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CONSTRUCTIONS OF INTERPOLATION CURVES FROM GIVEN SUPPORTING ELEMENTS (I)

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This paper deals with the constructions of interpolation curves which pass through given supporting points (nodes) and touch supporting tangent vectors given at only some of these points or, as the case may be, at all these points. The mathematical kernel of these constructions is based on Lienhard's interpolation method.

Keywords. Interpolation, curves.

1. ADJUSTMENT OF LIENHARD'S INTERPOLATION METHOD

Our approach is based on the papers [1], [2]. The original Lienhard method will be modified so that instead of polynomials of the fifth degree we shall use polynomials of the third degree which are "more stable" from the viewpoint of the behaviour of the interpolation curves. In the text which follows the method applied is briefly referred to as method I.

Let $n \geqq 3$ be an integer. In the space $\mathbf{R}^m$ $(m > 1$ integer$)$ let n different points $P_i = x_j^{(i)}$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ be given. The symbol $x_j^{(i)}$ denotes also the corresponding ordered m-tuple of coordinates, or rather the vector which has these coordinates. Thus, the elements of the set $\mathbf{R}^m$ are either points or vectors, according to which of the notions corresponds more to our conception in the given context. As a rule, we use the notion of a point in situations when location in the space $\mathbf{R}^m$ is discussed while the notion of a vector indicates that we are interested in the direction. Also, bold types will be sometimes used to denote vectors.

We shall look for polynomials in the real variable t (of degree at most K, not determined more precisely at the moment)

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{K} a_{jk}^{(i)} t^k \quad (i = 1, \ldots, n - 1) \tag{1.1}$$

such that

$$P_{x_j}^{(i)}(-1) = x_j^{(i)}\,, \quad P_{x_j}^{(i)}(1) = x_j^{(i+1)}\,, \tag{1.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{x_j}^{(i)}(1) = \frac{\mathrm{d}}{\mathrm{d}t} P_{x_j}^{(i+1)}(-1)\,. \tag{1.3}$$

Conditions (1.2) guarantee that the interpolation arc parametrized with the aid of the functions $P_{x_j}^{(i)}(t)$ $(j = 1, \ldots, m)$ passes through the points P_i, P_{i+1} while conditions (1.3) guarantee fluent transition from arc to arc. To satisfy conditions (1.3) we have to know the values of the functions $\mathrm{d}P_{x_j}^{(i)}(t)/\mathrm{d}t$ at the points P_i, P_{i+1}:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{x_j}^{(i)}(-1) = \mathrm{D}x_j^{(i)}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t} P_{x_j}^{(i)}(1) = \mathrm{D}x_j^{(i+1)}\,; \tag{1.4}$$

$\mathrm{D}x_j^{(i)}, \mathrm{D}x_j^{(i+1)}$ is the notation used for these values. The manner of determining these values will be discussed later. By (1.2), (1.3), four determining conditions are given for every polynomial (1.1). With their aid each of the polynomials is uniquely determined as a polynomial of degree at most $K = 3$:

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{3} a_{jk}^{(i)} t^k\,. \tag{1.5}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{x_j}^{(i)}(t) = \sum_{k=1}^{3} k a_{jk}^{(i)} t^{k-1}\,. \tag{1.6}$$

If we substitute the values $t = -1, 1$ into (1.5), (1.6), we obtain (taking (1.2), (1.4) into account) the following system of four linear equations for the four unknown coefficiencts $a_{jk}^{(i)}$ of polynomial (1.5):

$$\begin{aligned} \sum_{k=0}^{3} (-1)^k a_{jk}^{(i)} &= x_j^{(i)}\,, \\ \sum_{k=1}^{3} (-1)^{k-1} k a_{jk}^{(i)} &= \mathrm{D}x_j^{(i)}\,, \\ \sum_{k=0}^{3} a_{jk}^{(i)} &= x_j^{(i+1)}\,, \\ \sum_{k=1}^{3} k a_{jk}^{(i)} &= \mathrm{D}x_j^{(i+1)}\,. \end{aligned} \tag{1.7}$$

We introduce the matrices

$$\boldsymbol{A}_{ij} = (a_{j0}^{(i)}, a_{j1}^{(i)}, a_{j2}^{(i)}, a_{j3}^{(i)})\,, \tag{1.8}$$

$$\boldsymbol{X}_{ij} = (x_j^{(i)}, \mathrm{D}x_j^{(i)}, x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)})\,. \tag{1.9}$$

The matrix of coefficients of system (1.7), which is necessarily regular in view of the uniqueness of the determination of the desired polynomials, is denoted by $\boldsymbol{A}$. Then the solution of system (1.7) is represented in matrix notation by the equality

$$\mathbf{A}_{ij}^{\mathrm{T}} = \mathbf{A}^{-1} \circ \mathbf{X}_{ij}^{\mathrm{T}}, \tag{1.10}$$

where the superscript T denotes the transposed matrices to matrices (1.8), (1.9) while $\mathbf{A}^{-1}$ denotes the inverse matrix to matrix $\mathbf{A}$.

