max_{t\in I} \|\dot{v}^*(x, t)\|_j \quad (j = 0, 1, 2) .$$

Estimates $(3.19)_2$ and (3.37) imply $(3.34)_1$. Relation (3.36) follows from the finite element interpolation theorem and (3.37). Relation $(3.34)_2$ follows from the estimates

$$\|w^r\|_j = \Delta t_n^{-1} \left\| \int_{t_{r-1}}^{T} [w^*(x, s) - w^*(x, T)]\, ds \right\|_j \leqq$$

$$\leqq \Delta t_n^{-1} \int_{t_{r-1}}^{T} |s - T| \,.\, \|\dot{w}^*(x, \tilde{s})\|_j\, ds \leqq \Delta t_n \max_{t\in I} \|\dot{w}^*(x, t)\|_j$$

and relation $(3.35)_1$ follows from

$$\|w^1 - w^*(0)\|_j = \Delta t_n^{-1} \left\| \int_0^{t_1} [w^*(s) - w^*(0)]\, ds \right\|_j \leqq$$

$$\leqq \max_{s \in [0,t_1]} \|w^*(s) - w^*(0)\|_j \quad (0 \leqq j \leqq 2)\,.$$

Relations $(3.34)_3$ and $(3.35)_2$ can be proved in the same way.

Passing to the limit for $m \to \infty$ in relations (3.26), (3.27), which are considered only for the subsequence $\{m\}$ of the sequence $\{n\}$, we obtain, according to (3.5), (3.6), (3.15), (3.16), (3.19), (3.24), (3.25), (3.28)–(3.36):

$$(3.38)\qquad \int_0^T D(\vartheta(t), w^*(t))\,\mathrm{d}t - c_1 \int_0^T (\vartheta(t), \dot{w}^*(t))\,\mathrm{d}t - c_1(\vartheta_0, w^*(0)) -$$
$$- c_2 \int_0^T (\operatorname{div} u(t), \dot{w}^*(t))\,\mathrm{d}t - c_2(\operatorname{div} u_0, w^*(0)) +$$
$$+ \beta \int_0^T (\vartheta(t), w^*(t))_\vartheta\,\mathrm{d}t = \int_0^T (Q(t), w^*(t))\,\mathrm{d}t + \beta \int_0^T (g(t), w^*(t))_\vartheta\,\mathrm{d}t \quad \forall w^* \in \mathscr{S}_1\,,$$

$$(3.39)\qquad \int_0^T a(u(t), v^*(t))\,\mathrm{d}t - c_4 \int_0^T (\dot{u}(t), \dot{v}^*(t))\,\mathrm{d}t - c_4(v_0, v^*(0)) -$$
$$- c_3 \int_0^T (\vartheta(t), \operatorname{div} v^*(t))\,\mathrm{d}t = \int_0^T [(f(t), v^*(t)) + (p(t), v^*(t))_u]\,\mathrm{d}t \quad \forall v^* \in \mathscr{S}_2\,.$$

Relations (3.38), (3.39) together with Lemma 9 imply relations (3.2), (3.3).

B) Property a) is proved in Lemma 8.

C) Now we prove the uniqueness of the solution. As the problem is linear it suffices to prove that the homogeneous problem

$$(3.40)\qquad \int_0^T D(\vartheta(t), w(t))\,\mathrm{d}t - c_1 \int_0^T (\vartheta(t), \dot{w}(t))\,\mathrm{d}t -$$
$$- c_2 \int_0^T (\operatorname{div} u(t), \dot{w}(t))\,\mathrm{d}t + \beta \int_0^T (\vartheta(t), w(t))_\vartheta\,\mathrm{d}t = 0 \quad \forall w \in \mathscr{F}_1\,,$$

$$(3.41)\qquad \int_0^T a(u(t), v(t))\,\mathrm{d}t - c_4 \int_0^T (\dot{u}(t), \dot{v}(t))\,\mathrm{d}t -$$
$$- c_3 \int_0^T (\vartheta(t), \operatorname{div} v(t))\,\mathrm{d}t = 0 \quad \forall v \in \mathscr{F}_2\,,$$

$$(3.42)\qquad u(0) = 0$$

has only the trivial solution $\vartheta(t) \equiv 0$, $u(t) \equiv 0$.

