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SOME ITERATIVE POISSON SOLVERS APPLIED TO NUMERICAL SOLUTION OF THE MODEL FOURTH-ORDER ELLIPTIC PROBLEM

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The numerical solution of the model fourth-order elliptic boundary value problem in two dimensions is presented. The iterative procedure in which the biharmonic operator is splitted into two Laplace operators is used. After formulating the finite-difference approximation of the procedure, a formula for the evaluation of the transformed iteration vectors is developed. The Jacobi semi-iterative, Richardson and A. D. I. iterative Poisson solvers are applied to compute one transformed iteration vector. By the efficient use of the decomposition property of the corresponding iteration matrices, the fast Fourier transform algorithm needs to be applied twice in the evaluation of one iteration vector. The asymptotic number of operations for the sequential computation is $5n^2 \log_2 n$, where n^2 is the number of interior grid points in the unit square. The result of $7 \log_2 n$ parallel steps for the parallel computation on an SIMD machine with n^2 processors is so far the best one.

1. INTRODUCTION

During the recent years the growth of interest in the numerical solution of the first biharmonic boundary value problem in a bounded domain has been observed [6]. Some numerical algorithms based on splitting the biharmonic operator into a pair of Laplace operators have been proposed for solving this problem [2, 4–9, 11–13]. If the problem is considered on the unit square covered by a grid consisting of $n \times n$ mesh points, there is an iterative procedure [4] where the solution of two discrete Poisson equations is involved in one iteration. The boundary conditions for the first Poisson equation are prescribed, while the boundary conditions for the second need to be approximated [9]. By the respective use of the five-point formula and a simple

differerence formula for the approximation of the Laplace operator and of the unknown boundary conditions, two linear algebraic systems of order n^2 in a special symmetric blocktridiagonal form are obtained. The convergence of the iterative method is improved by an SOR parameter which has been proposed in [4]. Thus, the solutions of the coupled blocktridiagonal systems and two smoothing formulas have to be evaluated in one iteration. Since, on the basis of the careful analysis in [4], the number of iterations for the optimal smoothing parameter and for the accuracy $O(n^{-2})$ has been estimated by $O(n^{1/2} \log n)$, our attention will be devoted to fast computation of one iteration. (Here and throughout the paper $\log n \equiv \lceil \log_2 n \rceil$, while the natural logarithm of n will be denoted by $\ln n$).

Most of the approaches for solving this problem utilize the fast direct Poisson solvers to compute the two Poisson equations. The matrix decomposition and the cyclic odd-even reduction methods [1] have been used as Poisson solvers for computational experiments in [2]. The algorithm of Hockney [10] has been applied in [4, 5]. The number of operations per iteration is $O(n^2 \log n)$ for these approaches with the exception of the matrix – decomposition that requires $O(n^3)$ operations. Also the improved $O(n^2)$ technique proposed in [12] is based on the direct methods. The SOR iterative method has been used in [7]. An interesting approach has been proposed by the same authors in [8], where the both Poisson equations have been solved by the SOR method as one linear system. Unfortunately, the rate of convergence for this scheme has not been estimated.

As given in [3], the best of the iterative methods, i.e. the A.D.I. method, requires $O(n^2(\log n)^2$ operations for the solution of the Poisson equation. Hence, a comparison with the fast direct Poisson solvers shows that it is not efficient to use an iterative method for our problem in a straightforward manner.

The purpose of this paper is to apply the Jacobi semi-iterative, Richardson and A.D.I. methods for solving the two coupled linear systems in an efficient way. Thus, an iterative method is used for the evaluation of one iteration vector in the outer iteration procedure. The iteration matrices of the methods considered can be decomposed by the eigenvalue-eigenvector decomposition. The inner iteration process can be simplified if the outer iteration process is performed for the vectors transformed by the eigenvector matrix. It is shown that such strategy does not increase the asymptotic number of outer iterations. The evaluation of one iteration will include some steps with $O(n^2)$ complexity and $2n$ real Fourier transforms of vectors of the length n. Hence, if the procedure FOUR 67 of Hockney is adopted [10], the asymptotic number of operations per iteration is $5n^2 \log n$.

For parallel computation, there are results $24 \log n$ [11] and $14 \log n$ [13] steps per iteration, if an SIMD-type machine with n^2 processors is taken into account. The application of the proposed strategy results in $7 \log n$ steps if the same machine is considered.

