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OPTIMAL DESIGN OF AN ELASTIC BEAM ON AN ELASTIC BASIS

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Summary. An elastic simply supported beam of given volume and of constant width and length, fixed on an elastic base, is considered. The design variable is taken to be the thickness of the beam; its derivatives of the first order are bounded both above and below. The load consists of concentrated forces and moments, the weight of the beam and of the so called continuous load. The cost functional is either the H^2-norm of the deflection curve or the L^2-norm of the normal stress in the extreme fibre of the beam.

Existence of solutions of optimization problems in both the primary and dual formulations of the state problem is proved. For both formulations, approximate problems are introduced and convergence of their solutions to those of the continuous problem is established. Theoretical conclusions are corroborated by an illustrative example.

INTRODUCTION

The theme of this work stems from the paper [6], the assumptions imposed on the beam being, however, of different physical nature.

An elastic beam of constant width, length and weight is considered. This beam is fixed on an elastic base and its ends are hinged with fixed supports. Its own weight, the concentrated forces and moments and the so called continuous load act on the beam.

The thickness of the beam is taken for the design variable and the Lipschitz functions bounded simultaneously from below and from above form the set of admissible functions.

The cost function depends either on the deflection curve or on the normal stress in the extreme fibre of the beam.

In Section 1 we formulate an optimal design problem and prove the existence of a solution. Section 2 deals with a dual variational formulation as well as with the existence of a solution. In Sections 3 and 4 we study finite element approximations to the optimization problem and their convergence to the solutions from the preceding sections. An illustrative example and some comments are given in Section 5.

1.1. Formulation of the problem

Let us consider an elastic beam lying on an elastic basis. The classical theory (see e.g. [9]) yields the relation for the deflection function y:

$$\frac{d^2}{dx}\left(E\frac{s}{12}e^3\frac{d^2y}{dx^2}\right) = q - a_0 y\,,$$

where E is the Young modulus, s is the width of the beam, e is the thickness of the beam, q is a continuous loading, $a_0 > 0$ is the constant of elasticity of the basis.

Now let the thickness e be a function depending on x and let the length of the beam be equal to 1. For simplicity we shall write the product $E(s/12)$ only as E. Under these conditions the differential operator of the deflection function may be expressed in the form

$$D_e(w) = \frac{d^2}{dx^2}\left(E\,e^3(x)\,\frac{d^2w(x)}{dx^2}\right) + a_0\,w(x)\,, \quad x \in \langle 0, 1\rangle\,.$$

Let us define a bilinear form

$$A(e; v, w) = \int_\Omega E\,e^3(x)\,v''(x)\,w''(x)\,dx + \int_\Omega a_0\,v(x)\,w(x)\,dx\,,$$

where $\Omega = (0, 1)$ and the derivatives are denoted by primes. The parametric function $e \in U_{ad}$,

$$U_{ad} = \Big\{e \in C^{0,1}(\bar{\Omega})\,; \quad 0 < e_{min} \leqq e(x) \leqq e_{max}\,, \quad x \in \bar{\Omega}\,;$$

$$\left|\frac{de}{dx}\right| \leqq C_1\,; \quad \int_\Omega e(x)\,dx = C_2\,; \quad C_1, C_2 > 0\Big\}\,,$$

where $C_1, C_2, e_{min}, e_{max}$ are given constants and $C^{0,1}(\bar{\Omega})$ is the set of Lipschitz functions on $\bar{\Omega}$. The variable functions $v, w \in V = \{v \in H^2(\Omega);\ v(0) = v(1) = 0\}$. $H^k(\Omega)$ denotes the Sobolev space $W^{k,2}(\Omega)$, $k = 1, 2, \ldots$, with the usual norm $\|\cdot\|_k$. The seminorm $(\int_\Omega (v^{(k)})^2\,dx)^{1/2}$ will be denoted by $|\cdot|_k$. It is obvious that V is a Hilbert space.

The differential equation

(1.1) $$D_e(w) = F$$

with boundary conditions $w(0) = w(1) = w''(0) = w''(1) = 0$ corresponds to the hinge joint of the beam ends.

Let us replace (1.1) by

(1.2) $$A(e; u, v) = \langle F(e), v\rangle \quad \forall v \in V\,,$$

where $u \in V$, $e \in U_{\text{ad}}$.

$$F(e) = \sum_{i=1}^{i_0} (P_i \delta(x - \bar{X}_i) - M_i \delta'(x - \bar{X}_i)) + f_0 - pe$$

represents the load of the beam (positive acting upwards). Here P_i, M_i and $p > 0$ are given constants, $\bar{X}_i \in \langle 0, 1 \rangle$ are prescribed points, δ is the Dirac measure and $f_0 \in L_1(\Omega)$. Then

$$\langle F(e), v \rangle = \sum_{i=1}^{i_0} (P_i\, v(\bar{X}_i) + M_i\, v'(\bar{X}_i)) + \int_\Omega (f_0 - pe)\, v \,\mathrm{d}x\,.$$

The state problem (1.2) is the variational formulation of (1.1). Furthermore, let us define

$$j_1(e, u) = \|u\|_2^2\,, \quad j_2(e, u) = \int_\Omega e^2(u'')^2 \,\mathrm{d}x\,.$$

The functional j_1 corresponds to the magnitude of the deflection curve, j_2 is proportional to the normal stress in the extreme fibre of the elastic beam.

We shall solve the *optimization problem*:
Let $u(e) \in V$ be a solution of the state problem (1.2), $e \in U_{\text{ad}}$ a given function. Let us define $\mathcal{J}_i(e) = j_i(e, u(e))$, $i = 1, 2$. Find a control $e_i^0 \in U_{\text{ad}}$ such that

$$\mathcal{J}_i(e_i^0) = \min_{e \in U_{\text{ad}}} \mathcal{J}_i(e)\,, \quad i = 1, 2\,. \tag{1.3}$$

1.2. Existence Theorem

The proof of our existence theorem is based on a general result for a class of optimal design problems (see [5]):

Let U be a Banach space of controls and $U_{\text{ad}} \subset U$ a set of admissible design variables. Assume that U_{ad} is compact in U.

Let a Hilbert space V with a norm $\|\cdot\|$ be given. Consider a bilinear form $A(e; \cdot, \cdot)$ and a linear continuous functional $\langle F(e), \cdot \rangle$ on V, both depending on a parameter $e \in U$. Assume that there exist positive constants α_0, α_1 and a subset U^0, $U_{\text{ad}} \subset U^0 \subset U$, independent of e, v, w and such that

$$|A(e; v, w)| \leqq \alpha_1 \|v\|\, \|w\|\,, \tag{1.4}$$

$$A(e; w, w) \geqq \alpha_0 \|w\|^2 \tag{1.5}$$

hold for all $e \in U^0$ and $v, w \in V$.

Moreover, assume that:

(1.6) if $e, e_n \in U^0$, $e_n \to e$ in U and $v_n \rightharpoonup v$ (weakly) in V for $n \to \infty$,

then $A(e_n; v_n, w) \to A(e; v, w)$ $\forall w \in V$;

(1.7) if $e, e_n \in U^0$, $e_n \to e$ in U, then $\langle F(e_n), v\rangle \to \langle F(e), v\rangle \quad \forall v \in V$;

(1.8) there exists a positive constant γ, independent of e, v and such that

$$|\langle F(e), v\rangle| \leqq \gamma\|v\|$$

holds for all $e \in U^0$ and $v \in V$.

We consider the following state problem:
for $e \in U_{\mathrm{ad}}$ find $u(e) \in V$ such that

$$A(e; u(e), v) = \langle F(e), v\rangle \quad \forall v \in V. \tag{1.9}$$

Under the assumptions (1.4), (1.5), (1.8) the state problem (1.9) is uniquely solvable for any $e \in U^0$.

Let a functional $j: (U \times V) \to R$ be given, which satisfies the following condition:

$$\text{if } e_n, e \in U^0,\ e_n \to e \text{ in } U,\ u_n \rightharpoonup u \text{ in } V(\text{weakly}) \Rightarrow \tag{1.10}$$
$$\Rightarrow \liminf_{n\to\infty} j(e_n, u_n) \geqq j(e, u).$$

Denoting the cost functional by $\mathscr{J}(e) = j(e, u(e))$, where $u(e)$ denotes the solution of (1.9), we may consider the optimal design problem:

find $e^0 \in U_{\mathrm{ad}}$ such that

$$\mathscr{J}(e^0) \leqq \mathscr{J}(e) \quad \forall e \in U_{\mathrm{ad}}. \tag{1.11}$$

Theorem 1.1. *Under the assumptions* (1.4) *to* (1.8) *and* (1.10), *the optimal design problem* (1.11) *has at least one solution.*

Proof. See [5].

Theorem 1.2. *The problem* (1.3) *has a least one solution for* $i = 1, 2$.

Proof. It is sufficient to verify the assumptions of Theorem 1.1. Let us introduce $U = C(\langle 0, 1\rangle)$, $U^0 = \{e \in U;\ e_{\min} \leqq e(x) \leqq e_{\max}\ \forall x \in \langle 0, 1\rangle\}$. The set U_{ad} is bounded and closed in $C(\bar{\Omega})$ and, moreover, consists of uniformly continuous functions. The theorem of Arzelà implies the compactness of U_{ad} in $C(\bar{\Omega})$.

The inequality (1.4) can be easily established.