The values of the first derivative at the points P_i, P_{i+1} (see (1.4)) are determined as follows. According to Fig. 1

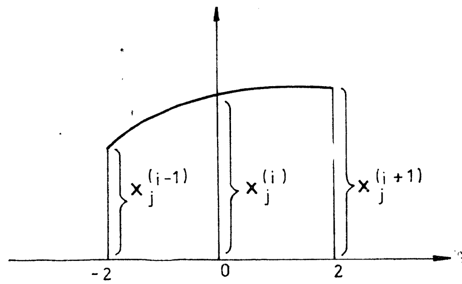


Fig.1

the points $(2h, x_j^{(i+h)})$ $(-1 \leqq h \leqq 1$, h integer) uniquely determine a polynomial of at most second degree

$$R_{x_j}^{(i)}(t) = \sum_{k=0}^{2} b_{jk}^{(i)} t^k. \tag{1.11}$$

With its aid we put

$$\mathrm{D}x_j^{(i)} = \frac{\mathrm{d}}{\mathrm{d}t} R_{x_j}^{(i)}(0) = b_{j1}^{(i)}. \tag{1.12}$$

The originality of Lienhard's interpolation method consists precisely in this manner of determining the values (1.4), where the "missing" values of the first derivatives are obtained from the auxiliary polynomials (1.11). Since every coefficient of polynomial (1.11) is a certain linear combination of the values $x_j^{(i-1)}$, $x_j^{(i)}$, $x_j^{(i+1)}$, the same is true for the derivative $\mathrm{D}x_j^{(i)}$. Therefore there exists a matrix $\mathbf{B}$ of type (1, 3) such that we have

$$\mathrm{D}x_j^{(i)} = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}) \circ \mathbf{B}^{\mathrm{T}}. \tag{1.13}$$

Here we have identified the type (1, 1) matrix $(\mathrm{D}x_j^{(i)})$ with the element $\mathrm{D}x_j^{(i)}$; this will be done always in the sequel. Then we have

$$(x_j^{(i)}, \mathrm{D}x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & \\ 1 & \mathbf{B}^{\mathrm{T}} \\ 0 & \\ 0 & 0 \end{bmatrix}. \tag{1.14}$$

Analogously we obtain

$$(1.15)\qquad \left(x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}\right) = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}\right) \circ \begin{bmatrix} 0 & 0 \\ 0 & \\ 1 & \boldsymbol{B}^{\mathrm{T}} \\ 0 & \end{bmatrix};$$

in this case the number i was replaced by the number $i + 1$ in Fig. 1. By (1.14), (1.15), the matrix (1.9) can be represented in the form

$$(1.16)\qquad \boldsymbol{X}_{ij} = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}\right) \circ \begin{bmatrix} 0 & & 0 & 0 \\ 1 & \boldsymbol{B}^{\mathrm{T}} & 0 & \\ 0 & & 1 & \boldsymbol{B}^{\mathrm{T}} \\ 0 & 0 & 0 & \end{bmatrix}.$$

After substituting (1.16) into (1.10) we have

$$(1.17)\qquad \boldsymbol{A}_{ij}^{\mathrm{T}} = \boldsymbol{C} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix},$$

where

$$(1.18)\qquad \boldsymbol{C} = \boldsymbol{A}^{-1} \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ & \boldsymbol{B} & & 0 \\ 0 & 0 & 1 & 0 \\ 0 & & \boldsymbol{B} & \end{bmatrix}.$$

A simple computation yields

$$(1.19)\qquad \boldsymbol{A}^{-1} = \tfrac{1}{4} \begin{bmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix},$$

$$(1.20)\qquad \boldsymbol{B} = \tfrac{1}{4}(-1, 0, 1).$$

With the aid of (1.18), (1.19), (1.20) it is then possible to represent (1.17) in the form

$$(1.21)\qquad 16\,\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

From the given nodes $P_1, \ldots, P_n$ we form groups consisting of four points each (see (1.21)) as follows:

1-st group:	P_2, P_1, P_2, P_3 ,
2-nd group:	P_1, P_2, P_3, P_4 ,
3-rd group:	P_2, P_3, P_4, P_5 ,
....	
$(n-1)$-st group:	$P_{n-2}, P_{n-1}, P_n, P_{n-1}$.

If formula (1.21) is applied to the first group, we obtain polynomials $P_{x_j}^{(1)}(t)$ $(j = 1, \ldots, m)$ which parametrize the arc P_1P_2. Similarly, if formula (1.21) is applied to the second group of points, we obtain polynomials $P_{x_j}^{(2)}(t)$ $(j = 1, \ldots, m)$ which parametrize the arc P_2P_3. Finally, if formula (1.21) is applied to the $(n-1)$-st group of points, we obtain polynomials $P_{x_j}^{(n-1)}(t)$ $(j = 1, \ldots, m)$ which parametrize the arc $P_{n-1}P_n$. These arcs constitute the desired unclosed interpolation curve $P_1P_2 \ldots P_{n-1}P_n$. For instance, for $n = 3$ we form the following groups consisting of four points each:

1-st group:	P_2, P_1, P_2, P_3 ,
2-nd group:	P_1, P_2, P_3, P_2 .

In case that we are looking for a closed interpolation curve $P_1P_2 \ldots P_nP_1$ we group the nodes in the following manner:

1-st group:	P_n, P_1, P_2, P_3 ,
2-nd group:	P_1, P_2, P_3, P_4 ,
3-rd group:	P_2, P_3, P_4, P_5 ,
....	
n-th group:	P_{n-1}, P_n, P_1, P_2 .

For instance, for $n = 3$ we have:

1-st group:	P_3, P_1, P_2, P_3 ,
2-nd group:	P_1, P_2, P_3, P_1 ,
3-rd group:	P_2, P_3, P_1, P_2 .

Example 1. In the plane $\mathbf{R}^2$ we consider the points $P_1 = (0, 0)$, $P_2 = (2, 3)$, $P_3 = (15, -6)$, $P_4 = (2, -10)$, $P_5 = (10, 5)$. For the individual arcs of the unclosed planar interpolation curve $P_1P_2P_3P_4P_5$ we have, by (1.21), the following parametric equations:

$$(2.1) \qquad P_1P_2 \ \ldots \quad \begin{aligned} P_{x_1}^{(1)}(t) &= 0{\cdot}0625 + 0{\cdot}5625t + 0{\cdot}9375t^2 + 0{\cdot}4375t^3 , \\ P_{x_2}^{(1)}(t) &= 1{\cdot}875 + 2{\cdot}625t - 0{\cdot}375t^2 - 1{\cdot}125t^3 , \end{aligned}$$