2. THE METHOD FOR SOLVING THE MODEL FOURTH-ORDER ELLIPTIC PROBLEM

Under the fourth-order elliptic model problem we shall understand the biharmonic equation

$$\Delta\Delta v(x, y) = f_1(x, y) \quad \text{in} \quad Q \tag{1a}$$

with given boundary conditions

$$\begin{aligned} v(x, y) &= f_2(x, y) \\ v_n(x, y) &= f_3(x, y) \end{aligned} \quad \text{on} \quad \bar{Q} \tag{1b}$$

where $\bar{Q}$ is the boundary of the unit square Q and v_n is the derivative in the direction of the outward normal. The equation (1a) can be represented in terms of the Poisson equations as

$$\begin{aligned} \Delta v(x, y) &= w(x, y), \\ \Delta w(x, y) &= f_1(x, y). \end{aligned} \tag{2}$$

Let us construct an $(n + 1) \times (n + 1)$ grid of mesh points over $Q \cup \bar{Q}$, where $(n + 1) = h^{-1}$ for a discretization parameter $h > 0$. Then the Laplace operators and the boundary conditions can be approximated by simple difference formulas, as used in [4]. For the numerical solution of the finite-difference analogue of (2), (1b) there is an efficient procedure

$$\begin{aligned} \mathbf{M}\hat{\mathbf{v}}^{(k+1)} &= h^2\mathbf{w}^{(k)} + \mathbf{c}_1 \\ \mathbf{v}^{(k+1)} &= \omega\mathbf{v}^{(k)} + (1 - \omega)\,\hat{\mathbf{v}}^{(k+1)} \\ \mathbf{M}\hat{\mathbf{w}}^{(k+1)} &= -2/h^2\,\mathbf{C}\mathbf{v}^{(k+1)} + \mathbf{c}_2 \\ \mathbf{w}^{(k+1)} &= \omega\mathbf{w}^{(k)} + (1 - \omega)\,\hat{\mathbf{w}}^{(k+1)} \quad k = 0, 1, \ldots \end{aligned} \tag{3}$$

which starts from the initial vectors $\mathbf{v}^{(0)} = \mathbf{w}^{(0)} = \mathbf{0}$. The matrix $\mathbf{M}$ is blocktridiagonal of order n^2

$$\mathbf{M} = (-\mathbf{I}, \mathbf{M}_0, -\mathbf{I}), \tag{4}$$

where $\mathbf{M}_0$ is the tridiagonal matrix $(-1, 4, -1)$ of order n and $\mathbf{I}$ is the identity matrix of the same order. The matrix $\mathbf{C} = \operatorname{diag}(\mathbf{C}_0 + \mathbf{I}, \mathbf{C}_0, \ldots, \mathbf{C}_0, \mathbf{C}_0 + \mathbf{I})$ consists of n diagonal blocks where the matrix $\mathbf{C}_0 = \operatorname{diag}(1, 0, \ldots, 0, 1)$ is of order n. The vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ arise from the discretization of the given functions $f_i(x, y)$, $i = 1, 2, 3$ in (1). The optimal convergence parameter $\omega \in (0, 1)$ is a result of the analysis in [4].

In most of the algorithms for the computation of this procedure a direct [2, 4, 5, 11–13] or an iterative [7, 8] Poisson solver is applied to compute one iteration (3) straightforwardly. In this paper we shall follow the approach presented in [12].

The iteration formula (3) can be rewritten in terms of $\mathbf{v}$ as

$$\mathbf{v}^{(k+1)} = \left[2\omega\mathbf{I} - 2(1-\omega)^2\,\mathbf{M}^{-2}\mathbf{C}\right]\mathbf{v}^{(k)} - \omega^2\mathbf{v}^{(k-1)} + (1-\omega)^2\,\mathbf{c} \tag{5}$$

$$k = 1, 2, \ldots$$

with $\mathbf{v}^{(0)} = \mathbf{0}$, $\mathbf{v}^{(1)} = (1-\omega)\,\mathbf{M}^{-1}\mathbf{c}_1$, $\mathbf{c} = \mathbf{M}^{-1}\mathbf{c}_1 + h^2\,\mathbf{M}^{-2}\mathbf{c}_2$.
The iterations are tested by the criterion

$$\left\|\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}\right\|_\infty < \delta \tag{6}$$