Owing to the inequality

$$\int_\Omega (v')^2\,\mathrm{d}x = -\int_\Omega v''v\,\mathrm{d}x \leqq |v|_2\,|v|_0 \leqq \tfrac{1}{2}(|v|_2^2 + |v|_0^2)$$

the assumption (1.5) is fulfilled.

The validity of (1.6) is an easy consequence of the boundedness of the sequence $\{v_n\}_{n=1}^\infty$ in $H^2(\Omega)$ and of the uniform convergence of e_n.

It is readily seen that the functional $F(e) \in V^*$ and (1.8) holds, because of the continuous embedding $H^2(\Omega) \subset C^1(\bar{\Omega})$.

The condition (1.7) is obvious.

Let us verify (1.10). The norm squared is a weakly lower semicontinuous functional and the case of j_1 is solved. For any fixed $e \in U_{ad}$ the functional $j_2(e, \cdot): V \to R^+$ is weakly lower semicontinuous. Next, $j_2(e_n, u_n) = j_2(e, u_n) + I$ and

$$|I| = \left| \int_\Omega (e_n^2 - e^2)(u_n'')^2 \, dx \right| \leqq \|e_n^2 - e\|_{C(\bar{\Omega})} \|u_n\|_2^2 \to 0 \,, \quad n \to \infty \,,$$

since all the norms $\|u_n\|_2$ are bounded. Hence

$$\liminf_{n \to \infty} j_2(e_n, u_n) \geqq \liminf_{n \to \infty} j_2(e, u_n) + \lim_{n \to \infty} I \geqq j_2(e, u) \,,$$

Q.E.D.

2. DUAL VARIATIONAL FORMULATION OF THE STATE PROBLEM

The state problem (1.2) will be transformed into the dual variational formulation, called the principle of minimum complementary energy.

2.1. Derivation of the Dual Variational Formulation

First we derive the functional of complementary energy by means of the Friedrichs transform (see [3]).

Denoting $a(e) = Ee^3$, $g = f_0 - pe$, we can introduce the functional of potential energy

$$\mathscr{L}(v) = \frac{1}{2} \int_\Omega [a(e)(v'')^2 + a_0 v^2] \, dx - \langle F(e), v \rangle \,. \tag{2.1}$$

Let us define $\eta_1 = v''$, $\eta_2 = v$ and the Lagrange multipliers $\lambda_1(x)$, $\lambda_2(x)$. Thus we obtain the functional

$$\mathscr{H}(v, \eta_1, \eta_2, \lambda_1, \lambda_2) =$$

$$= \frac{1}{2} \int_\Omega [a(e)\eta_1^2 + a_0 \eta_2^2] \, dx - \langle F(e), v \rangle + \int_\Omega \lambda_1 (v'' - \eta_1) \, dx + \int_\Omega \lambda_2 (v - \eta_2) \, dx \,.$$

Using integration by parts and setting the variations with respect to v, η_1, η_2 equal to 0, we deduce the following conditions:

$$\lambda_1(0) = 0 \,, \quad \lambda_1(1) = 0 \,, \tag{2.2}$$

$$\lambda_1'' + \lambda_2 - F(e) = 0 \,, \tag{2.3}$$

$$a(e)\eta_1 = \lambda_1 \,, \tag{2.4}$$

$$a_0 \eta_2 = \lambda_2 \,. \tag{2.5}$$

Let us denote $b(e) = (a(e))^{-1}$, $b_0 = a_0^{-1}$. Using (2.4), (2.5), (2.3) leads to the functional

$$\mathscr{S}_1(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega (b(e)\,\lambda_1^2 + b_0\lambda_2^2)\,\mathrm{d}x - \int_\Omega (b(e)\,\lambda_1^2 + b_0\lambda_2^2)\,\mathrm{d}x =$$

$$= -\frac{1}{2}\int_\Omega (b(e)\,\lambda_1^2 + b_0\lambda_2^2)\,\mathrm{d}x\,.$$

Let us introduce $H = [L_2(\Omega)]^2$ and the bilinear form on $H \times H$:

$$(\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}})_H = \int_\Omega (b(e)\,\lambda_1\tilde{\lambda}_1 + b_0\lambda_2\tilde{\lambda}_2)\,\mathrm{d}x\,. \tag{2.6}$$

It is easily seen that H with the scalar product (2.6) is a Hilbert space.

Let us define

$$B(\boldsymbol{\lambda}, v) = \int_\Omega (\lambda_1 v'' + \lambda_2 v)\,\mathrm{d}x \quad \forall \boldsymbol{\lambda} \in H\,, \quad v \in H^2(\Omega)\,;$$

$$H_1 = \{\boldsymbol{\lambda} \in H;\ \exists v \in V,\ \lambda_1 = a(e)\,v'',\ \lambda_2 = a_0 v\}\,,$$

$$H_2 = \{\boldsymbol{\lambda} \in H;\ B(\boldsymbol{\lambda}, v) = 0\ \forall v \in V\}\,,$$

$$\Lambda_F = \{\boldsymbol{\lambda} \in H;\ B(\boldsymbol{\lambda}, v) = \langle F(e), v\rangle\ \forall v \in V\}\,.$$

Any vector $\boldsymbol{\lambda} \in \Lambda_F$ satisfies (2.2), (2.3) in the weak sense. We shall write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(v)$ iff $\lambda_1 = a(e)\,v''$, $\lambda_2 = a_0 v$.

Let us remark that λ_1 is the bending moment of the beam and λ_2 is the reaction of the elastic basis.

Theorem 2.1. (Principle of minimum complementary energy.) *Let $\mathscr{S}(\boldsymbol{\lambda}) = \frac{1}{2}\|\boldsymbol{\lambda}\|_H^2$ be given. Then*

$$\mathscr{S}(\boldsymbol{\lambda}^0) = \min_{\boldsymbol{\lambda} \in \Lambda_F} \mathscr{S}(\boldsymbol{\lambda}) \Leftrightarrow \boldsymbol{\lambda}^0 = \boldsymbol{\lambda}^0(u)\,, \tag{2.7}$$

where u is the solution of the problem (1.2).

Proof. The proof is based on the method of orthogonal projection. The space H_1 is orthogonal to H_2. Let $\boldsymbol{\lambda} \in H_1$, $\tilde{\boldsymbol{\lambda}} \in H_2$. Then we have $(\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}})_H = B(\tilde{\boldsymbol{\lambda}}, v) = 0$. Evidently $\boldsymbol{\lambda}(u) \in H_1$. For $\boldsymbol{\lambda} \in \Lambda_F$ we have $\boldsymbol{\lambda} - \boldsymbol{\lambda}(u) \in H_2$, since

$$B(\boldsymbol{\lambda} - \boldsymbol{\lambda}(u), v) = \int_\Omega [(\lambda_1 - \lambda_1(u))\,v'' + (\lambda_2 - \lambda_2(u))\,v]\,\mathrm{d}x = \langle F(e), v\rangle -$$

$$- \int_\Omega (a(e)\,u''v'' + a_0 uv)\,\mathrm{d}x = 0 \quad \forall v \in V\,.$$

Consequently, for $\boldsymbol{\lambda} \in \Lambda_F$ we may write

$$\|\boldsymbol{\lambda}\|_H^2 = \|\boldsymbol{\lambda} - \boldsymbol{\lambda}(u) + \boldsymbol{\lambda}(u)\|_H^2 = \|\boldsymbol{\lambda} - \boldsymbol{\lambda}(u)\|_H^2 + \|\boldsymbol{\lambda}(u)\|_H^2$$

and the proof is complete.

Q.E.D.

Remark. It is not difficult to prove that H_1 and H_2 are closed subspaces of H and $H = H_1 \oplus H_2$.

Next we shall employ the structure of the affine hyperplane Λ_F. Let $\boldsymbol{\lambda}^0 \in \Lambda_F$ be a particular element. Then we may write $\Lambda_F = \boldsymbol{\lambda}^0 + H_2$. Hence for $\boldsymbol{\chi} \in H_2$ we have

$$\mathscr{S}(\boldsymbol{\lambda}) = \mathscr{S}(\boldsymbol{\lambda}^0 + \boldsymbol{\chi}) = \tfrac{1}{2}\|\boldsymbol{\lambda}^0\|_H^2 + \tfrac{1}{2}\|\boldsymbol{\chi}\|_H^2 + (\boldsymbol{\chi}, \boldsymbol{\lambda}^0)_H\,.$$

Thus the principle of minimum complementary energy has the form

$$(2.8) \qquad \Phi(\boldsymbol{\chi}_0) = \tfrac{1}{2}\|\boldsymbol{\chi}_0\|_H^2 + (\boldsymbol{\chi}_0, \boldsymbol{\lambda}^0)_H = \min_{\boldsymbol{\chi} \in H_2} \Phi(\boldsymbol{\chi}) \Leftrightarrow \boldsymbol{\chi}_0 = \boldsymbol{\lambda}(u) - \boldsymbol{\lambda}^0\,,$$

where u is a solution of (1.2).