$$
P_2P_3 \;\ldots\quad \begin{aligned} P^{(2)}_{x_1}(t) &= 9{\cdot}4375 + 8{\cdot}8125t - 0{\cdot}9375t^2 - 2{\cdot}3125t^3\,, \\ P^{(2)}_{x_2}(t) &= -1{\cdot}0625 - 5{\cdot}5625t - 0{\cdot}4375t^2 + 1{\cdot}0625t^3\,, \end{aligned}
$$

$$
P_3P_4 \;\ldots\quad \begin{aligned} P^{(3)}_{x_1}(t) &= 8{\cdot}8125 - 9{\cdot}4375t - 0{\cdot}3125t^2 + 2{\cdot}9375t^3\,, \\ P^{(3)}_{x_2}(t) &= -9{\cdot}5 - 2{\cdot}875t + 1{\cdot}5t^2 + 0{\cdot}875t^3\,, \end{aligned}
$$

$$
P_4P_5 \;\ldots\quad \begin{aligned} P^{(4)}_{x_1}(t) &= 5{\cdot}6875 + 6{\cdot}3125t + 0{\cdot}3125t^2 - 2{\cdot}3125t^3\,, \\ P^{(4)}_{x_2}(t) &= -1{\cdot}8125 + 10{\cdot}5625t - 0{\cdot}6875t^2 - 3{\cdot}0625t^3\,. \end{aligned}
$$

The interpolation curve is shown in Fig. 2.

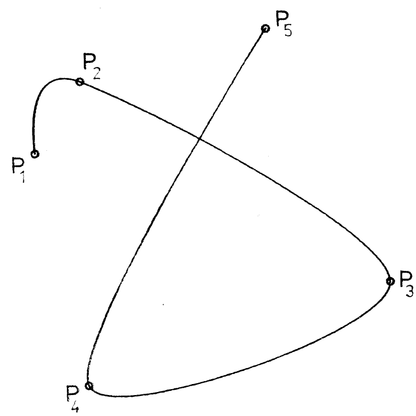


Fig .2

3. MODIFICATION OF THE ADJUSTED LIENHARD METHOD

When constructing interpolation polynomials in Section 1 we did not consider the mutual distances between the nodes. "Better" behaviour of the resulting interpolation curve may be expected if these distances are taken into account. In his paper [1] Lienhard mentions the possibility of a modification of his method which would take into account the mutual distances of the nodes. In [2] this modification is elaborated in detail for interpolation polynomials of the fifth and seventh degrees. Here we shall work out this modification for the adjusted Lienhard method from Section 1, i.e., for interpolation polynomials of the third degree. In the text which follows we briefly speak of method II.

We shall proceed in the same way as in Section 1, but with the difference that the values $-2, 0, 2$ (see Fig. 1) of the variable t are replaced by the values $-2q_{i,-1}/q_{i,0}$, 0, $2q_{i,1}/q_{i,0}$, where we denote $q_{i,-1} = |P_{i-1}P_i|$, $q_{i,1} = |P_iP_{i+1}|$ (the distances of the respective points), $q_{i,0} = (q_{i,-1} + q_{i,1})/2$. Exploiting the corresponding polynomial (1.11) and condition (1.12) we then have (cf. (1.13))

$$\mathrm{D}x_j^{(i)} = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}\right) \circ \mathbf{B}_i^{\mathrm{T}}\,, \tag{3.1}$$

where

$$\mathbf{B}_i = \tfrac{1}{4}\left(-r_i, r_i - r_i^{-1}, r_i^{-1}\right), \quad r_i = q_{i,1}/q_{i,-1}\,. \tag{3.2}$$

Then (cf. (1.14))

$$\left(x_j^{(i)}, \mathrm{D}x_j^{(i)}\right) = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}\right) \circ \begin{bmatrix} 0 & \\ 1 & \mathbf{B}_i^{\mathrm{T}} \\ 0 & \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

and similarly (cf. (1.15))

$$\left(x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}\right) = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}\right) \circ \begin{bmatrix} 0 & 0 \\ 0 & \\ 1 & \mathbf{B}_{i+1}^{\mathrm{T}} \\ 0 & \end{bmatrix}. \tag{3.4}$$

By (3.3), (3.4) it is possible to represent matrix (1.9) in the form

$$\mathbf{X}_{ij} = \left(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}\right) \circ \begin{bmatrix} 0 & & 0 & 0 \\ 1 & \mathbf{B}_i^{\mathrm{T}} & 0 & \\ 0 & & 1 & \mathbf{B}_{i+1}^{\mathrm{T}} \\ 0 & 0 & 0 & \end{bmatrix}. \tag{35}$$

After substituting (3.5) into (1.10) we have

$$\mathbf{A}_{ij}^{\mathrm{T}} = \mathbf{C}_i \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}, \tag{3.6}$$

where

$$\mathbf{C}_i = \mathbf{A}^{-1} \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ & \mathbf{B}_i & & 0 \\ 0 & 0 & 1 & 0 \\ 0 & & \mathbf{B}_{i+1} & \end{bmatrix}. \tag{3.7}$$

To sum up: If in the unmodified case, i.e., when the mutual distances of the nodes are not considered, the matrix $\mathbf{B}$ (cf. (1.20)) in formula (1.18) is constant for all interpolation arcs P_iP_{i+1}, then in the modified case, i.e., when the mutual distances of the nodes are considered, this matrix changes from arc to arc (and passes into matrix (3.2)). Simultaneously the matrix $\mathbf{C}$ from formula (1.18), which passes into matrix (3.7), also changes.