for a given $\delta > 0$, where $\|\cdot\|_\infty$ denotes the maximum norm of a real vector. If the formula (5) is multiplied by the matrix $\mathbf{U}$ whose columns $\mathbf{U}^{(k,l)}$ $k, l = 1, \ldots, n$ are defined as

$$\mathbf{U}_{i,j}^{(k,l)} = \frac{2}{n+1}\sin\frac{ik\pi}{n+1}\sin\frac{jl\pi}{n+1}, \quad i, j = 1. \ldots, n, \tag{7}$$

we get

$$\bar{\mathbf{v}}^{(k+1)} = 2\omega\bar{\mathbf{v}}^{(k)} - 2(1-\omega)^2\,\mathbf{U}\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)}) - \omega^2\bar{\mathbf{v}}^{(k-1)} + (1-\omega)^2\,\bar{\mathbf{c}}$$

$$k = 0, 1, \ldots \tag{8}$$

where $\bar{\mathbf{c}} = \mathbf{U}\mathbf{c}$ and

$$\bar{\mathbf{v}}^{(k)} = \mathbf{U}\mathbf{v}^{(k)} \quad \text{for} \quad k = 0, 1, \ldots \tag{9}$$

Analogously to [12], if the condition

$$\left\|\bar{\mathbf{v}}^{(k+1)} - \bar{\mathbf{v}}^{(k)}\right\|_\infty < \frac{\delta}{\|U\|_\infty} \tag{10}$$

is satisfied, the condition (6) is also satisfied for the same δ. Here,

$$\|\mathbf{U}\|_\infty = \max_{k,l}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left|U_{i,j}^{(k,l)}\right|\right]. \tag{11}$$

The matrix $\mathbf{U}$ can be expressed in the form

$$\mathbf{U} = \mathbf{FPF} \tag{12}$$

where

$$\mathbf{F} = \operatorname{diag}(\mathbf{F}_0, \ldots, \mathbf{F}_0),\ (\mathbf{F}_0)_{i,j} = \sqrt{[2/(n+1)]}\sin ij\pi/(n+1) \quad i, j = 1, \ldots, n$$

and where the permutation matrix $\mathbf{P}$ is composed from n^2 blocks $\mathbf{P}_{ij}$ $i, j = 1, \ldots, n$ of order n, each of them containing 1 exactly in the position (j, i) and 0's otherwise. The matrix $\mathbf{P}$ enables us permute elements of the n^2 vector $(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{nn})$ into the vector $(x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, \ldots, x_{nn})$.
Since both the matrices $\mathbf{F}$ and $\mathbf{P}$ are symmetric and orthogonal, the property of orthogonality

$$\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{FPF}(\mathbf{FPF})^{\mathrm{T}} = \mathbf{I}$$

is satisfied also for $\mathbf{U}$.

By using the result given in [12], the norm (11) can be estimated by

$$\|\mathbf{U}\|_\infty \leqq \|\mathbf{F}\|_\infty \|\mathbf{P}\|_\infty \|\mathbf{F}\|_\infty = \|\mathbf{F}_0\|_\infty^2 = \left[\frac{\sqrt{\left(\frac{2}{n+1}\right)}}{\operatorname{tg}\frac{\pi}{2(n+1)}}\right]^2 < n\,.$$

The number of iterations required to reduce the initial error of the process (8) by $\varepsilon = O(h^2)$ is

$$\bar{q} \geqq \frac{1}{R_\infty} \ln \frac{1}{\varepsilon} \tag{13}$$

where R_∞ denotes the asymptotic rate of convergence. The iteration matrices of the processes (5) and (8) are similar [13], hence $R_\infty = O(h^{1/2})$ [5].

Thus, the process (8) can be performed instead of the original process (5) without increasing the asymptotic number of iterations $O(n^{1/2} \log n)$ for $\varepsilon = O(n^{-2})$.

In (8), the solution of the two Poisson equations is involved in the term $\mathbf{U}(\mathbf{M}^{-2} \cdot (\mathbf{C}\mathbf{v}^{(k)}))$ which may be obtained in the following computational phases:

1. To evaluate $\mathbf{C}\mathbf{v}^{(k)} = \mathbf{C}\mathbf{U}\bar{\mathbf{v}}^{(k)}$.
2. To solve $\mathbf{M}\mathbf{v}_0^{(k+1)} = \mathbf{C}\mathbf{v}^{(k)}$.
3. To solve $\mathbf{M}\mathbf{v}_1^{(k+1)} = \mathbf{v}_0^{(k+1)}$.
4. To evaluate $\mathbf{U}\mathbf{v}_1^{(k+1)} = \mathbf{U}(\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)}))$.