The condition (2.3) will be exploited now. For Λ_F we have $\lambda_1'' = F(e) - \lambda_2$ in the sense of distributions. Consequently, for $\boldsymbol{\lambda} \in \Lambda_F$, $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]$ we can define a functional $G(\boldsymbol{\lambda}) \in [H^2(\Omega)]^*$ by means of the relation

$$\langle G(\boldsymbol{\lambda}), w\rangle = \int_\Omega (\lambda_1 w'' - \lambda_1'' w)\,\mathrm{d}x\,, \quad w \in H^2(\Omega)\,.$$

For any $\boldsymbol{\chi} \in H_2 = \Lambda_0$ we have $\langle G(\boldsymbol{\chi}), v\rangle = B(\boldsymbol{\chi}, v) = 0\ \forall v \in V$. The equality $\langle G(\boldsymbol{\chi}), v\rangle = 0$ implies $\chi_1(1)\,v'(1) - \chi_1(0)\,v'(0) = 0$ as we find by integration by parts. From these conditions, $\chi_1(0) = \chi_1(1) = 0$ follows. We see that

$$(2.9) \qquad \bar{\chi} \in V \Rightarrow \boldsymbol{\chi} = [\bar{\chi}, -\bar{\chi}''] \in H_2\,;$$

$$(2.10) \qquad \boldsymbol{\chi} \in H_2 \Rightarrow \boldsymbol{\chi} = [\bar{\chi}, -\bar{\chi}''], \quad \bar{\chi} \in V\,.$$

Let us define the scalar product $(\chi, \tilde{\chi})_Q = \int_\Omega (b(e)\,\chi\tilde{\chi} + b_0\chi''\tilde{\chi}'')\,\mathrm{d}x$ and the norm $\|\chi\|_Q = (\chi, \chi)_Q^{1/2}$; $\chi, \tilde{\chi} \in H^2(\Omega)$.

Now we specify the particular element $\boldsymbol{\lambda}^0 = [\lambda_1^0, \lambda_2^0] \in \Lambda_F$. The functions are defined as follows:

$$(2.11) \qquad \lambda_2^0 = 0\,,$$

$$\lambda_1^0(x) = \lambda_1^0(P, M; x) + \int_0^x (f_0 - pe)(t)\,(x - t)\,\mathrm{d}t - x\int_0^1 (f_0 - pe)(t)\,(1 - t)\,\mathrm{d}t\,,$$

where

$$\lambda_1^0(P, M; x) = \sum_{i=1}^{i_0}\left[P_i\int_0^x H(t - \bar{X}_i)\,\mathrm{d}t - M_i H(x - \bar{X}_i)\right] - x\sum_{i=1}^{i_0}[P_i(1 - \bar{X}_i) - M_i]$$

and H denotes the Heaviside function. Integration by parts yields

$$\int_\Omega \lambda_1^0 v'' \,\mathrm{d}x = \langle F(e), v \rangle \quad \forall v \in V .$$

Conditions $\lambda_1^0(0) = \lambda_1^0(1) = 0$ are also fulfilled. The function λ_1^0 is the bending moment of the simply supported beam ($\lambda_2^0 = 0$).

Theorem 2.2. (Equivalent version of the principle of minimum complementary energy.)

Let us define a functional on V:

$$\Psi(\bar{\chi}) = \tfrac{1}{2}\|\bar{\chi}\|_Q^2 + \int_\Omega b(e)\,\bar{\chi}\lambda_1^0 \,\mathrm{d}x .$$

Then

$$\Psi(\bar{\chi}^0) = \min_{\bar{\chi}\in V} \Psi(\bar{\chi}) \tag{2.12}$$

if and only if $\boldsymbol{\chi}^0 = [\bar{\chi}^0, -(\bar{\chi}^0)''] = \boldsymbol{\lambda}(u) - \boldsymbol{\lambda}^0$, *where* u *is the solution of* (1.2).

Proof. Let $\boldsymbol{\chi} = [\chi_1, \chi_2] = [\bar{\chi}, -\bar{\chi}'']$, $\bar{\chi} \in V$,

$$\Psi(\bar{\chi}) = \frac{1}{2}\int_\Omega [b(e)\,\bar{\chi}^2 + b_0(\bar{\chi}'')^2] \,\mathrm{d}x + \int_\Omega b(e)\,\bar{\chi}\lambda_1^0 \,\mathrm{d}x =$$

$$= \tfrac{1}{2}\|\boldsymbol{\chi}\|_H^2 + \int_\Omega b(e)\,\bar{\chi}\lambda_1^0 \,\mathrm{d}x = \Phi(\boldsymbol{\chi}) .$$

We have used $\lambda_2^0 = 0$. The first inequality $\min_{\bar{\chi}\in V} \Psi(\bar{\chi}) \geqq \min_{\boldsymbol{\chi}\in H_2} \Phi(\boldsymbol{\chi})$ holds according to (2.9). The second inequality $\min_{\bar{\chi}\in V} \Psi(\bar{\chi}) \leqq \min_{\boldsymbol{\chi}\in H_2} \Phi(\boldsymbol{\chi})$ is an easy consequence of (2.10) and $\boldsymbol{\lambda}(u) - \boldsymbol{\lambda}^0 \in H_2$. Finally, the assertion (2.12) follows from (2.8).

Q.E.D.

2.2. Existence Theorem

We shall apply the above results to the functionals $\mathscr{J}_1$ and $\mathscr{J}_2$.

Let $\boldsymbol{\lambda}(u) = [\lambda_1(u), \lambda_2(u)]$ solve (2.7). Then $u'' = (a(e))^{-1}\lambda_1(u)$, $u = a_0^{-1}\lambda_2(u)$. Let $\bar{\chi}^0$ be the solution of (2.12), i.e. $[\lambda_1(u), \lambda_2(u)] = [\lambda_1^0, 0] + [\bar{\chi}^0, -(\bar{\chi}^0)'']$. Inserting this into $\mathscr{J}_i$, we have

$$\mathscr{J}_1(e) = a_0^{-2}\|(\bar{\chi}^0)''\|_2^2 , \quad \mathscr{J}_2(e) = E^{-2}\int_\Omega e^{-4}(\lambda_1^0 + \bar{\chi}^0)^2 \,\mathrm{d}x .$$

Thus we obtain the *equivalent version of the optimization problem*:

find a control $\tilde{e}_i \in U_{\mathrm{ad}}$, $i = 1, 2$, such that

$$\mathscr{J}_1(\tilde{e}_1) = \min_{e\in U_{\mathrm{ad}}} \|(\bar{\chi}^0(e))''\|_2^2 , \tag{2.13}$$

(2.14) $$\mathscr{J}_2(\tilde{e}_2) = \min_{e \in U_{ad}} \int_\Omega e^{-4}(\lambda_1^0(e) + \bar{\chi}^0(e))^2 \, dx ,$$

where $\bar{\chi}^0(e)$ is the solution of the problem (2.12) which is taken with a parametric function e and the function $\lambda_1^0(e)$ also depending on e.

Theorem 2.3. *Both the problem* (2.13) *and the problem* (2.14) *have at least one solution.*

Proof. We know (Theorem 1.2) that there exist $e_i \in U_{ad}$, $i = 1, 2$, such that $\|u(e_1)\|_2^2 \leqq \|u(e)\|_2^2 \ \forall e \in U_{ad}$ and $\|e_2 \, u''(e_2)\|_0^2 \leqq \|e \, u''(e)\|_0^2 \ \forall e \in U_{ad}$, $u(e)$ solves (1.2) with a parameter e. The latter relations result in solutions of (2.12) and (2.13). In fact, $u''(e) = \lambda_1(e)/Ee^3$ and $u(e) = \lambda_2(e)/a_0$ according to (2.7).

Q.E.D.

Remark. Let us notice that the solution $\bar{\chi}^0(e) \in V$ even satisfies $\bar{\chi}^0(e) \in H^4(\Omega)$.

3. APPROXIMATIONS OF THE PRIMAL APPROACH

The following assumptions are considered a basis for our forthcoming considerations:

(I) Let N be an integer and $\mathscr{T}_h$ a partition of the interval $\langle 0, 1\rangle$ into N subintervals $\Delta_j = \langle X_{j-1}, X_j \rangle$ of the length h; $j = 1, 2, \ldots, N(h)$; $X_0 = 0$, $X_N = 1$. Let $P_k(\Delta)$ be the set of polynomials the order of which is at most k. We define

$$U_{ad}^h = \{e \in U_{ad}, \ e|_{\Delta j} \in P_1(\Delta_j) \ \forall_j\} ,$$

$$V_h = \{v \in V, \ v|_{\Delta j} \in P_3(\Delta_j) \ \forall_j\} .$$

(II) Instead of $A(e_h; v_h, w_h)$ we shall use the form

$$A_h(e_h; v_h, w_h) = \sum_{j=1}^{N(h)} \left(Ee_h^3(\xi_j) \int_{\Delta j} v_h'' w_h'' \, dx + \int_{\Delta j} a_0 v_h w_h \, dx \right),$$

where $\xi_j = \frac{1}{2}(X_{j-1} + X_j)$; $e_h \in U_{ad}^h$; $v_h, w_h \in V_h$.

(III) Assume that there exist open subintervals D_k such that $\bigcup_{k=1}^{k_0} \bar{D}_k = \langle 0, 1 \rangle$, $D_k \cap D_m = \emptyset$, $k \neq m$, $k_0 \geqq 1$, and for any k the function f_0 is extensible from D_k onto $\bar{D}_k$ in such a way that $f_0 \in C^1(\bar{D}_k)$. Let $\mathscr{T}_h^*$ denote the mesh $\mathscr{T}_h$ refined by the points $Y_k = \bar{D}_k \cap \bar{D}_{k+1}$, $k = 1, \ldots, k_0 - 1$.

