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ACTIONS WITH THE CONSERVATION PROPERTY

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1. INTRODUCTION

In their theory of actions on thermodynamical systems, Coleman and Owen [1, 2] introduce the important concepts of "upper potential" and "potential". Both the upper potential and the potential are functions of state related in a specific way to an underlying action; the entropy function provides an example of an upper potential for a certain action while the energy function is a potential for another action. Coleman and Owen [1, 2] derived necessary and sufficient conditions for an action to have an upper potential or a potential and discussed in detail the uniqueness and regularity of these functions of state.

The main results of Coleman & Owen [1] concerning potentials may be summarized as follows[1]): If an action for a system has a potential then it has the conservation property on a dense set of states; if, conversely, an action has the conservation property at one state, then it has the conservation property on a dense set of states and admits a potential which is defined and continuous on a dense set of states; moreover, two potentials for a given action can differ by at most a constant on the intersection of their domains. Coleman and Owen deduced these results as corollaries of their former results in the more general theory of actions with the Clausius property. Here I give a direct, explicit construction of the potential which leads to a sharpening of the above results:

If an action for a system has the conservation property at one state, then it has the conservation property at *every* state and admits an *everywhere* defined continuous potential; any potential for the action differs from this potential by a constant on its domain and hence can be extended to the entire state space.

[1]) The definitions of the concepts used in this introduction are found in the subsequent sections of the paper.

I also present extensions of these results appropriate for the very general semi-systems introduced in [2].

It turns out that the natural substitute for the conservation property in this more general setting is the path-independence defined in Section 3. For systems, the path-independence is equivalent to the conservation property but for semi-systems it is stronger than the conservation property. When an action for a semi-system is path-independent at one base state, it must be path-independent at each state and admits an everywhere defined continuous potential. Any potential whose domain contains at least one base state differs from this potential by a constant on its domain.

The actions with the conservation property and path-independent actions arise naturally as consequences of a primitive version of the first law of thermodynamics. A postulate given in [3, 4] states the first law in such a way that the proportionality of work and heat in cyclic processes figures as a consequence. As stated in [3, 4], the result is applicable only to systems with perfect accessibility, but a natural generalization of the postulate of [3, 4] enables one to prove that the difference of the actions giving the heat gained and the work done in a process is conservative. The result of the present paper then yields the existence of an everywhere defined energy function of the system. A future paper will treat these questions as well as some simplifications and generalizations of the concepts of the thermodynamical system and action. Some of these have already been announced in [5].

I believe that the reader will find this paper self-contained in the sense that all necessary concepts are defined. For the motivation and applications of the theory I suggest that he examine the original papers of Coleman and Owen [1, 2, 6, 7] and my appendix in the second edition of Truesdell's "Rational Thermodynamics" [8].

2. SEMI-SYSTEMS, SYSTEMS AND ACTIONS

This section recalls the basic concepts of the theory of systems (see [1], [2]).

Definition 2.1. Let (Σ, Π) be an ordered pair in which Σ is a topological space and Π a set of objects such that each P in Π determines a continuous mapping ϱ_P of a non-empty open subset $\mathscr{D}(P)$ of Σ onto a subset $\mathscr{R}(P)$ of Σ. If (Σ, Π) has the properties [I] and [II] below, then (Σ, Π) is called a *semi-system*, each element σ of Σ is called a *state*, each element P of Π is called a *process*, and ϱ_P is called the *transformation induced by P*.

[I] There is at least one element σ_0 of Σ for which the set $\Pi\sigma_0$, defined by

$$\Pi\sigma_0 := \{\varrho_P\sigma_0 \colon P \in \Pi,\ \sigma_0 \in \mathscr{D}(P)\},$$

is dense in Σ.

[II] On the set $\mathscr{P}$, defined by

$$\mathscr{P} := \{(P'', P') \in \Pi \times \Pi \colon \mathscr{D}(P'') \cap \mathscr{R}(P') \neq \emptyset\},$$

there is a function $\mathscr{P} \to \Pi$, written $(P'', P') \mapsto P''P'$, such that

$$\mathscr{D}(P''P') = \varrho_{P'}^{-1}(\mathscr{D}(P'') \cap \mathscr{R}(P'))\} ,$$

and for each σ in $\mathscr{D}(P''P')$,

$$\varrho_{P''P'}\sigma = \varrho_{P''}\varrho_{P'}\sigma .$$

For the set of all ordered pairs (P, σ) with P in Π and σ in $\mathscr{D}(P)$, one writes $\Pi \square \Sigma$, i.e.

$$\Pi \square \Sigma := \{(P, \sigma) \in \Pi \times \Sigma \colon \sigma \in \mathscr{D}(P)\} .$$

A state σ_0 such that $\Pi\sigma_0$ is dense in Σ is called a base state. If σ_0 is a base state then for each non-empty open subset $\mathscr{O}$ of Σ there is a process P such that (P, σ_0) is in $\Pi \square \Sigma$ and $\varrho_P\sigma_0$ is in $\mathscr{O}$.

A *system* is a semi-system (Σ, Π) with the property $[\mathrm{I}^+]$ below, which is a strengthened version of [I].