3. APPLICATION OF SOME ITERATIVE POISSON SOLVERS

We shall consider three classical iterative methods for solving the discrete Poisson equation in order to evaluate the vector $\mathbf{U}(\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)}))$ according to the computational phases 1–4. To recall these methods we first solve a general linear system

$$\mathbf{M}\mathbf{x} = \mathbf{y} \tag{14}$$

with the same matrix $\mathbf{M}$ as in (4).

If we denote the Jacobi iterative matrix corresponding to $\mathbf{M}$ as

$$\mathbf{B} = \mathbf{I} - \tfrac{1}{4}\mathbf{M}$$

then the Jacobi semi-iteractive method for solving the system (14) is expressed by

$$\hat{\mathbf{x}}^{(m+1)} = \mathbf{T}_{JS}^{(m+1)}\hat{\mathbf{x}}^{(m)} + \hat{\mathbf{y}} \tag{15}$$

where

$$\hat{\mathbf{x}}^{(m+1)} = \begin{bmatrix} \mathbf{x}^{(m+1)} \\ \mathbf{x}^{(m)} \end{bmatrix}, \quad \hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{4}\mathbf{y} \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{T}_{JS}^{(m+1)} = \begin{bmatrix} a_{m+1}\mathbf{B} & (1 - a_{m+1})\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad m = 0, 1, \ldots .$$

The starting vector is

$$\hat{\mathbf{x}}^{(0)} = \begin{bmatrix} \mathbf{B}\mathbf{x}^{(0)} + \frac{1}{4}\mathbf{y} \\ \mathbf{x}^{(0)} \end{bmatrix}$$

and the parameters a are given by $a_1 = 1$, $a_2 = 2/(2 - \varrho^2)$, $a_{i+1} = (1 - \frac{1}{4}\varrho^2 a_i)$, $i = 2, 3, \ldots$, where $\varrho = \cos(\pi/(n+1))$ is the spectral radius of the matrix $\mathbf{B}$ [14]. In order to reduce the initial error by the factor h^2, the number of iterations (15) is $p_{JS} = (2n/\pi) \ln n$ [14]. If setting $\mathbf{x}^{(0)} = \mathbf{0}$, the solution $\hat{\mathbf{x}}^{(p_{JS})}$ can be written as

$$\hat{\mathbf{x}}^{(p_{JS})} = \Big(\prod_{i=1}^{p_{JS}} \mathbf{T}_{JS}^{(i)} + \prod_{i=2}^{p_{JS}} \mathbf{T}_{JS}^{(i)} + \ldots + \mathbf{I}\Big)\hat{\mathbf{y}} \equiv \mathbf{T}_{JS}\hat{\mathbf{y}} . \tag{16}$$

In the Richardson method, where p convergence parameters (p depends on the accuracy of the solution) are used in a cyclic order [15, p. 364], the iterations are computed by

$$\mathbf{x}^{(m+1)} = \mathbf{T}_R^{(m+1)}\mathbf{x}^{(m)} + b_{m+1}\mathbf{y} , \quad m = 0, \ldots, p . \tag{17}$$

Here, the iteration matrix is

$$\mathbf{T}_R^{(m+1)} = \mathbf{I} - b_{m+1}\mathbf{M}$$

and the convergence parameters are

$$b_m = \left[\frac{\lambda_0 - \lambda_1}{2}\cos\vartheta_l + \frac{\lambda_0 + \lambda_1}{2}\right]^{-1}$$

with

$$\lambda_0 = \frac{8}{h^2}\cos^2\left(\frac{\pi h}{2}\right), \quad \lambda_1 = \frac{8}{h^2}\sin^2\left(\frac{\pi h}{2}\right) \tag{18}$$

and

$$\vartheta_l = \frac{(2l-1)}{2p_R} \quad \text{for} \quad l = 1, 2, \ldots, p .$$