With the aid of (3.7), (1.19), (3.2) it is then possible to represent (1.17) in the form

$$(3.8) \qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} =$$

$$= \begin{bmatrix} -r_i & 8 + r_i - r_i^{-1} + r_{i+1} & 8 + r_i^{-1} - r_{i+1} + r_{i+1}^{-1} & -r_{i+1}^{-1} \\ r_i & -12 - r_i + r_i^{-1} + r_{i+1} & 12 - r_i^{-1} - r_{i+1} + r_{i+1}^{-1} & -r_{i+1}^{-1} \\ r_i & - r_i + r_i^{-1} - r_{i+1} & - r_i^{-1} + r_{i+1} - r_{i+1}^{-1} & r_{i+1}^{-1} \\ -r_i & 4 + r_i - r_i^{-1} - r_{i+1} & -4 + r_i^{-1} + r_{i+1} - r_{i+1}^{-1} & r_{i+1}^{-1} \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Example 2. Let us consider the same nodes P_1, P_2, P_3, P_4, P_5 as in Example 1. For the individual arcs of the unclosed planar interpolation curve $P_1P_2P_3P_4P_5$ we then have, by (3.8), the following parametric equations:

$$P_1P_2 \ \ldots \quad \begin{aligned} P_{x_1}^{(1)}(t) &= 0{\cdot}26656 + 0{\cdot}76656t + 0{\cdot}73344t^2 + 0{\cdot}23344t^3, \\ P_{x_2}^{(1)}(t) &= 0{\cdot}80603 + 1{\cdot}55603t + 0{\cdot}69397t^2 - 0{\cdot}05603t^3, \end{aligned}$$

$$P_2P_3 \ \ldots \quad \begin{aligned} P_{x_1}^{(2)}(t) &= 9{\cdot}47902 + 9{\cdot}26213t - 0{\cdot}97902t^2 - 2{\cdot}76213t^3, \\ P_{x_2}^{(2)}(t) &= -0{\cdot}03153 - 6{\cdot}66947t - 1{\cdot}46847t^2 + 2{\cdot}16947t^3, \end{aligned}$$

$$P_3P_4 \ \ldots \quad \begin{aligned} P_{x_1}^{(3)}(t) &= 8{\cdot}86989 - 8{\cdot}88896t - 0{\cdot}36989t^2 + 2{\cdot}38896t^3, \\ P_{x_2}^{(3)}(t) &= -9{\cdot}21212 - 2{\cdot}66312t + 1{\cdot}21212t^2 + 0{\cdot}66312t^3, \end{aligned}$$

$$P_4P_5 \ \ldots \quad \begin{aligned} P_{x_1}^{(4)}(t) &= 5{\cdot}38453 + 6{\cdot}61547t + 0{\cdot}61547t^2 - 2{\cdot}61547t^3, \\ P_{x_2}^{(4)}(t) &= -2{\cdot}06238 + 10{\cdot}81238t - 0{\cdot}43762t^2 - 3{\cdot}31238t^3. \end{aligned}$$

The interpolation curve is shown in Fig. 3.

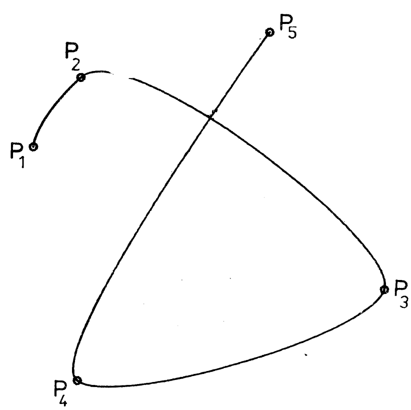


Fig. 3

If we compare the behaviour of the interpolation curves in Figs. 2 and 3, we see that the "smaller" curvature of the arc P_1P_2 combined with the "larger" curvature of the arc P_2P_3 in Fig. 3 is "more favourable" for the overall behaviour of the interpolation curve than the "larger" curvature of the arc P_1P_2 combined with the "smaller" curvature of the arc P_2P_3 in Fig. 2. From this point of view, the modified interpolation method is thus more advantageous in the given example.

A more convincing justification of the advantages of the modified interpolation method consists in the following facts: When constructing the auxiliary polynomials (1.11) with whose aid the "missing" values of the first derivatives are determined (see (1.12)), the points $-2, 0, 2$ are equidistantly distributed on the t axis (see Fig. 1). This is "equivalent" to the fact that mutual distances between nodes are not taken into account (as if these distances were the same). If the construction of the auxiliary polynomials is performed under the mentioned nonequidistant distribution of the points on the t axis, this is "equivalent" to the fact that the mutual distances of the nodes are taken into account. The "missing" values of the first derivatives obtained from these polynomials have an intuitively "better" chance to render "better" overall behaviour of the resulting interpolation curve.

4. COMPUTATION OF THE TANGENT VECTOR.

In this and the following sections we follow method I. Consider the nodes P_i, P_{i+1}, P_{i+2} and look for the tangent vector at the point P_{i+1} with respect to the interplation arc P_iP_{i+1}. Since we require that condition (1.3) hold this tangent vector is equal to the tangent vector at the point P_{i+1} with respect to the interpolation arc $P_{i+1}P_{i+2}$. By (1.21) the relation

$$16P_{x_j}^{(i)}(t) = (1, t, t^2, t^3) \circ 16\boldsymbol{A}_{ij}^{\mathrm{T}}$$

yields, by differentiation,

$$(4.1) \qquad 16P_{x_j}^{\prime(i)}(t) = (1, t, t^2, t^3) \circ \begin{bmatrix} 1 & -11 & 11 & -1 \\ 2 & -2 & -2 & 2 \\ -3 & 9 & -9 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

For $t = 1$ we obtain, from (4.1),

$$(4.2) \qquad 4P_{x_j}^{\prime(i)}(1) = (0, -1, 0, 1) \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} = x_j^{(i+2)} - x_j^{(i)},$$

i.e., the tangent vector at the point P_{i+1} with respect to the interpolation arc P_iP_{i+1} is collinear with the vector $\overrightarrow{P_iP_{i+2}}$, its length being four times smaller than that of the vector $\overrightarrow{P_iP_{i+2}}$.