(IV) Instead of $F(e)$ the following functional $F_h(e)$ will be used:

$$F_h(e_h)(v_h) = \langle F(e_h), v_h \rangle_h =$$

$$= \sum_{i=1}^{i_0} (P_i \, v_h(\bar{X}_i) + M_i \, v_h'(\bar{X}_i)) - \int_0^1 pe_h v_h \, dx + \left\{ \int_0^1 f_0 v_h \, dx \right\}_h ,$$

where $e_h \in U^h_{\text{ad}}$, $\bar{X}_i$ are prescribed points and $\{\ \}_h$ denotes the approximate value of the integral, obtained by means of the trapezoidal rule on the mesh $\mathcal{T}^*_h$.

Lemma 3.1. *Let the assumptions* (I)–(IV) *be satisfied. Then*

$$|A_h(e_h; v_h, w_h) - A(e_h; v_h, w_h)| \leqq c\, h \|v_h\|_2 \, \|w_h\|_2 \quad \forall e_h \in U^h_{\text{ad}}\,,\ v_h, w_h \in H^2(\Omega)\,, \tag{3.1}$$

$$|\langle F(e_h), v_h\rangle - \langle F(e_h), v_h\rangle_h| \leqq c(f_0)\, h \|v_h\|_2 \quad \forall v_h \in H^2(\Omega)\,. \tag{3.2}$$

Proof. Studying the left hand sides of these inequalities on the intervals Δ_j and $\Delta_j^* = \langle X_{j-1}, Y_k\rangle$, we can easily derive (3.1), (3.2); for the details see e.g. [6].

Q.E.D.

The form $A_h(e_h; \cdot, \cdot)$ on $V_h \times V_h$ and the functional $F_h(e_h) \in V_h^*$ comply with the assumptions of the Lax-Milgram Theorem. Consequently,

$$A_h(e_h; u_h, v_h) = \langle F(e_h), v_h\rangle_h \quad \forall v_h \in V_h \tag{3.3}$$

is uniquely solvable for any fixed $e_h \in U^h_{\text{ad}}$.

Lemma 3.2. *Let the assumptions* (I)–(IV) *hold. Furthermore, let a sequence* $\{e_h\}$, $e_h \in U^h_{\text{ad}}$ *converge to a function* e *uniformly on the interval* $\langle 0, 1\rangle$ *for* $h \to 0_+$. *Finally, let* $u(e)$ *be the solution of* (1.2) *and let* $u_h(e_h)$ *be the solution of* (3.3). *Then*

$$\|u(e) - u_h(e_h)\|_2 \to 0\,, \quad h \to 0_+\,.$$

Proof. Let us denote for brevity $u_h = u_h(e_h)$. The sequence $\{u_h\}$ is bounded, since $\alpha\|u_h\|_2^2 \leqq A_h(e_h; u_h, u_h) = \langle F(e_h), v_h\rangle_h \leqq (c(f_0)\, h + c)\, \|u_h\|_2$. The space V is convex and closed, i.e. weakly closed and $u_h \rightharpoonup u^* \in V$ (weakly) in V; the picked subsequence is denoted $\{u_h\}$ again.

It is not difficult to derive by means of regularization (see e.g. [7]) that there exist functions $v_\varkappa \in C^\infty(\bar{\Omega})$, $v_\varkappa \to v$ in V for $\varkappa \to 0_+$. Denoting the Hermite cubic interpolate of $v_\varkappa \in C^\infty(\bar{\Omega})$ over the partition $\mathcal{T}_h$ as $v_h \in V_h$, we easily obtain

$$\|v_h - v\|_2 \to 0\,, \quad h \to 0_+\,.$$

The following auxiliary assertions are derived from easy estimates or a weak convergence.

$$|A(e_h; u_h, v_h) - A(e; u_h, v_h)| \to 0\,, \qquad h \to 0_+\,; \tag{3.4}$$

$$|A(e; u_h, v_h) - A(e; u_h, v)| \to 0\,, \qquad h \to 0_+\,; \tag{3.5}$$

$$|A(e; u_h, v) - A(e; u^*, v)| \to 0\,, \qquad h \to 0_+\,. \tag{3.6}$$

Combining (3.1), (3.4), (3.5), (3.6), we have

$$|A_h(e_h; u_h, v_h) - A(e; u^*, v)| \to 0\,, \quad h \to 0_+\,. \tag{3.7}$$

Similarly

$$(3.8)\qquad |\langle F(e_h), v_h\rangle - \langle F(e), v_h\rangle| \to 0\,, \quad h \to 0_+\,;$$

$$(3.9)\qquad |\langle F(e), v_h\rangle - \langle F(e), v\rangle| \to 0\,, \quad h \to 0_+$$

and

$$(3.10)\qquad |\langle F(e_h), v_h\rangle_h - \langle F(e), v\rangle| \to 0\,, \quad h \to 0_+$$

by virtue of (3.2), (3.8), (3.9).

Passing to the limit with $h \to 0_+$ and using (3.7), (3.10), we conclude that $A(e; u^*, v) = \langle F(e), v\rangle\ \forall v \in V$. The uniqueness of $u(e)$ yields both $u(e) = u^*$ and the weak convergence of the primary sequence $\{u_h\}$ to the function u^*. It remains to prove the strong convergence.

Combining $|\langle F(e_h), u_h\rangle - \langle F(e), u(e)\rangle| \to 0$, $h \to 0_+$ and (3.3), (1.2), (3.2), we may write

$$(3.11)\qquad |A_h(e_h; u_h, u_h) - A(e; u(e), u(e))| \to 0\,, \quad h \to 0_+\,.$$

Let us introduce the scalar product $A(e; u, v)$ in V, the norm being $\|u\|_A = [A(e; u, u)]^{1/2}$. The following estimate is a consequence of (3.1), (3.4), (3.11):

$$\big|\,\|u_h\|_A^2 - \|u(e)\|_A^2\big| \leqq |A(e; u_h, u_h) - A_h(e_h; u_h, u_h)| + \\ + |A_h(e_h; u_h, u_h) - A(e; u(e), u(e))| \to 0\,, \quad h \to 0_+\,.$$

Using the weak convergence $u_h \rightharpoonup u(e)$, we accomplish the proof. In fact,

$$\alpha_0\|u_h - u(e)\|_2^2 \leqq (u_h - u(e),\ u_h - u(e))_A \to 0\,, \quad h \to 0\,.$$

Q.E.D.

Definition 3.1. *Let the approximate optimal design problem $\mathscr{P}_{hi}$, $i = 1, 2$, be defined in this way:*

find $e_{hi}^0 \in U_{\text{ad}}^h$ such that

$$\mathscr{J}_i(e_{hi}^0) = j_i(e_h^0, u_h(e_h^0)) = \min_{e_h \in U_{\text{ad}}^h} \mathscr{J}_i(e_h)\,,$$

where $u_h(e_h)$ solves (3.3).

Remark. For brevity we shall write $e_{hi}^0 \equiv e_h^0$. This convention will be used in Theorem 3.1 and Section 4.2.

Lemma 3.3. *The problem $\mathscr{P}_{hi}$ has at least one solution for any sufficiently small and positive h; $i = 1, 2$.*

Proof. We employ Theorem 1.1 again. Let us choose $U = C(\bar{\Omega})$, $V = V_h$, $U^0 = \{e \in C(\bar{\Omega});\ 0 < e_{\min} \leqq e(x) \leqq e_{\max}\ \forall x \in \bar{\Omega}\}$. It is evident that $U_{\text{ad}}^h \subset U$ is a compact set and that the form A_h and the functional $F_h(e_h)$ fulfil (1.4), (1.5) and (1.8), respectively.

Let us verify (1.6). Let us assume $e, e_n \in U^0$, $e_n \to e$ in U and $v_n \rightharpoonup v$ (weakly) in V_h for $n \to \infty$. The dimension of the space V_h is finite, therefore the convergence $v_n \to v \in V_h$ in $H^2(\Omega)$ is strong. Then

$$|A_h(e_n; v_n, w) - A_h(e; v, w)| \leqq$$
$$\leqq \sum_{j=1}^{N(h)} \left[E \left| \int_{\Delta j} (e_n^3(\xi_j)\, v_n'' - e^3(\xi_j)\, v_n'' + e^3(\xi_j)\, v_n'' - e^3(\xi_j)\, v'')\, w \, dx \right| + \right.$$
$$+ \int_{\Delta j} a_0 |(v_n - v)\, w| \, dx \to 0 \quad n \to \infty \quad \forall w \in V_h .$$

The condition (1.7) is a consequence of the equality

$$|\langle F(e_n), v \rangle_h - \langle F(e), v \rangle_h| = \left| \int_0^1 pv(e - e_n) \, dx \right| .$$

Finally, the condition (1.10) is fulfilled in virtue of the proof of Theorem 1.2, since $V_h \subset V = H^2(\Omega) \cap H_0^1(\Omega)$.

Q.E.D.

Lemma 3.4. *Let us consider the approximate problem $\mathscr{P}_{hi}$. Assume that a sequence $\{e_h\}$, $e_h \in U_{ad}^h$, converges to a function e uniformly on the interval $\langle 0, 1 \rangle$ for $h \to 0_+$. Then*

$$\lim_{h \to 0+} \mathscr{J}_i(e_h) = \mathscr{J}_i(e) , \quad i = 1, 2 .$$

Proof. The case of $i = 1$ immediately results from Lemma 3.2. Indeed,

$$| \|u_h(e_h)\|_2 - \|u(e)\|_2 | \leqq \|u_h(e_h) - u(e)\|_2 .$$

Combining the boundedness of the sequence $\{u_h(e_h)\}$ and the assumptions of the lemma, we derive

$$|\mathscr{J}_2(e_h) - \mathscr{J}_2(e)| \to 0 , \quad h \to 0_+ .$$

Q.E.D.