One can obtain from [14, p. 140] that the number of iterations required for the accuracy h^2 is $p = p_R = n/\pi(2 \ln n + \ln 2)$. Assuming $\mathbf{x}^{(0)} = \mathbf{0}$ we get from (17)

$$\mathbf{x}^{(p_R)} = \mathbf{T}_R^{(p_R)} \ldots \mathbf{T}_R^{(2)} b_1 \mathbf{y} + \ldots + \mathbf{T}_R^{(p_R)} b_{p_R - 1}\mathbf{y} + b_{p_R}\mathbf{y} \equiv \mathbf{T}_R\mathbf{y} . \tag{19}$$

One iteration of the *A.D.I.* method is computed by

$$\mathbf{x}^{(m+1)} = \mathbf{T}_A^{(m+1)}\mathbf{x}^{(m)} + \mathbf{g}_{m+1} , \quad m = 0, 1, \ldots \tag{20}$$

where

$$\mathbf{T}_A^{(m+1)} = (\mathbf{S} + r_{m+1}\mathbf{I})^{-1}(r_{m+1}\mathbf{I} - \mathbf{H})(\mathbf{H} + r_{m+1}\mathbf{I})^{-1}(r_{m+1}\mathbf{I} - \mathbf{S})$$

and

$$g_{m+1}(\mathbf{y}) = (\mathbf{S} + r_{m+1}\mathbf{I})^{-1}\left[(r_{m+1}\mathbf{I} - \mathbf{H})(\mathbf{H} + r_{m+1}\mathbf{I})^{-1} + \mathbf{I}\right]\mathbf{y}$$

with

$$\mathbf{H} = (-\mathbf{I}, 2\mathbf{I}, -\mathbf{I})$$

$$\mathbf{S} = \operatorname{diag}(\mathbf{S}_0, \mathbf{S}_0, \ldots, \mathbf{S}_0), \quad \mathbf{S}_0 = (-1, 2, -1).$$

In terms of the values of (18), the Peaceman-Rachford iteration parameters are estimated by

$$r_i = \lambda_0 \left(\frac{\lambda_1}{\lambda_0}\right)^{(2i-1)/2s} \quad \text{for} \quad i = 1, \ldots, s$$

where

$$(0{,}414)^{2s} \leqq \frac{\lambda_1}{\lambda_0}, \quad [15].$$

If we take $\mathbf{x}^{(0)} = \mathbf{0}$ and $p_A = \frac{1}{5}\log^2 n$ iterations to reduce the initial error by h^2 [3] the solution of (14) can be written as

$$\mathbf{x} \approx \mathbf{x}^{(p_A)} = g_{p_A}(\mathbf{y}) + \mathbf{T}_A^{(p_A)} g_{p_A-1}(\mathbf{y}) + \ldots + \mathbf{T}_A^{(p_A)}\mathbf{T}_A^{(p_A-1)} \ldots \tag{21}$$

$$\ldots \mathbf{T}_A^{(2)} g_1(\mathbf{y}) \equiv \mathbf{T}_A \mathbf{y}.$$

As known [14], the matrix $\mathbf{M}$ and hence the matrices $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{S}$ can be diagonalized by the eigenvector matrix $\mathbf{U}$ given in (7). Thus, the matrices $\mathbf{T}_i$, $i = A, R, JS$ can be expressed as

$$\mathbf{T}_i = \mathbf{U}\mathbf{D}_i\mathbf{U}, \quad i = JS, R, A \tag{22}$$

where the diagonal eigenvector matrices $\mathbf{D}_i$ for $i = JS$, R, A can be evaluated respectively according to (16), (19), (21) from the eigenvalues of $\mathbf{T}_{JS}^{(m)}$, $\mathbf{T}_R^{(m)}$ and $\mathbf{T}_A^{(m)}$ which are explicitly known.

The above methods can be applied to the evaluation of the computational phases 2 and 3, using the iteration matrices in the decomposed form (22). The solution $\mathbf{v}_0^{(k+1)}$ of the phase 2 can be computed by

$$\mathbf{v}_0^{(k+1)} = \mathbf{T}_i\mathbf{C}\mathbf{U}\bar{\mathbf{v}}^{(k)}$$

where $i = A$, R or JS.

Then the vector $\mathbf{v}_1^{(k+1)}$ can be obtained by

$$\mathbf{v}_1^{(k+1)} = \mathbf{T}_i\mathbf{v}_0^{(k+1)} = \mathbf{T}_i^2\mathbf{C}\mathbf{U}\bar{\mathbf{v}}^{(k)}$$

when one of the methods is considered.