Let us set

$$(3.43)\qquad \xi(t) = \int_0^t \vartheta(\tau)\,\mathrm{d}\tau\,,$$

let us choose $s \in I$ arbitrarily and let us define

$$(3.44) \qquad w_s(t) = \begin{cases} \int_s^t \xi(\tau)\,\mathrm{d}\tau & \text{if } \ 0 \leqq t \leqq s \\ 0 & \text{if } \ s < t \leqq T, \end{cases}$$

$$(3.45) \qquad v_s(t) = \begin{cases} \int_s^t u(\tau)\,\mathrm{d}\tau & \text{if } \ 0 \leqq t \leqq s \\ 0 & \text{if } \ s < t \leqq T. \end{cases}$$

It is evident that $w_s \in \mathscr{F}_1$, $v_s \in \mathscr{F}_2$.

According to (3.42)–(3.45), we have

$$(3.46) \qquad \int_0^T D(\vartheta(t), w_s(t))\,\mathrm{d}t = \int_0^s D(\vartheta(t), w_s(t))\,\mathrm{d}t =$$

$$= \int_0^s D(\dot{\xi}(t), w_s(t))\,\mathrm{d}t = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} D(\xi(t), w_s(t))\,\mathrm{d}t -$$

$$- \int_0^s D(\xi(t), \dot{w}_s(t))\,\mathrm{d}t = D(\xi(s), w_s(s)) - D(\xi(0), w_s(0)) -$$

$$- \int_0^s D(\xi(t), \xi(t))\,\mathrm{d}t = - \int_0^s D(\xi(t), \xi(t))\,\mathrm{d}t\,,$$

$$(3.47) \qquad \int_0^T (\vartheta(t), \dot{w}_s(t))\,\mathrm{d}t = \int_0^s (\dot{\xi}(t), \xi(t))\,\mathrm{d}t = \tfrac{1}{2}\|\xi(s)\|_0^2\,,$$

$$(3.48) \qquad \int_0^T (\operatorname{div} u(t), \dot{w}_s(t))\,\mathrm{d}t = - \int_0^s (\operatorname{div} u(t), \xi(t))\,\mathrm{d}t\,,$$

$$(3.49) \qquad \int_0^T (\vartheta(t), w_s(t))_\vartheta\,\mathrm{d}t = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} (\xi(t), w_s(t))_\vartheta\,\mathrm{d}t -$$

$$- \int_0^s (\xi(t), \dot{w}_s(t))_\vartheta\,\mathrm{d}t = - \int_0^s (\xi(t), \xi(t))_\vartheta\,\mathrm{d}t\,,$$

$$(3.50) \qquad \int_0^T a(u(t), v_s(t))\,\mathrm{d}t = \int_0^s a(\dot{v}_s(t), v_s(t))\,\mathrm{d}t =$$

$$= \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} a(v_s(t), v_s(t))\,\mathrm{d}t = - \frac{1}{2} a(v_s(0), v_s(0))\,,$$

$$(3.51) \qquad \int_0^T (\dot{u}(t), \dot{v}_s(t))\,\mathrm{d}t = \int_0^s (\dot{u}(t), u(t))\,\mathrm{d}t = \tfrac{1}{2}\|u(s)\|_0^2\,,$$

$$(3.52) \qquad \int_0^T (\vartheta(t), \operatorname{div} v_s(t))\,\mathrm{d}t = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} (\xi(t), \operatorname{div} v_s(t))\,\mathrm{d}t -$$

$$-\int_0^s (\xi(t), \operatorname{div} \dot{v}_s(t))\,dt = -\int_0^s (\xi(t), \operatorname{div} u(t))\,dt\,.$$

Let us set $w = w_s$ in (3.40) and $v = v_s$ in (3.41) and let us express the integrals appearing in (3.40) and (3.41) by means of (3.46) – (3.52). Then let us add the resulting relations up (for simplicity we assume $c_2 = c_3 = 1$). We obtain

$$(3.53)\qquad \int_0^s \{D(\xi(t), \xi(t)) + \beta(\xi(t), \xi(t))_\vartheta\}\,dt +$$
$$+ \tfrac{1}{2}a(v_s(0), v_s(0)) + \tfrac{1}{2}c_4\|u(s)\|_0^2 + \tfrac{1}{2}c_1\|\xi(s)\|_0^2 = 0\,.$$

As $s \in I$ is arbitrary relations (3.53), (1.17) imply

$$\|\xi(s)\|_0 = \|u(s)\|_0 = 0 \quad \forall s \in I\,.$$

Thus $\vartheta(t) = 0$, $u(t) = 0$ almost everywhere in I and the uniqueness of the solution is proved.