Theorem 3.1. *Let $\{e_h^0\}$, $h \to 0_+$, be a sequence of solutions of the approximate problems $\mathscr{P}_{hi}$, $i = 1, 2$. Then there exists a subsequence $\{e_{\bar h}^0\}$ such that for $\bar h \to 0_+$ $e_{\bar h}^0 \to e^0$ in $C(\bar\Omega)$, $u_{\bar h}(e_{\bar h}^0) \to u(e^0)$ in $H^2(\Omega)$, where $e^0 \in U_{ad}$ is the solution of the optimization problem P_i and $u(e^0) \in V$ is the corresponding solution of (1.2).*

Proof. Let $\eta \in U_{ad}$ be an arbitrary function. There exists a sequence $\{\eta_h\}$, $\eta_h \in U_{ad}^h$, such that $\eta_h \to \eta$ in $C(\bar\Omega)$ for $h \to 0_+$ (see [2]). Let us denote by $u_h(\eta_h)$ the solution of (3.3) where e_h is replaced by η_h. Since U_{ad} is compact in $C(\bar\Omega)$, there exists a subsequence $\{e_{\bar h}^0\} \subset \{e_h^0\}$ such that $e_{\bar h}^0 \to e^0$ uniformly on $\bar\Omega$ for $\bar h \to 0_+$ so that $e^0 \in U_{ad}$. Having in mind the definition of the problem P_{hi}, we arrive at the inequality $\mathscr{J}_i(e_{\bar h}^0) \leqq \mathscr{J}_i(\eta_{\bar h})$. Passing to the limit with $h \to 0_+$ and applying Lemma 3.4, we derive $\mathscr{J}_i(e^0) \leqq \mathscr{J}_i(\eta)$. Hence e^0 is a solution of the problem $\mathscr{P}_i$. The remaining part of the assertion is essentially Lemma 3.2.

Q.E.D.

4. APPROXIMATIONS OF THE DUAL VARIATIONAL FORMULATION

In this section we adopt the assumptions (I)–(IV) from Section 3, however, we distinguish $\{\ \}_h$ and $\{\ \}_{h*}$, i.e. the approximate values of the integral obtained by means of the trapezoidal rule on the mesh $\mathscr{T}_h$ and on the mesh $\mathscr{T}_h^*$, respectively.

4.1. Approximations of the Principle of the Minimum Complementary Energy

The further part of this paper deals with the numerical analysis based on Theorem 2.2. A parametric function e is indicated in the functional Ψ.

Lemma 4.1. *Let $\chi^0 \in V$, $\chi_h^0 \in V_h$, $\bar{\chi}_h^0 \in V_h$ be such functions that*

$$\Psi(e; \chi^0) = \min_{\chi \in V} \Psi(e, \chi)\,, \quad \Psi(e; \chi_h^0) = \min_{\chi_h \in V_h} \Psi(e; \chi_h)$$

and

$$\Psi(e_h; \bar{\chi}_h^0) = \min_{\chi_h \in V_h} \Psi(e_h; \chi_h)\,.$$

Let the sequence $\{e_h\}$, $e_h \in U_{ad}^h$, converge to a function $e \in U_{ad}$ uniformly on the interval $\langle 0, 1\rangle$ for $h \to 0_+$. Then

$$\Psi(e; \bar{\chi}_h^0) \to \Psi(e; \chi^0) \quad \textit{and} \quad \|\chi^0 - \bar{\chi}_h^0\|_2 \to 0 \quad \textit{for} \quad h \to 0_+\,.$$

Proof. V is a Hilbert space and the functional $\Psi(e, \cdot)$ has positively definite second Gâteaux derivative, therefore there exist unique functions χ^0, χ_h^0 and $\bar{\chi}_h^0$ (see e.g. [4], Theorem 6.4.5; similar theorems can be found also in [8]).

The sequence $\{\bar{\chi}_h^0\}$ is minimizing as we now derive with the assistance of the following three auxiliary assertions.

A) $$\Psi(e; \chi_h^0) \to \Psi(e; \chi^0)\,, \quad h \to 0_+\,.$$

In fact, let $\{v_h\}$, $v_h \in V_h$ be a sequence such that $\|\chi^0 - v_h\|_2 \to 0$, $h \to 0_+$. Then

$$0 \leqq \Psi(e; \chi_h^0) - \Psi(e; \chi^0) \leqq \Psi(e; v_h) - \Psi(e; \chi^0) \to 0\,, \quad h \to 0_+\,.$$

B) Using the uniform convergence $e_h \to e$, we derive for any bounded sequence $\{w_h\}$:

$$\Psi(e; w_h) - \Psi(e_h; w_h) \to 0\,, \quad h \to 0_+\,, \quad w_h \in V_h\,. \tag{4.1}$$

C) $$\Psi(e; \bar{\chi}_h^0) - \Psi(e; \chi_h^0) \to 0\,, \quad h \to 0_+\,.$$

The sequences $\{\|\chi_h^0\|_2\}$ and $\{\|\bar{\chi}_h^0\|_2\}$ are bounded for $h \to 0_+$. Indeed, if e.g. the sequence $\{\|\bar{\chi}_h^0\|_2\}$ is not bounded, then the inequality

$$\Psi(e_h; \bar{\chi}_h^0) \geqq K_0\|\bar{\chi}_h^0\|_2^2 - K_1\|\bar{\chi}_h^0\|_2, \quad K_0, K_1 > 0,$$

leads to a contradiction $\Psi(e_h, 0) = 0 < \Psi(e_h, \bar{\chi}_h^0)$ for sufficiently small h. Analogously for $\{\|\chi_h^0\|_2\}$.

Let us assume now that there exist subsequences, also denoted by $\{\bar{\chi}_h^0\}$ and $\{\chi_h^0\}$, with the property

$$\Psi(e; \bar{\chi}_h^0) - \Psi(e; \chi_h^0) \geqq \varepsilon \quad \forall h, \quad 0 < h \leqq \delta_1, \quad \varepsilon > 0.$$

The assertion B) implies that there exists δ_2, $0 < \delta_2 \leqq \delta_1$, such that

$$\Psi(e_h; \bar{\chi}_h^0) - \Psi(e; \chi_h^0) \geqq \tfrac{1}{2}\varepsilon \quad \forall h, \quad 0 < h \leqq \delta_2.$$

Using once more B), we have

$$\Psi(e_h; \bar{\chi}_h^0) - \Psi(e_h; \chi_h^0) \geqq \tfrac{1}{4}\varepsilon \quad \forall h, \quad 0 < h \leqq \delta_3 \leqq \delta_2.$$

Since $\Psi(e_h; \cdot)$ attains its minimum over V_h at the point $\bar{\chi}_h^0$, we have derived a contradiction.

Finally, applying A) and C), we conclude that

$$\Psi(e; \chi^0) - \Psi(e; \bar{\chi}_h^0) \quad \text{tends to zero for} \quad h \to 0_+.$$

The sequence $\{\bar{\chi}_h^0\}$, $h \to 0_+$, minimizes the coercive, weakly lower semicontinuous and strictly convex functional $\Psi(e; \cdot)$, hence ([4], Theorem 6.4.5)

$$\|\chi^0 - \bar{\chi}_h^0\|_2 \to 0, \quad h \to 0_+.$$

Q.E.D.

Using a numerical method to carry out the computations, we obtain only an approximate value of $\Psi(e_h, v_h)$. For this reason we shall introduce a functional Ψ_h.

First, let us approximate the function λ_1^0 (see (2.11)) by a function λ_h. Let us define

$$\lambda_h^*(X_j) = \left\{\int_0^{X_j} (X_j - t)\,(f_0(t) - p\, e_h(t)\, \mathrm{d}t)\right\}_{h*}, \quad j = 0, \ldots, N(h).$$

Next, let us construct the Lagrange linear interpolate $I\lambda_h^*$ of λ_h^* on the mesh $\mathscr{T}_h$.

Let the function λ_h be given by

$$\lambda_h(x) = I\,\lambda_h^*(x) - x\left\{\int_0^1 (1 - t)\,(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*} + \lambda_1^0(P, M; x), \quad x \in \langle 0, 1\rangle,$$

where

$$\lambda_1^0(P, M; x) = \sum_{i=1}^{i_0}\left[P_i \int_0^x H(t - \bar{X}_i)\, \mathrm{d}t - M_i H(x - \bar{X}_i)\right] - x\sum_{i=1}^{i_0}[P_i(1 - \bar{X}_i) - M_i].$$

Now we can define the functional

$$\Psi_h(e_h; v_h) = \frac{1}{2}\int_\Omega b_0(v_h'')^2 \,\mathrm{d}x + \frac{1}{2}\sum_{j=1}^{N(h)} b(e_h)(\xi_j)\int_{\Delta j} v_h^2 \,\mathrm{d}x + \left\{\int_\Omega b(e_h)\, v_h\lambda_h \,\mathrm{d}x\right\}_h,$$

$$\xi_j = \tfrac{1}{2}(X_{j-1} + X_j).$$

Lemma 4.2. *There exists just one function $v_h^0 \in V_h$ for which*

$$\Psi_h(e_h; v_h^0) = \min_{v_h \in V_h} \Psi_h(e_h; v_h). \tag{4.2}$$

Proof. The functional $\Psi_h(e_h; \cdot)$ is coercive, weakly lower semicontinuous and strictly convex on V_h.