If the matrix $\mathbf{U}$ is considered in the form (12) and one of the above methods is aapplied, then the formula for the required vector $\mathbf{U}(\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)}))$ is

$$\mathbf{U}(\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)})) = \mathbf{FPF}(\mathbf{FPFD}_i^2\mathbf{FPF}(\mathbf{CFPF}\bar{\mathbf{v}}^{(k)})) = \mathbf{D}_i^2\mathbf{FPF}(\mathbf{CFPF}\ \bar{\mathbf{v}}^{(k)}). \tag{23}$$

Thus, the computation of one iteration (8) depends on the fast evaluation of the formula (23). After the evaluation of the new iteration vector $\bar{\mathbf{v}}^{(k+1)}$ the criterion (10) is examined. If the condition (10) holds for some vector $\bar{\mathbf{v}}^{(k+1)}$, the solution vector $\mathbf{v}^{(k+1)}$ is computed from (9) by a completing calculation. Since the elements of the diagonal matrices $\mathbf{D}_i$ depend on the acceleration parameters and on the explicitly known eigenvalues, these can be computed, as well as the matrix $\mathbf{F}$ and the constant vector $\mathbf{c}$, before starting the iterative procedure.

4. THE COMPLEXITY FOR SERIAL AND PARALLEL REALIZATIONS

In the evaluation of one iteration (8), the computation of the vector $\mathbf{U}(\mathbf{M}^{-2}(\mathbf{C}\mathbf{v}^{(k)}))$ is the most expensive from the point of view of arithmetical operations. Following the formula (23) we can obtain the vector $\mathbf{F}\bar{\mathbf{v}}^{(k)}$ as a result of n Fourier transforms of real vectors, each of the length n. Since the sparse matrix $\mathbf{C}$ in (3) is of the form $\mathbf{C} = \operatorname{diag}(\mathbf{C}_0, \mathbf{C}_0, \ldots, \mathbf{C}_0) + \operatorname{diag}(\mathbf{I}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{I}) \equiv \mathbf{C}_1 + \mathbf{C}_2$, in the computation of the vector $\mathbf{CF}(\mathbf{PF}\bar{\mathbf{v}}^{(k)})$ there are $2n$ products required for the evaluation of $\mathbf{C}_1\mathbf{F} \cdot (\mathbf{PF}\bar{\mathbf{v}}^{(k)})$. Two transforms by $\mathbf{F}_0$ in the computation of the expression $\mathbf{C}_2\mathbf{F}(\mathbf{PF}\bar{\mathbf{v}}^{(k)})$ need not be performed because these will be eliminated, in view of the orthogonality of $\mathbf{F}_0$, in the subsequent computation of the vector $\mathbf{F}(\mathbf{CF}(\mathbf{PF}\bar{\mathbf{v}}^{(k)}))$. Since the sparse vector $\mathbf{C}_1\mathbf{F}(\mathbf{PF}\bar{\mathbf{v}}^{(k)})$ involves $2n$ nonzero elements, its multiplication by the matrix $\mathbf{F}$ is performed in a classical manner. Finally, Fourier transform of n real vectors of the length n is required to evaluate the vector $\mathbf{FPF}(\mathbf{CF}(\mathbf{PF}\bar{\mathbf{v}}^{(k)}))$.

For a sequential implementation, the contribution to the asymptotic number of operations will arise from the two Fourier transform applications. For our purpose, the procedure FOUR 67 for the Dirichlet boundary conditions will be suitable. The number of operations is $\frac{5}{2}n^2 \log n$ [10], hence $5n^2 \log n + O(n^2)$ operations are required for an effective serial computation of one iteration (8). The number of steps for the complete solution of (3) is $5n^2 \log n \,.\, q$, where $q = O(n^{1/2} \log n)$ is estimated by (13).

Our approach can be implemented on a parallel computer with the following SIMD characteristics [11]:

- n^2 processors are available;
- each processor may either perform one of the four arithmetic operations in one time step or be idle;
- no memory or data alignment time penalties are incurred.