5. CONSTRUCTION OF INTERPOLATION CURVES WITH PRESCRIBED TANGENT VECTORS (METHOD $\mathscr{S}$)

With the aid of formula (4.2) it is possible to construct an interpolation arc which passes through two adjacent nodes one of which is provided with a supporting tangent vector (see Fig. 4).

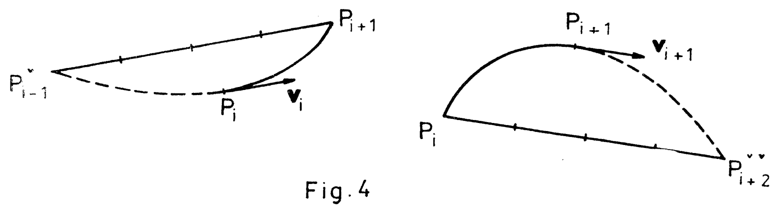


Fig. 4

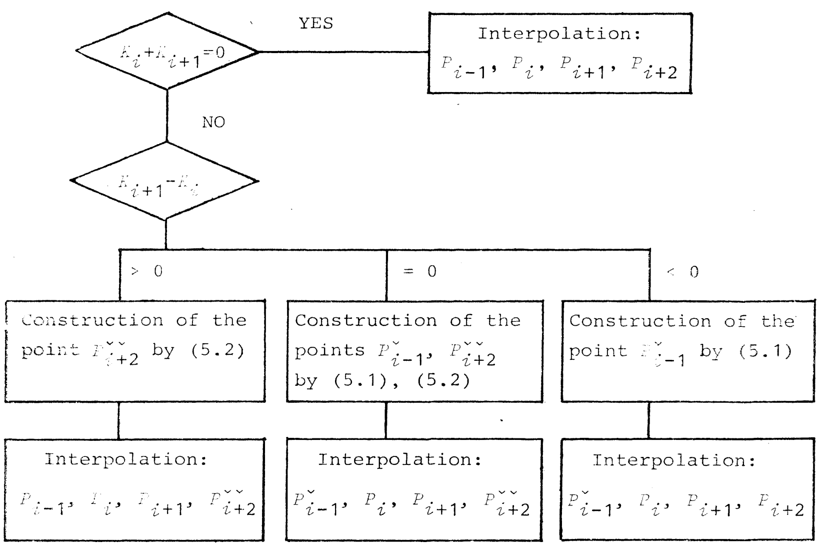
For this purpose we construct, in the first case, the auxiliary point

$$P^{\vee}_{i-1} = P_{i+1} - 4\mathbf{v}_i\,, \tag{5.1}$$

and, in the second case, the auxiliary point

$$P^{\vee\vee}_{i+2} = P_i + 4\mathbf{v}_{i+1}\,. \tag{5.2}$$

Now we assign to every node P_i with a supporting tangent vector (or without a supporting tangent vector) the number $K_i = 1$ (or $K_i = 0$, respectively). Then every interpolation arc is processed according to the following scheme:



In this scheme "Interpolation" stands for the procedure of the adjusted Lienhard method (method I), as applied to the respective quadruples of points (see (1.21)).

Example 3. Let us choose the same nodes $P_1 = (0, 0)$, $P_2 = (2, 3)$, $P_3 = (15, -6)$, $P_4 = (2, -10)$, $P_5 = (10, 5)$ as in Example 1. At the points P_3, P_4 let us consider the supporting tangent vectors $\mathbf{v}_3 = (1, -2)$, $\mathbf{v}_4 = (0, 3)$. In accordance with the above scheme we construct a planar unclosed interpolation curve $P_1P_2P_3P_4P_5$ which takes account of the given supporting tangent vectors.

We start with the interpolation arc P_1P_2. Since $K_1 + K_2 = 0 + 0 = 0$, we apply formula (1.21) to the points P_2, P_1, P_2, P_3. This leads to the following parametric equations of the arc P_1P_2 (cf. (2.1)):

$$P_1P_2 \quad \ldots \quad \begin{aligned} P^{(1)}_{x_1}(t) &= 0{\cdot}0625 + 0{\cdot}5625t + 0{\cdot}9375t^2 + 0{\cdot}4375t^3\,, \\ P^{(1)}_{x_2}(t) &= 1{\cdot}875 + 2{\cdot}625t - 0{\cdot}375t^2 - 1{\cdot}125t^3\,. \end{aligned}$$

Next we treat the arc P_2P_3. Since $K_2 + K_3 = 0 + 1 = 1 \neq 0$, $K_3 - K_2 = 1 - 0 = 1 > 0$, we construct by (5.2) the point $P_4^{\vee\vee} = P_2 + 4\mathbf{v}_3 = (2, 3) + 4(1, -2) = (6, -5)$. We apply (1.21) to the points $P_1, P_2, P_3, P_4^{\vee\vee}$, which leads to the following parametric equations of the arc P_2P_3:

$$P_2P_3 \quad \ldots \quad \begin{aligned} P^{(2)}_{x_1}(t) &= 9{\cdot}1875 + 8{\cdot}5625t - 0{\cdot}6875t^2 - 2{\cdot}0625t^3\,, \\ P^{(2)}_{x_2}(t) &= -1{\cdot}375 - 5{\cdot}875t - 0{\cdot}125t^2 + 1{\cdot}375t^3\,. \end{aligned}$$