D) The fact that the sequences $\{U_n\}$, $\{\bar{\Theta}_n\}$ can be arbitrary follows from the uniqueness of the solution. Theorem 5 is proved.

4. THREE SPECIAL (DEGENERATE) CASES

Problem (1.1) – (1.14) includes formally three special (or degenerate) cases. We obtain them if we set $c_1 = 0$, or $c_4 = 0$, or $c_1 = c_4 = 0$. In the case $c_1 = 0$ we do not prescribe initial condition (1.7) and in the case $c_4 = 0$ initial conditions (1.8), (1.9) are substituted by the initial condition

$$(4.1)\qquad (\operatorname{div} u)(x_1, x_2, 0) = \varphi(x_1, x_2)\,, \quad (x_1, x_2) \in \Omega$$

where $\varphi(x_1, x_2)$ is a given function.

The physical meaning of the case $c_1 = 0$, $c_4 > 0$ is unknown; thus we shall not discuss this situation. Problem (1.1) – (1.7), (4.1), (1.10) – (1.14) (when $c_1 > 0$, $c_4 = 0$) describes the quasistatical thermoeleasticity.

Problem (1.1) – (1.6), (4.1) (when $c_1 = c_4 = 0$) is one of the models of consolidation of clay. In this case boundary condition (1.4) has a modificated form

$$(4.2)\qquad \partial\vartheta/\partial\nu = g(x_1, x_2, t)\,, \quad (x_1, x_2) \in \Gamma_{2\vartheta}\,, \quad t > 0$$

and relation (1.10) is changed to

$$(4.3)\qquad \sigma_{ij} = \sigma_{ij}(u, \vartheta) \equiv D_{ijkm}\,\varepsilon_{km}(u) - \vartheta\delta_{ij}\,.$$

Thus $c_3 = 1$. The function ϑ has the meaning of pore water pressure in this case. (For more detail, see [15].)

We call these three cases degenerate because it is impossible to repeat (or modify) the considerations from Sections 1 and 2. The reason is simple: Considerations introduced in Section 1 are based on the assumption that there exist functions $S^0 \in H^2$, $R^0 \in H$ which satisfy (1.27) and (1.28). In the case $c_1 = c_4 = 0$ we must substitute this assumption by the following one: There exist functions $\vartheta_0 \in W$, $u_0 \in V^2$, $U^{-1} \in V^2$, where U^{-1} depends on Δt_n, such that

(4.4) $$\operatorname{div} u_0 = \varphi\,,$$

(4.5) $$D(\vartheta_0, w) + c_2 \Delta t_n^{-1}(\operatorname{div}(u_0 - U^{-1}), w) = \tilde{G}_0(w) \quad \forall w \in W\,,$$

(4.6) $$a(u_0, v) - (\vartheta_0, \operatorname{div} v) = F_0(v) \quad \forall v \in V^2$$

where $\tilde{G}_0(w) = (Q(0), w) + (g(0), w)_\vartheta$ and $F_0(v)$ is the same as in (1.28). There is no reasonable physical situation where these conditions can be satisfied. (Even in the simplest and most frequent case $\varphi \equiv 0$, which expresses the assumption that the pore water is incompressible, we are not able to succeed: we can set $u_0 \equiv 0$; then assuming $(2.4)_2$ we obtain from (4.6)

(4.7) $$\operatorname{grad} \vartheta_0 = f(0) \quad \text{in} \quad H^2\,.$$

Requirement (4.7) restricts essentially the choice of f. For example, as $\vartheta_0 \in W$ the quite natural datum $f(x, t) = \text{const.}$ is eliminated.)

In the case $c_1 > 0$, $c_4 = 0$ assumptions (1.27), (1.28) must be substituted by the following assumption: There exist functions $R^0 \in H$, $u_0 \in V^2$, $U^{-1} \in V^2$, where U^{-1} depends on Δt_n, such that relations (4.4), (4.6) and

(4.8) $$c_1(R^0, w) + D(\vartheta_0, w) + c_2 \Delta t_n^{-1}(\operatorname{div}(u_0 - U^{-1}), w) = \\ = (Q(0), w) + (g(0), w) \quad \forall w \in W$$

are satisfied. In this case $\vartheta_0 \in W$ is a given function appearing in the initial condition (1.7). Requirements (4.4), (4.6), (4.8) again essentially restrict the choice of data.