Q.E.D.

Lemma 4.3. *Let a sequence $\{e_h\}$, $e_h \in U_{\mathrm{ad}}^h$, converge to a function $e \in U_{\mathrm{ad}}$ uniformly on the interval $\bar{\Omega}$ for $h \to 0_+$. Then $\|\bar{\chi}^0 - v_h^0\|_2 \to 0$, $h \to 0_+$, where $\bar{\chi}^0 \in V$ is a solution of (2.12) and $v_h^0 \in V_h$ solves (4.2).*

Proof. The proof is based on the following convergence. If $\{\|v_h\|_2\}$ is a bounded sequence, then

$$\Psi(e_h; v_h) - \Psi_h(e_h; v_h) \to 0, \quad h \to 0_+. \tag{4.3}$$

Now we are going to verify this assertion.

$$\Psi(e_h; v_h) - \Psi_h(e_h; v_h) = I_1 + I_2,$$

where

$$I_1 = \frac{1}{2}\int_\Omega b(e_h)\, v_h^2 \,\mathrm{d}x - \frac{1}{2}\sum_{j=1}^{N(h)} b(e_h)(\xi_j)\int_{\Delta j} v_h^2 \,\mathrm{d}x,$$

$$I_2 = \int_\Omega b(e_h)\, v_h\lambda_1^0 \,\mathrm{d}x - \left\{\int_0^1 b(e_h)\, v_h\lambda_h \,\mathrm{d}x\right\}_h.$$

It is easy to estimate I_1:

$$|I_1| = \left|\frac{1}{2}\sum_{j=1}^{N(h)}\int_{\Delta j} [b(e_h)(x) - b(e_h)(\xi_j)]\, v_h^2 \,\mathrm{d}x\right| \leqq \frac{1}{2}\sum_{j=1}^{N(h)}\frac{1}{E}\int_{\Delta j} 3h\|e_h^{-4}e_h'\|_{C(\Delta j)}\, v_h^2 \,\mathrm{d}x \leqq$$

$$\leqq \frac{3}{2E}\max_j \|e_h^{-4}e_h'\|_{C(\Delta j)}\, \|v_h\|_2^2\, h.$$

We may write $I_2 = I_3 + I_4$, where

$$I_3 = \int_\Omega b(e_h)\, v_h\lambda_1^0 \,\mathrm{d}x - \int_\Omega b(e_h)\, v_h\lambda_h \,\mathrm{d}x,$$

$$I_4 = \int_\Omega b(e_h)\, v_h\lambda_h \,\mathrm{d}x - \left\{\int_0^1 b(e_h)\, v_h\lambda_h \,\mathrm{d}x\right\}_h.$$

The crucial point of the estimation of I_3 consists in

$$\lambda_1^0(x) - \lambda_h(x) = xI_5 - xI_6 + I_7(x) - I\lambda_h^*(x),$$

where

$$I_5 = \left\{\int_0^1 (1 - t)(f_0(t) - p\, e_h(t))\, dt\right\}_{h^*},$$

$$I_6 = \int_0^1 (1 - t)(f_0(t) - p\, e_h(t))\, dt,$$

$$I_7(x) = \int_0^x (x - t)(f_0(t) - p\, e_h(t))\, dt.$$

The assumption $\Delta_l^* = \langle Y_{k-1}, Y_k \rangle$ may be introduced without loss of generality. Then

$$\left|\int_{\Delta_l^*} \{(1 - t)(f_0(t) - p\, e_h(t) - \tfrac{1}{2}[(1 - Y_{k-1})(f_0(Y_{k-1}) - p\, e_h(Y_{k-1}) + \right.$$

$$\left. + (1 - Y_k)(f_0(Y_k) - p\, e_h(Y_k))]\}\right| dt \leqq \left\|\frac{d}{dt}((1 - t)(f_0(t) - p\, e_h(t)))\right\|_{C(\Delta_l^*)} \int_{\Delta_l^*} h\, dt.$$

Thus we obtain

$$|I_5 - I_6| \leqq h \max_k \|(1 - t)(f_0(t) - p\, e_h(t))\|_{C^1(\bar{D}_k)}. \tag{4.4}$$

Now we shall deal with $I_7 - I\lambda_h^*$.

$$|I_7(X_j) - I\lambda_h^*(X_j)| = \left|\int_0^{X_j} (X_j - t)(f_0(t) - p\, e_h(t))\, dt - \right.$$

$$\left. - \left\{\int_0^{X_j} (X_j - t)(f_0(t) - p\, e_h(t))\, dt\right\}_{h^*}\right| \leqq$$

$$\leqq \max_{k,j} \left\|\frac{d}{dt}((X_j - t)(f_0(t) - p\, e_h(t)))\right\|_{C(\bar{D}_k)} \int_0^{X_j} h\, dx \leqq 2 \max_k \|f_0 - pe_h\|_{C^1(\bar{D}_k)}\, h.$$

For $x \in \langle X_{j-1}, X_j \rangle$ we derive

(4.5)

$$|I_7(x) - I\, \lambda_h^*(x)| = \left|I_7(x) - \frac{1}{h}[\lambda_h^*(X_j) - \lambda_h^*(X_{j-1})](x - X_{j-1}) - \lambda_h^*(X_{j-1})\right| \leqq$$

$$\leqq |I_7(X_{j-1}) - \lambda_h^*(X_{j-1})| + \left|\int_{X_{j-1}}^x (x - t)(f_0(t) - p\, e_h(t))\, dt\right| +$$

$$+ \left|(x - X_{j-1}) \int_0^{X_{j-1}} (f_0(t) - p\, e_h(t))\, dt\right| +$$

$$+\left|\frac{x - X_{j-1}}{h}\left[\left\{\int_{X_{j-1}}^{X_j} (X_j - t)(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*} +\right.\right.$$

$$\left.\left. + \left\{\int_0^{X_{j-1}} (X_j - X_{j-1})(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*}\right]\right| \leqq 6 \max_k \|f_0 - pe_h\|_{C^1(\bar{D}_k)}\, h\,.$$

The term I_3 tends to zero by virtue of (4.4), (4.5) but the term I_4 is left to analyse. Let us introduce $\tilde{\lambda}_h \equiv \lambda_h - \lambda_1^0(P, M; \cdot)$. It is seen that the functions $\tilde{\lambda}_h$ are bounded idependently of the meshes $\mathscr{T}_h$ and $\mathscr{T}_h^*$. The derivative $\tilde{\lambda}_h'$ satisfies

$$\tilde{\lambda}_h'(x) = \frac{1}{h}(\tilde{\lambda}_h(X_j) - \tilde{\lambda}_h(X_{j-1})) = \frac{1}{h}\left[\left\{\int_{X_{j-1}}^{X_j} (X_j - t)(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*} +\right.$$

$$\left. + \left\{\int_0^{X_{j-1}} (X_j - X_{j-1})(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*}\right] - \left\{\int_0^1 (1 - t)(f_0(t) - p\, e_h(t))\, \mathrm{d}t\right\}_{h*},$$

hence all the norms $\|\tilde{\lambda}_h\|_{C^1(\Delta j)}$ are bounded independently of the partition. Then

$$G_j = \left|\int_{\Delta j} \{b(e_h)(x)\, v_h(x)\, \tilde{\lambda}_h(x) - \tfrac{1}{2}[b(e_h)(X_{j-1})\, v_h(X_{j-1})\, \tilde{\lambda}_h(X_{j-1}) +\right.$$

$$\left. + b(e_h)(X_j)\, v_h(X_j)\, \tilde{\lambda}_h(X_j)]\}\, \mathrm{d}x\right| \leqq h^2 \max_j \|b(e_h)\, v_h \tilde{\lambda}_h\|_{C^1(\Delta_j)} \leqq K_1 h^2\,,$$

the continuous embedding $H^2(\Omega) \subset C^1(\bar{\Omega})$ being applied to the function v_h. The estimate for

$$G_j(P, M) = \left|\int_{\Delta j} \{b(e_h)(x)\, v_h(x)\, \lambda_1^0(P, M; x) -\right.$$

$$\left. - \tfrac{1}{2}[b(e_h)(X_{j-1})\, v_h(X_{j-1})\, \lambda_1^0(P, M; X_{j-1}) + b(e_h)(X_j)\, v_h(X_j)\, \lambda_1^0(P, M; X_j)]\}\right| \mathrm{d}x$$

proceeds by two different cases:

Firstly, $\bar{X}_i \notin \Delta_j$, $i = 1, \ldots, i_0$, i.e. a differentiable function is integrated and $G_j(P, M) \leqq K_2 h^2$, $K_2 > 0$, independently of i, j.

Secondly, $\bar{X}_i \in \Delta_j$. Functions $b(e_h), v_h, \lambda_1^0(P, M; \cdot)$ are bounded and $G_j(P, M) \leqq K_3 h$, $K_3 > 0$.