Then we treat the arc P_3P_4. Since $K_3 + K_4 = 1 + 1 = 2 \neq 0, K_4 - K_3 = 1 - 1 = 0$, we construct by (5.1) the point $P_2^{\vee} = P_4 - 4\mathbf{v}_3 = (2, -10) - 4(1, -2) = (-2, -2)$ and by (5.2) the point $P_5^{\vee\vee} = P_3 + 4\mathbf{v}_4 = (15, -6) + 4(0, 3) = (15, 6)$. To the points $P_2^{\vee}, P_3, P_4, P_5^{\vee\vee}$ we apply (1.21) and obtain the following parametric equations of the arc P_3P_4:

$$P_3P_4 \quad \ldots \quad \begin{aligned} P^{(3)}_{x_1}(t) &= 8{\cdot}75 - 10t - 0{\cdot}25t^2 + 3{\cdot}5t^3\,, \\ P^{(3)}_{x_2}(t) &= -9{\cdot}25 - 3{\cdot}25t + 1{\cdot}25t^2 + 1{\cdot}25t^3\,. \end{aligned}$$

It remains to treat the arc P_4P_5. Since $K_4 + K_5 = 1 + 0 = 1$, $K_5 - K_4 = 0 - 1 = -1 < 0$, we construct by (5.1) the auxiliary point $P_3^{\vee} = P_5 - 4\mathbf{v}_4 = (10,5) - 4(0,3) = (10, -7)$. If we apply (1.21) to the quadruple of points $P_3^{\vee}, P_4, P_5, P_4$, then we obtain the following parametric equations of the arc P_4P_5:

$$P_4P_5 \quad \ldots \quad \begin{aligned} P^{(4)}_{x_1}(t) &= 6 + 6t - 2t^3\,, \\ P^{(4)}_{x_2}(t) &= -1{\cdot}75 + 10{\cdot}5t - 0{\cdot}75t^2 - 3t^3\,. \end{aligned}$$

The desired interpolation curve is shown in Fig. 5.

Example 4. In the space $\mathbf{R}^3$ consider the points $P_1 = (0, 0, 0)$, $P_2 = (10, 5, 5)$, $P_3 = (0, 10, 15)$, $P_4 = (-5, 3, 8)$. At the points P_1, P_3 let us consider the supporting tangent vectors $\mathbf{v}_1 = (4, 0, 0)$, $\mathbf{v}_3 = (-2, -2, 2)$. We construct a spatial closed interpolation curve $P_1P_2P_3P_4P_1$ which takes into account the given supporting tangent vectors. We shall proceed in accordance with the above scheme.

We start from the interpolation arc P_1P_2. Since $K_1 + K_2 = 1 + 0 = 1 \neq 0$, $K_2 - K_1 = 0 - 1 = -1 < 0$, we construct by (5.1) the auxiliary point $P_0^{\vee} = P_2 - 4\mathbf{v}_1 = (10, 5, 5) - 4(4, 0, 0) = (-6, 5, 5)$. To the points $P_0^{\vee}, P_1, P_2, P_3$ we then apply (1.21) and obtain the following parametric equations of the arc P_1P_2:

$$P_1P_2 \quad \ldots \quad \begin{aligned} P_{x_1}^{(1)}(t) &= 6 + 6{\cdot}5t - t^2 - 1{\cdot}5t^3\,, \\ P_{x_2}^{(1)}(t) &= 1{\cdot}875 + 3{\cdot}125t + 0{\cdot}625t^2 - 0{\cdot}625t^3\,, \\ P_{x_3}^{(1)}(t) &= 1{\cdot}5625 + 2{\cdot}8125t + 0{\cdot}9375t^2 - 0{\cdot}3125t^3\,. \end{aligned}$$

We continue with the arc P_2P_3. Since $K_2 + K_3 = 0 + 1 = 1 \neq 0$, $K_3 - K_2 = 1 - 0 = 1 > 0$, we construct by (5.2) the auxiliary point $P_4^{\vee\vee} = P_2 + 4\mathbf{v}_3 = (10, 5, 5) + 4(-2, -2, 2) = (2, -3, 13)$. To the points $P_1, P_2, P_3, P_4^{\vee\vee}$ we then apply (1.21) and obtain the following parametric equations of the arc P_2P_3:

$$P_2P_3 \quad \ldots \quad \begin{aligned} P_{x_1}^{(2)}(t) &= 5{\cdot}5 - 7t - 0{\cdot}5^2 + 2t^3\,, \\ P_{x_2}^{(2)}(t) &= 8{\cdot}625 + 3{\cdot}625t - 1{\cdot}125t^2 - 1{\cdot}125t^3\,, \\ P_{x_3}^{(2)}(t) &= 10{\cdot}4375 + 6{\cdot}0625t - 0{\cdot}4375t^2 - 1{\cdot}0625t^3\,. \end{aligned}$$

We continue with the arc P_3P_4. Since $K_3 + K_4 = 1 + 0 = 1 \neq 0$, $K_4 - K_3 = 0 - 1 = -1 < 0$, we construct by (5.1) the point $P_2^{\vee} = P_4 - 4\mathbf{v}_3 = (-5, 3, 8) - 4(-2, -2, 2) = (3, 11, 0)$. To the points $P_2^{\vee}, P_3, P_4, P_1$ we then apply (1.21) which leads to the following parametric equations of the arc P_3P_4:

$$P_3P_4 \quad \ldots \quad \begin{aligned} P_{x_1}^{(3)}(t) &= -3 - 3{\cdot}25t + 0{\cdot}5t^2 + 0{\cdot}75t^3\,, \\ P_{x_2}^{(3)}(t) &= 6{\cdot}625 - 4{\cdot}125t - 0{\cdot}125t^2 + 0{\cdot}625t^3\,, \\ P_{x_3}^{(3)}(t) &= 12{\cdot}9375 - 4{\cdot}8125t - 1{\cdot}4375t^2 + 1{\cdot}3125t^3\,. \end{aligned}$$