On the contrary, the weaker variational formulation is quite appropriate for the cases $c_1 = c_4 = 0$ and $c_1 > 0$, $c_4 = 0$ and all considerations from Section 3 can be easily modified. We mention here briefly the case $c_1 = c_4 = 0$.

We restrict our considerations to functions φ with the following property: To a given $\varphi \in H$ there exists such a vector $u_0 \in V^2 \cap [W_2^2(\Omega)]^2$ that condition (4.4) is satisfied. (In the case $\varphi \equiv 0$, which is most important for applications, we can set $u_0 = 0$.)

Inspecting the proofs from Section 3 we see that we can prove the following result: Let $c_1 = c_4 = 0$, let $\operatorname{mes} \Gamma_{1u} > 0$, $\operatorname{mes} \Gamma_{1\vartheta} > 0$ and let the assumptions of Lemma 8 be satisfied. Then the solution of PC - 2 exists and is unique and we have

$$\bar{U}_n \rightharpoonup u \text{ in } L_2(I; V^2)\,, \quad \bar{\Theta}_n \rightharpoonup \vartheta \text{ in } L_2(I; W)$$

where u, ϑ is the solution of PC - 2 and $\{\bar{U}_n\}$ and $\{\bar{\Theta}_n\}$ are arbitrary sequences of functions (3.13) and (3.14), respectively, which are generated by the unique solution of PD - 3.

Let us note that the variational problem considered in [15] is a special case of the problem from Definition 2.

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Souhrn

ANALÝZA PŘIBLIŽNÝCH ŘEŠENÍ SDRUŽENÉ DYNAMICKÉ TERMOELASTICITY A PŘÍBUZNÝCH PROBLÉMŮ

Jozef Kačur a Alexander Ženíšek

V článku jsou studovány otázky existence a jednoznačnosti řešení různých variačních formulací sdruženého problému dynamické termoelasticity a konvergence přibližných řešení těchto problémů.

V části 1 je definováno semidiskrétní přibližné řešení, které je získáno časovou diskretizací variačního problému z definice 1 pomocí Eulerovy zpětné formule. Za předpokladu, že data jsou dostatečně hladká (viz (1.21)—(1.24) a (1.27), (1.28)) je dokázána existence a jednoznačnost řešení (věta 1) a rychlost konvergence $O(\Delta t^{1/2})$ Rotheho funkcí v prostoru $C(I; W_2^1(\Omega))$ pro složky posunutí a v prostoru $C(I; L_2(\Omega))$ pro teplotu (věta 2); regulárnost řešení je studována ve větách 1 a 3.

V části 2 je definováno plně diskretizované řešení variačního problému z definice 1 pomocí Eulerovy zpětné formule a nejjednodušších konečných prvků. Konvergence tohoto přibližného řešení je dokázána ve větě 4.

V části 3 jsou na data položeny co možná nejslabší požadavky. Tomu odpovídá jiná definice variačního řešení. Ve větě 5 je dokázána existence a jednoznačnost tohoto řešení a konvergence plně diskretizovaného řešení.

Резюме

АНАЛИЗ ПРИБЛИЖЁННЫХ РЕШЕНИЙ СОПРЯЖЁННОЙ ДИНАМИЧЕСКОЙ ТЕРМОЭЛАСТИЧНОСТИ И РОДСТВЕННЫХ ПРОБЛЕМ

Jozef Kačur a Alexander Ženíšek

В статье изучаются вопросы существования и единственности решений различных вариационных формулировок сопряжённой проблемы динамической термоэластичности и сходимость приближённых решений этих проблем.

В части 1 определено полудискретное приближенное решение, которое получено временной дискретизацией вариационной проблемы из определения 1 при помощи обратной формулы Эйлера. При предположении, что данные достаточно гладки (см. (1.21)—(1.24) и (1.27), (1.28), доказаны существование и единственность решения (теорема 1) и скорость сходимости $O(\Delta t^{1/2})$ функций Ротэ в пространстве $C(I; W_2^1(\Omega))$ для компонент смещений и в пространстве $C(I; L_2(\Omega))$ для температуры (теорема 2). Регулярность решений изучена в теоремах 1 и 3.

В части 2 определено полностью дискретизированное решение вариационной проблемы из определения 1 при помощи обратной формулы Эйлера и простейших конечных элементов. Сходимость этого приближенного решения доказана в теореме 4.

В части 3 на данные накладываются как можно слабейшие условия. Этому соответствует другая формулировка вариационного решения. В теореме 5 доказаны существование и единственность этого решения и сходимость полностью дискретизированного решения.

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