Finally, $|I_4| \leqq \sum_{j=1}^{N(h)} (K_1 + K_2)\, h^2 + \sum_{i=1}^{i_0} K_3 h = O(h), h \to 0_+$. We have proved that $I_2 \to 0$, $h \to 0_+$, hence $\Psi(e_h; v_h) - \Psi_h(e_h; v_h) \to 0$, $h \to 0_+$.

With regard to the properties of $\Psi(e; \cdot)$, for the strong convergence of $\{v_h^0\}$ it is sufficient to show that this sequence is minimizing (cf. the proof of Lemma 4.1).

Taking into consideration the coerciveness of Ψ_h on $H^2(\Omega)$, we see that the sequence $\{\|v_h^0\|_2\}$, $h \to 0_+$, is bounded.

Further, we shall proceed in the same way as in Lemma 4.1, part C). Let the following inequality hold for a subsequence of $\{v_h^0\}$ (also denoted by $\{v_h^0\}$):

$$\Psi(e; v_h^0) - \Psi(e; \bar{\chi}^0) \geqq \varepsilon_1 > 0\,, \quad 0 < h \leqq \delta_1\,.$$

Inserting $\bar{\chi}^0 \equiv \chi^0$ into Lemma 4.1, we derive

$$\Psi(e; v_h^0) - \Psi(e; \bar{\chi}_h^0) \geqq \varepsilon_2 > 0\,, \qquad \varepsilon_2 < \varepsilon_1\,, \quad 0 < h \leqq \delta_2 \leqq \delta_1\,;$$

$$(4.1) \Rightarrow \quad \Psi(e_h; v_h^0) - \Psi(e; \bar{\chi}_h^0) \geqq \varepsilon_3 > 0\,, \qquad \varepsilon_3 < \varepsilon_2\,, \quad 0 < h \leqq \delta_3 \leqq \delta_2\,;$$

$$(4.1) \Rightarrow \quad \Psi(e_h; v_h^0) - \Psi(e_h; \bar{\chi}_h^0) \geqq \varepsilon_4 > 0\,, \qquad \varepsilon_4 < \varepsilon_3\,, \quad 0 < h \leqq \delta_4 \leqq \delta_3\,;$$

$$(4.3) \Rightarrow \quad \Psi_h(e_h; v_h^0) - \Psi(e_h; \bar{\chi}_h^0) \geqq \varepsilon_5 > 0\,, \qquad \varepsilon_5 < \varepsilon_4\,, \quad 0 < h \leqq \delta_5 \leqq \delta_4\,;$$

$$(4.3) \Rightarrow \quad \Psi_h(e_h; v_h^0) - \Psi_h(e_h; \bar{\chi}_h^0) \geqq \varepsilon_6 > 0\,, \qquad \varepsilon_6 < \varepsilon_5\,, \quad 0 < h \leqq \delta_6 \leqq \delta_5\,.$$

Since v_h^0 is the point of minimum of Ψ_h, we arrive at a contradiction. Thus $\{v_h^0\}$ is a minimizing sequence for $h \to 0_+$. Hence

$$\|v_h^0 - \bar{\chi}^0\|_2 \to 0\,, \quad h \to 0_+\,.$$

Q.E.D.

4.2. **Approximations of the Optimization Problem**

We know (Theorem 2.3, Remark) that $\bar{\chi}^0(e) \in H^4(\Omega)$. It is logical to use the functional $\mathscr{J}_{1h}(e_h) = \|(v_h^0)''\|_2^2$ as a numerical approximation of $\mathscr{J}_1(e)$, v_h^0 solving (4.2), but since generally the norm $\|(v_h^0)''\|_2$ does not exist, it would be necessary to use polynomials of higher order for the approximation of the space V. This difficulty can be avoided by introducing the cost function $j_3(e, u) = |u|_2^2$, because $|u|_2$ and $\|u\|_2$ are equivalent norms in the space V. Then our optimization problem can be transformed from (2.13) to the following problem:

find a control $\tilde{e}_3 \in U_{\mathrm{ad}}$ such that

$$(4.6) \qquad \mathscr{J}_3(\tilde{e}_3) = \min_{e \in U_{\mathrm{ad}}} \int_\Omega e^{-6}\,(\lambda_1^0(e) + \bar{\chi}^0(e))^2\,\mathrm{d}x\,.$$

Since (4.6) is similar to (2.14), we shall study only the case of $\mathscr{J}_2$.

Definition 4.1. *Let the approximate optimal design problem* $\mathscr{P}_{h2}^h$ *be defined in this way:*

find $e_h^0 \in U_{\mathrm{ad}}^h$ *such that*

$$\mathscr{J}_{2h}(e_h^0) = j_{2h}(e_h^0, v_h^0(e_h^0)) = \min_{e_h \in U_{\mathrm{ad}}{}^h} \mathscr{J}_{2h}(e_h)\,,$$

where $v_h^0(e_h^0) \in V_h$ *solves* (4.2) *with a parametric function* e_h^0 *and*

$$j_{2h}(e_h, v_h) = \left\{\int_0^1 e_h^{-4}(\lambda_h(e_h) + v_h)^2\,\mathrm{d}x\right\}_h\,, \quad v_h \in V_h\,.$$

Lemma 4.4. *The problem $\mathscr{P}^h_{h2}$ has at least one solution for any positive h.*

Proof. Let $\{e_h^n\}_{n=1}^{\infty} \subset U_{ad}^h$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} \mathscr{J}_{2h}(e_h^n) = \inf_{e_h \in U_{ad}{}^h} \mathscr{J}_{2h}(e_h).$$

It is seen that the sequence $\{v_h^0(e_h^n)\}_{n=1}^{\infty}$ is bounded in $H^2(\Omega)$ since $\lambda_h(e_h^n)$ is bounded. Hence there exists a weakly convergent subsequence $v_h^0(e_h^m) \rightharpoonup \bar{w}_h \in V_h$, $m \to \infty$. The linear dimension of V_h is finite and therefore $v_h^0(e_h^m) \to \bar{w}_h$ in $H^2(\Omega)$, $m \to \infty$. By virtue of compactness of U_{ad}^h in $C(\bar{\Omega})$ there exists a uniformly convergent subsequence $\{e_h^k\}_{k=1}^{\infty} \subset \{e_h^m\}_{m=1}^{\infty}$ with a limit $e_h^0 \in U_{ad}^h$. Combining this convergence and boundedness of $v_h^0(e_h^k)$ in $H^2(\Omega)$, we have $\lambda_h(e_h^k) \to \lambda_h(e_h^0)$ in $L_\infty(\Omega)$ for $k \to \infty$. Hence

$$\Psi_h(e_h^0; \bar{w}_h) - \Psi_h(e_h^k; \bar{w}_h) \to 0, \quad k \to \infty, \quad \bar{w}_h \in V_h. \tag{4.7}$$

Taking the strong convergence $\|v_h^0(e_h^k) - \bar{w}_h\|_2 \to 0$, $k \to \infty$, we obtain

$$\Psi_h(e_h^k; \bar{w}_h) - \Psi_h(e_h^k; v_h^0(e_h^k)) \to 0, \quad k \to \infty. \tag{4.8}$$

By means of (4.7), (4.8) we derive

$$\Psi_h(e_h^0; \bar{w}_h) - \Psi_h(e_h^k; v_h^0(e_h^k)) \to 0, \quad k \to \infty.$$

For any $v_h \in V_h$ we have $\Psi_h(e_h^k; v_h^0(e_h^k)) \leqq \Psi_h(e_h^k; v_h)$. Passing to the limit with $k \to \infty$, we conclude that

$$\Psi_h(e_h^0; \bar{w}_h) \leqq \Psi_h(e_h^0; v_h) \quad \forall v_h \in V_h.$$

The uniqueness of the solution $\bar{w}_h \in V_h$ is a consequence of the properties of Ψ_h. According to Lemma 4.2 we may write $\bar{w}_h \equiv v_h^0(e_h^0)$.

Inserting the term $j_{2h}(e_h^k, v_h^0(e_h^0))$, we arrive at the convergence

$$\mathscr{J}_{2h}(e_h^0) - \mathscr{J}_{2h}(e_h^k) \to 0, \quad k \to \infty.$$

Using the following equalities, we prove the assertion of the lemma:

$$\inf_{e_h \in U_{ad}{}^h} \mathscr{J}_{2h}(e_h) = \lim_{k\to\infty} \mathscr{J}_{2h}(e_h^k) = \mathscr{J}_{2h}(e_h^0)$$

Q.E.D.

Lemma 4.5. *Let functions $e_h \in U_{ad}^h$ converge to a function $e \in U_{ad}$ uniformly on the interval $\bar{\Omega}$ for $h \to 0_+$. Then*

$$\lim_{h\to 0_+} \mathscr{J}_{2h}(e_h) = \mathscr{J}_2(e).$$

Proof. Denoting $I = \int_\Omega e_h^{-4}(\lambda_h(e_h) + v_h^0(e_h))^2 \, dx$, we estimate the expression

$$k(h) = |I - \mathscr{J}_{2h}(e_h)| = \left| \sum_{j=1}^{N(h)} \int_{\Delta j} \{e_h^{-4}(\lambda_h(e_h) + v_h^0(e_h))^2 - \tfrac{1}{2}[e_h^{-4}(X_{j-1}) (\lambda_h(e_h)(X_{j-1}) + \right.$$

$$\left. + v_h^0(e_h)(X_{j-1}))^2 + e_h^{-4}(X_j)(\lambda_h(e_h)(X_j) + v_h^0(e_h)(X_j))^2]\} \, dx \right| .$$

Two cases are possible:

(1) $\bar{X}_i \notin \Delta_j$, $i = 1, \ldots, i_0$. The function λ_h is differentiable and the functions e_h^{-4}, λ_h, v_h^0 are bounded in $C^1(\Delta_j)$ independently of j and h (for the boundedness of v_h^0 we apply Lemma 4.3 and the continuous embedding $H^2(\Omega) \subset C^1(\bar{\Omega})$). Hence the integral over Δ_j has the rate of convergence $O(h^2)$, $h \to 0_+$.