The arc P_4P_1 remains last. Since $K_4 + K_1 = 0 + 1 = 1 \neq 0$, $K_1 - K_4 = 1 - 0 = 1 > 0$, we construct by (5.2) the point $P_2^{\vee\vee} = P_4 + 4\mathbf{v}_1 = (-5, 3, 8) + 4(4, 0, 0) = (11, 3, 8)$. If we apply (1.21) to the points $P_3, P_4, P_1, P_2^{\vee\vee}$, we obtain the following parametric equations of the arc P_4P_1:

$$P_4P_1 \quad \ldots \quad \begin{aligned} P_{x_1}^{(4)}(t) &= -3{\cdot}5 + 2{\cdot}75t + t^2 - 0{\cdot}25t^3\,, \\ P_{x_2}^{(4)}(t) &= 0{\cdot}875 - 1{\cdot}625t + 0{\cdot}625t^2 + 0{\cdot}125t^3\,, \\ P_{x_3}^{(4)}(t) &= 3{\cdot}0625 - 5{\cdot}0625t + 0{\cdot}9375t^2 + 1{\cdot}0625t^3\,. \end{aligned}$$

The interpolation curve is shown in Fig. 6 in axonometric projection. For the sake of simplicity, the symbol P_i is also used here to denote the axonometric projection of a node while the symbol P_i' denotes its axonometric first projection. The similar holds for supporting vectors.

In the conclusion, we note that the spatial interpolation curve of Fig. 6 manifests the "deficiency" which consists in the fact that the osculation plane at the point P_i with respect to the arc $P_{i-1}P_i$ is generally different from the osculation plane at the same point with respect to the arc P_iP_{i+1} (provided these planes exist). Therefore

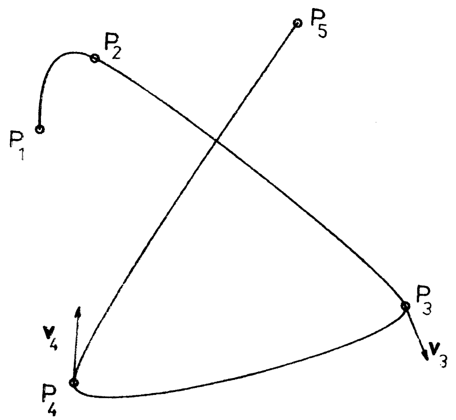


Fig.5

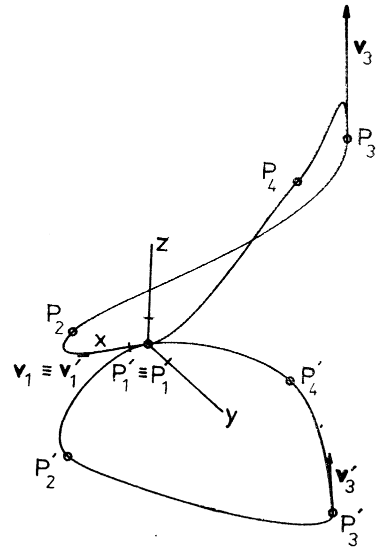


Fig .6

the question arises how to remove this deficiency; this problem we shall treat in a separate paper.

6. CONSTRUCTION OF INTERPOLATION CURVES WITH PRESCRIBED TANGENT VECTORS (METHOD $\mathscr{T}$)

We will again examine cases when at some nodes (or at all nodes) the supporting vectors are prescribed.

a) Let $K_i = 1$, $K_i = 0$. We put (see (1.4)) $\mathrm{D}x_j^{(i)} = v_j^{(i)}$. Then we have

$$(6.1) \qquad (x_j^{(i)}, \mathrm{D}x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, v_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4} \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Further, by (1.15), (1.20) we have

$$(6.2) \qquad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, v_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4} \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

With the aid of (6.1), (6.2) it is possible to represent the transposed matrix to matrix (1.9) in the form

$$(6.3)\qquad \boldsymbol{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (6.3) into (1.10), where $\boldsymbol{A}^{-1}$ stands for the matrix (1.19), we obtain

$$(6.4)\qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 0 & 9 & 4 & 8 & -1 \\ 0 & -11 & -4 & 12 & -1 \\ 0 & -1 & -4 & 0 & 1 \\ 0 & 3 & 4 & -4 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

b) Let $K_i = 0$, $K_{i+1} = 1$. By (1.14), (1.20) we have

$$(6.5)\qquad (x_j^{(i)}, \mathrm{D}x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, v_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4}\begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 4 & 1 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}.$$

Further, we put (see (1.14)) $\mathrm{D}x_j^{(i+1)} = v_j^{(i+1)}$. Then we have

$$(6.6)\qquad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, v_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}.$$

With the aid of (6.5), (6.6) it is possible to represent the transposed matrix to the matrix (1.9) in the form

$$(6.7)\qquad \boldsymbol{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (6.7) into (1.10), where $\boldsymbol{A}^{-1}$ stands for the matrix (1.19), we obtain

$$(6.8)\qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} -1 & 8 & 9 & -4 & 0 \\ 1 & -12 & 11 & -4 & 0 \\ 1 & 0 & -1 & 4 & 0 \\ -1 & 4 & -3 & 4 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

c) Let $K_i = K_{i+1} = 1$. By (6.1), (6.6) it is possible to represent the transposed matrix to matrix (1.9) in the form

$$(6.9)\qquad \mathbf{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (6.9) into (1.10), where $\mathbf{A}^{-1}$ stands for the matrix (1.19), we obtain

$$(6.10)\qquad 16\mathbf{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 0 & 8 & 4 & 8 & -4 & 0 \\ 0 & -12 & -4 & 12 & -4 & 0 \\ 0 & 0 & -4 & 0 & 4 & 0 \\ 0 & 4 & 4 & -4 & 4 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

We easily verify that cases a), b) can also be computed by applying formula (6.10) if we prescribe, in case a), the supporting tangent vector $v_j^{(i+1)} = (P_{i+2} - P_i)/4$ at the point P_{i+1}, or, in case b), the supporting tangent vector $v_j^{(i)} = (P_{i+1} - P_{i-1})/4$ at the point P_i (cf. Section 4). For instance, in case a) we thus obtain

$$(6.11)\qquad \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix};$$

substitution of (6.11) into (6.10) yields (6.4). Formula (6.10) can be further simplified to the form

$$(6.12)\qquad 4\mathbf{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \end{bmatrix}.$$