As given in [11], the coefficients of n Fourier transforms of n real vectors with n elements can be computed in $3 \log n$ steps on n^2 processors. Hence, $6 \log n$ steps are required for the two transforms by the matrix $\mathbf{F}$ in our approach. $2n$ vector products, required in the multiplication of the vector $\mathbf{PF}\bar{\mathbf{v}}^{(k)}$ by the matrix $\mathbf{C}_1\mathbf{F}$, can be obtained in $\log n + O(1)$ steps on n^2 processors. The argument for this number is that the

processors which become idle in the addition phase during the computation of the first n vector products can be utilized for the evaluation of the remaining vector products. Since the other phases in evaluation of the vector $\bar{v}^{(k+1)}$ require $O(1)$ operations on n^2 processors, one iteration (8) can be obtained in $7 \log n$ steps. The number of steps for the solution of (3), terminated by $O(n^{-2})$, is $7 \log n \,.\, q$. Defining the speedup of the parallel computation [11] over the serial computation time as

$$S_p = \frac{T_1}{T_p},$$

where T_1 is the serial computational time and T_p is the number of steps for the parallel realization on p processors, we get for the method proposed

$$S_{n^2} = \frac{5n^2 \log n \,.\, q}{7 \log n \,.\, q} = \frac{5}{7} n^2 .$$

If the efficiency of the parallel computation [11] is defined by

$$E_{n^2} = \frac{S_{n^2}}{n^2}$$

the result for our approach is $\frac{5}{7}$.

5. CONCLUSION

The classical procedure (3) for solving the biharmonic equation (1) has been replaced by the procedure (8) which can be evaluated efficiently by one of the three iterative methods of section 3. For solving the special blocktridiagonal systems, we have used the information about the eigenvector-eigenvalue decomposition of the corresponding iteration matrices.

Thus, the results obtained are mathematically equivalent to those which could be obtained by a straightforward application of these methods to the solution of the problem (1) by the procedure (3). The advantage of our approach is in a reduction of the number of arithmetical operations. If the two Poisson equations in (3) were solved by the direct inverse using the same information (22) as we did, the approach would require to perform $8n$ Fourier transforms of the length n in one iteration. Our approach reduces this requirement to $2n$ transforms only. Thus, the arithmetical complexity for our implementation of iterative methods is $5n^2 \log n$, i.e. the same as for the straightforward application of the most effective Hockney's direct Poisson solver [10] to the solution of two Poisson equations in (3).

Compared with $24 \log n$ and $14 \log n$ steps for the algorithms in [11] and [13], respectively, our result $7 \log n$ steps is the best one so far for the parallel computation on an SIMD-machine with n^2 processors.

Since the routines for the fast Fourier transform are well developed for the time being, the practical application of the methods needs no special procedures. The parallel implementation would call for an architecture which would allow to perform the fast Fourier transform and the inner product of real vectors in an efficient way.

As concerns the stability of numerical computation, the Richardson method in the form (17) is unstable [15]. A difficulty can be also due to the truncation error in the evaluation of the diagonal matrices $\mathbf{D}_i$. Since the crucial property used is the eigenvector-eigenvalue decomposition of the iteration matrices, the approach proposed is restricted for application to rectangualr domains only.

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Souhrn

NIEKTORÉ ITERATÍVNE METÓDY RIEŠENIA POISSONOVEJ ROVNICE APLIKOVANÉ NA NUMERICKÉ RIEŠENIE MODELOVEJ ELIPTICKEJ ÚLOHY ŠTVRTÉHO RÁDU

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V článku sa zaoberáme numerickým riešením modelovej eliptickej okrajovej úlohy štvrtého rádu na dvojrozmernej pravouholníkovej oblasti. Na jej riešenie je použitá iteratívna procedúra, ktorá je založená na rozštiepení biharmonického operátora na dvojicu Laplaceových operátorov. Pre jej konečno-diferenčnú aproximáciu je navrhnutá formula výpočtu iteračných vektorov v transformovanom tvare. Na výpočet týchto vektorov sú aplikované iteračné metódy numerického riešenia Poissonovej rovnice: Jacobiho semiiteračná, Richardsonova a A.D.I. metóda. Využijúc efektívnym spôsobom vlastnosť rozkladu príslušných matíc iterácie, treba pritom dvakrát aplikovať algoritmus rýchlej Fourierovej transformácie. Asymptotický počet operácií pre sériový výpočet jednej iterácie je $5n^2 \log_2 n$, kde n^2 je počet uzlových bodov siete nad jednotkovým štvorcom. Výsledok $7 \log_2 n$ paralelných krokov pre výpočet na n^2-procesorovom počítači typu SIMD predstavuje doteraz najnižšiu hodnotu aritmetickej paralelnej výpočtovej zložitosti pre túto úlohu.

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