(2) $\bar{X}_i \in \Delta_j$, $i \in \{1, \ldots, i_0\}$. In this case the rate of convergence is only $O(h)$, but the number of such intervals is at most i_0.
Combining (1) and (2), we conclude that $k(h)$ tends to zero for $h \to 0_+$.

An estimate of $\mathscr{J}_2 - I$ remains. Combining the definition of λ_1^0 and some parts of the proof of Lemma 4.3, we arrive at the uniform convergence $\lambda_h(e_h) \to \lambda_1^0(e)$ on $\bar{\Omega}$. From this and from the convergence of v_h^0, e_h^0 we obtain that also $\mathscr{J}_2 - I$ tends to zero for $h \to 0_+$.

Q.E.D.

Theorem 4.1. *Let $\{e_h^0\}$, $h \to 0_+$, be a sequence of solutions of the approximate problem $\mathscr{P}_{h2}^h$. Then there exists a subsequence $\{e_{\bar{h}}^0\}$ such that for $\bar{h} \to 0_+$, $e_{\bar{h}}^0 \to e^0$ in $C(\bar{\Omega})$, $v_{\bar{h}}^0(e_{\bar{h}}^0) \to \bar{\chi}^0(e^0)$ in $H^2(\Omega)$, where $v_h^0(e_h^0)$ is a solution of (4.2), $e^0 \in U_{ad}$ is a solution of (2.14) and $\bar{\chi}^0(e^0) \in V$ is the corresponding solution of (2.12).*

Proof. This proof is based on the same idea as that of Theorem 3.1.

Let us take $\eta \in U_{ad}$. Then there exists a sequence $\{\eta_h\}$, $\eta_h \in U_{ad}^h$, such that $\eta_h \to \eta$ in $C(\bar{\Omega})$ for $h \to 0$ (see [2]). Let us denote by $u_h(\eta_h)$ the solution of (4.2), where e_h is replaced by η_h.

The set U_{ad} is compact, hence there exists a uniformly convergent subsequence $\{e_{\bar{h}}^0\} \subset \{e_h^0\}$, $e_{\bar{h}}^0 \to e^0 \in U_{ad}$ in $C(\bar{\Omega})$ for $\bar{h} \to 0_+$. From the definition of the problem $\mathscr{P}_{h2}^h$ we conclude that $\mathscr{J}_{2h}(e_{\bar{h}}^0) \leqq \mathscr{J}_{2h}(\eta_{\bar{h}})$. Passing to the limit with $\bar{h} \to 0_+$ and applying Lemma 4.5, we derive $\mathscr{J}_2(e^0) \leqq \mathscr{J}_2(\eta)$. Consequently, e^0 is a solution of (2.14). The remaining part of the assertion is essentially Lemma 4.3.

Q.E.D.

Remark. The preceding lemmas and the theorem are easily applicable to the functional $\mathscr{J}_3$ and to the problem (4.6).

5.1. Example

The previous theory is illustrated by the following example: one concentrated force $P = -20$ acts on the beam at the point $X = 0{\cdot}5$, $E = 10\,000$, $a_0 = 100$, $p = 10$, $e_{\min} = 0{\cdot}05$, $e_{\max} = 0{\cdot}2$, the Lipschitz constant $0{\cdot}45$, $\int_0^1 e(x)\,\mathrm{d}x = 0{\cdot}1$ (see Section 1.1). The cost function is $j_2(e, u) = \int_0^1 e^2(u'')^2\,\mathrm{d}x$.

This problem was solved by means of numerical approximations of the primal approach (Section 3). From among the many methods of nonlinear programming, Rozen's algorithm of gradient projection was chosen (in detail see [1]). The gradient $\nabla\mathscr{J}_2(e_h)$ was evaluated by means of the adjoint state problem (see e.g. [5]).

Practical results were obtained for a partition of the interval $\langle 0, 1\rangle$ into 24 subintervals. Some of these results are represented by Fig. 1 which requires the following commentary:

The constant function e_h^0 is the first approximation, i.e. the initial choice of the thickness of the beam, $\mathscr{J}_2(e_h^0) = 2{\cdot}526$. For the second approximation e_h^1 we have $\mathscr{J}_2(e_h^1) = 0{\cdot}683$. The best result was shown in the tenth approximation, $\mathscr{J}_2(e_h^9) = 0{\cdot}392$.

All calculations were done by an HP 9825A calculator, one iteration (i.e. the step from e_h^n to e_h^{n+1}) took about 5 minutes.

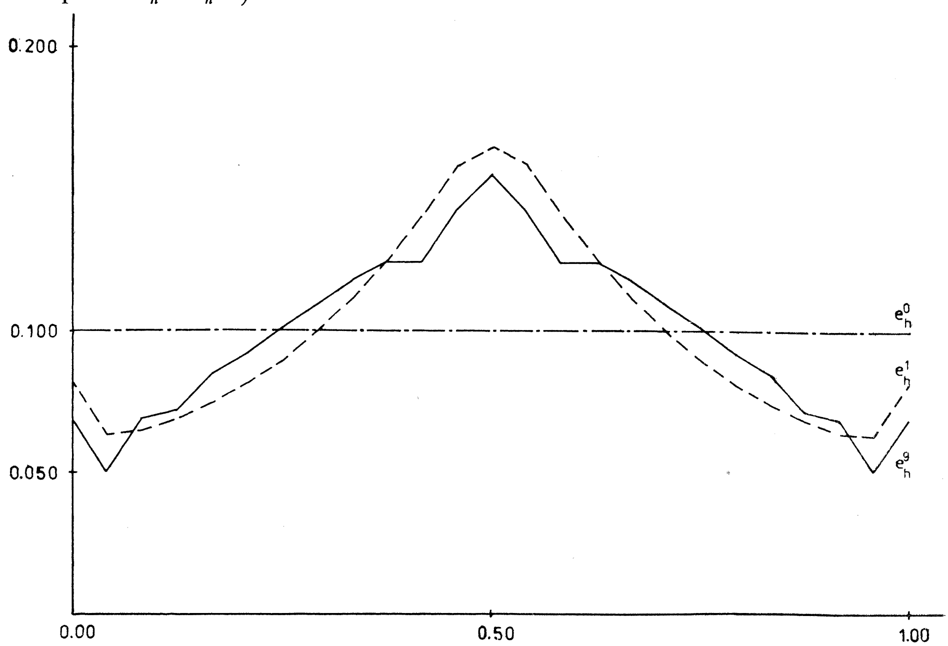

Fig. 1.

5.2. Comments

The dual variational formulation of the state problem does not simplify our optimization problems in contrast to the problems without an elastic foundation from the paper [6]. In addition, the cost function j_1 makes certain difficulties. On the other hand, we may expect that approximative results for j_2 will be better than those obtained by the primal approach. However, the numerical solution of $\mathscr{P}^h_{h2}$ appears somewhat more difficult than the solution of $\mathscr{P}_{h2}$.

The constant a_0 can be replaced by a positive function $a_0(x)$, $K_1 \leqq a_0(x) \leqq K_2$ $\forall x \in \bar{\Omega}, K_1, K_2 > 0$ being constants. This is a more general description of the reaction force of the elastic basis.

We may omit the assumption (II) in Section 3, then the value of $A(e_h; v_h, w_h)$ must be computed accurately. Simplifying the theory, we arrive at a little more complicated computer program.

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Souhrn

OPTIMÁLNÍ NÁVRH PRUŽNÉHO NOSNÍKU NA PRUŽNÉM PODKLADĚ

JAN CHLEBOUN

Uvažuje se pružný prostě podepřený nosník daného objemu a konstantní šířky i délky uložený na pružném podkladu. Za návrhovou proměnnou se bere funkce průběhu tloušťky nosníku, její derivace do 1. řádu jsou omezeny shora i zdola. Zatížení sestává z osamělých sil a momentů,

vlastní tíže a tzv. spojitého zatížení. Cenový funkcionál je buď integrál druhé mocniny průhybové čáry a její první a druhé derivace, nebo integrál druhé mocniny napětí v krajních vláknech nosníku.

Dokazuje se existence řešení optimalizačních problémů při primární i duální formulaci stavové úlohy. Pro obě formulace se zavádějí aproximační úlohy a je dokázána konvergence jejich řešení k řešení spojitého problému. Teoretické závěry jsou doplněny ilustračním příkladem.

Резюме

ОПТИМИЗАЦИЯ ФОРМЫ УПРУГОЙ БАЛКИ НА УПРУГОМ ОСНОВАНИИ

JAN CHLEBOUN

Рассматривается просто поддерживаемая балка данного объема и постоянной ширины и длины на упругом основании.

Доказано существование решений проблем оптимизации в первоначальной (перемещение) и дуальной (напряжение) формулировках. Для обеих вводятся приближённые проблемы и доказывается сходимость их решений к решению непрерывной проблемы. Теоретические исследования дополнены примером.

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