7. THE EQUIVALENCE OF METHODS $\mathscr{S}$, $\mathscr{T}$

Let, e.g., $K_i = K_{i+1} = 1$. According to method $\mathscr{S}$ (see the scheme in Section 5) we have $K_i + K_{i+1} = 1 + 1 = 2 \neq 0$, $K_{i+1} - K_i = 1 - 1 = 0$. Therefore we determine the auxiliary points $P_{i-1}^{\vee} = P_{i+1} - 4\mathbf{v}_i$, $P_{i+2}^{\vee\vee} = P_i + 4\mathbf{v}_{i+1}$ (see (5.1), (5.2)) and then apply formula (1.21) to the quadruple of points $P_{i-1}^{\vee}, P_i, P_{i+1}, P_{i+2}^{\vee\vee}$:

$$(7.1)\qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i+1)} - 4v_j^{(i)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i)} + 4v_j^{(i+1)} \end{bmatrix}.$$

We have

$$(7.2)\qquad \begin{bmatrix} x_j^{(i+1)} - 4v_j^{(i)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i)} + 4v_j^{(i+1)} \end{bmatrix} = \begin{bmatrix} 0 & -4 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \end{bmatrix},$$

so that the substitution of (7.2) into (7.1) yields

$$(7.3)\qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 8 & 4 & 8 & -4 \\ -12 & -4 & 12 & -4 \\ 0 & -4 & 0 & 4 \\ 4 & 4 & -4 & 4 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \end{bmatrix}.$$

However, (7.3) can also be written in the form

$$(7.4)\qquad 16\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 0 & 8 & 4 & 8 & -4 & 0 \\ 0 & -12 & -4 & 12 & -4 & 0 \\ 0 & 0 & -4 & 0 & 4 & 0 \\ 0 & 4 & 4 & -4 & 4 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

which is formula (6.10).

Let $K_i = K_{i+1} = 1$. Then formula (7.4), which can be modified to the form (7.3), holds by method $\mathscr{T}$ (see Section 6c)). We have

$$(7.5)\qquad \begin{bmatrix} x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \end{bmatrix} = 4 \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i+1)} - 4v_j^{(i)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i)} + 4v_j^{(i+1)} \end{bmatrix}.$$

Substituting (7.5) into (7.3) we obtain (7.1). This proves the equivalence of methods $\mathscr{S}$ and $\mathscr{T}$ for the case when $K_i = K_{i+1} = 1$. The equivalence is proved analogously for the other cases.

In conclusion we wish to add a few remarks. The interpolation method introduced by H. Lienhard in [1] for polynomials of at most fifth degree can be easily generalized. Polynomials of degree at most $2Q + 1$, where $Q > 1$ is an integer, are generally applied in [2]. Moreover, in this paper the case $Q = 1$ is investigated (method I, or its modification – method II). The auxiliary polynomial (cf. (1.11)) which is required to pass through the points $(2h, x_j^{(i+h)})$ $(-1 \leqq h \leqq 1,\ h$ integer) can pass in the general case, i.e. for $Q \geqq 1$, through the points $(2h, x_j^{(i+h)})$ $(-Q + p \leqq \leqq h \leqq Q - p$, p integer); here the fixed chosen number p satisfies the inequality $0 \leqq p \leqq Q - 1$. If $Q = 1$, then we necessarily have $p = 0$, and this leads to method I. Then the values of the "missing" derivatives of the first up to the Q-th orders depend on the choice of the parameter p as well. Generally, we see that the interpolation curve obtained loses "stability" with increasing Q. The reason is the possibility of the occurrence of a larger number of points of inflection. In addition, for a fixed chosen number Q the behaviour of the interpolation curve is the "better" the smaller is the paprameter p, i.e. the larger is the number of nodes which are used for the construction of the above mentioned auxiliary polynomials. For this reason the case $Q = 1$, $p = 0$ seems to be the most interesting.

The application of polynomials of the third degree for the interpolation has the advantage, as compared with polynomials of higher degree, that besides the mentioned smoother behaviour of the curve the computation of the coordinates of its points requires a smaller number of operations and, consequently, is faster. As compared with interpolation by cubic splines, the interpolation method presented is more advantageous in that it is possible to shape the curve better by determining its tangent vectors. Further, it is faster since the solution of the system of linear equations for the computation of the coefficients of the polynomials is not performed. On the other hand, the continuity of not only the first but also the second derivatives is the preferable property of the cubic splines. The freedom of the choice of some supporting tangents in the interpolation approach presented is its considerable advantage in comparison with, e.g., Ferguson's interpolation method where it is necessary to assign tangents at all nodes, or Akimov's interpolation method where tangents at the nodes are determined implicitly by the construction and cannot be prescribed.

The computation of the coordinates of the points of all interpolation curves was performed by computer. Spatial curves are drawn in axonometry, computer graphics was applied to the construction of all curves.

References

[1] *H. Lienhard:* Interpolation von Funktionswerten bei numerischen Bahnsteuerungen. Undated publication of CONTRAVES AG, Zürich.

[2] *J. Matušů:* The Lienhard interpolation method and some of its generalization (in Czech). Acta Polytechnica — Práce ČVUT, Prague, 3 (IV, 2), 1978.

Souhrn

KONSTRUKCE INTERPOLAČNÍCH KŘIVEK Z DANÝCH OPĚRNÝCH ELEMENTŮ (I)

JOSEF MATUŠŮ, JOSEF NOVÁK

Předmětem článku jsou konstrukce interpolačních křivek procházejících danými opěrnými body a dotýkajících se opěrných tečných vektorů v některých z těchto

bodů, popř. ve všech opěrných bodech. Matematickým jádrem těchto konstrukcí je Lienhardova interpolační metoda.

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