$[\mathrm{I}^+]$ For each σ in Σ, the set

$$\Pi\sigma := \{\varrho_P\sigma \colon P \in \Pi, \ \sigma \in \mathscr{D}(P)\}$$

is dense in Σ.

In other words, a system is a semi-system for which every state is a base state.

Definition 2.2. An *action* $\mathscr{a}$ for a semi-system (Σ, Π) is a real-valued function on $\Pi \square \Sigma$ with the following two properties:

(i) additivity – if (P'', P') is in $\mathscr{P}$ and σ is in $\mathscr{D}(P''P')$, then

$$\mathscr{a}(P''P', \sigma) = \mathscr{a}(P', \sigma) + \mathscr{a}(P'', \varrho_{P'}\sigma) ;$$

(ii) continuity – for each P in Π, the function $\mathscr{a}(P, \cdot) \colon \mathscr{D}(P) \to \mathbb{R}$ is continuous.

There are semi-systems for which two processes P', P'' can induce the same transformation $\varrho_{P'} = \varrho_{p''}$ but give different values to an action. Nonetheless, because the mapping $P \mapsto \varrho_P$ is single valued, one can simplify the notation and write $P\sigma$ for $\varrho_P\sigma$; equation (2.1) then becomes

$$\mathscr{a}(P''P', \sigma) = \mathscr{a}(P', \sigma) + \mathscr{a}(P'', P'\sigma) .$$

Henceforth, I adopt also the following useful convention from [1, 2]: whenever the symbol $\mathscr{a}(P, \sigma)$ or $P\sigma$ $(=\varrho_P\sigma)$ occurs, it is to be understood that σ is in $\mathscr{D}(P)$ (i.e., (P, σ) is in $\Pi\square\Sigma$), similarly if the symbol $P''P'$ occurs, it is to be understood that (P'', P') is in $\mathscr{P}$, for only under such circumstances the expressions $\mathscr{a}(P, \sigma)$, $P\sigma$, and $P''P'$ can be meaningful.

If $\mathscr{a}$ is an action for a semi-system (Σ, Π), P a process, and $\mathscr{O}$ a subset of Σ, then $\mathscr{a}(P, \mathscr{O})$ is the subset of $\mathbb{R}$ defined by

$$\mathscr{a}(P, \mathscr{O}) = \{\mathscr{a}(P, \sigma) \colon \sigma \in \mathscr{O} \cap \mathscr{D}(P)\}$$

and $P\mathscr{O}(= \varrho_P\mathscr{O})$ is the subset of Σ defined by

$$P\mathscr{O} := \{P\sigma = \varrho_P\sigma : \sigma \in \mathscr{O} \cap \mathscr{D}(P)\} .$$

3. PATH-INDEPENDENT ACTIONS AND ACTIONS WITH THE CONSERVATION PROPERTY

In [1], Coleman & Owen define the conservation property for actions within the context of systems and relate this property to the existence of a potential defined on a dense set of states. The definition of the conservation property can be extended without any change to semi-systems (Definition 3.2 below), but as is apparent from the results to be given below, for actions on general semi-systems the conservation property is too weak to imply the existence of potentials. It turns out that the path-independence of an action, as introduced in Definition 3.1 below, is a proper substitute for the conservation property in the context of semi-systems: while for systems it is equivalent to the conservation property, for general semi-systems it is stronger than the conservation property and leads to the existence of potentials. Namely, it is the principal result of this note that if an action on a semi-system is path-independent at one base state, then it is path-independent at each state and admits an everywhere defined, continuous, and essentially unique potential. As explained in Introduction, for systems this result gives a stronger version of the former results of Coleman & Owen [1].

Definition 3.1. Let α be an action for a semi-system (Σ, Π) and let σ_0 be a state. The action α is said to be *path-independent* at σ_0 if for each state σ and each $\varepsilon > 0$ there is a neighborhood $\mathcal{O}$ of σ such that

$$(3.1) \qquad P_1, P_2 \in \Pi\,, \quad P_1\sigma_0, P_2\sigma_0 \in \mathcal{O} \Rightarrow |\alpha(P_1, \sigma_0) - \alpha(P_2, \sigma_0)| < \varepsilon\,.$$

It is an immediate consequence of the above definition that if α is path-independent at σ_0 then

$$(3.2) \qquad P_1, P_2 \in \Pi\,, \quad P_1\sigma_0 = P_2\sigma_0 \Rightarrow \alpha(P_1, \sigma_0) = \alpha(P_2, \sigma_0)$$

but generally the path-independence in the sense of Definition 3.1 is a requirement stronger than (3.2).

Definition 3.2. Let α be an action for a semi-system (Σ, Π) and let σ_0 be a state. If for each $\varepsilon > 0$ there is a neighborhood $\mathcal{O}$ of σ_0 such that

$$P \in \Pi\,, \quad P\sigma_0 \in \mathcal{O} \Rightarrow |\alpha(P, \sigma_0)| < \varepsilon\,,$$

then the action α is said to have the *conservation property* at σ_0.

Proposition 3.1. *Let α be an action for a semi-system (Σ, Π) and let σ_0 be a state. If α is path-independent at σ_0 then it has the conservation property at σ_0.*

Proof. Let $\varepsilon > 0$. Since α is path-independent at σ_0, there exists a neighborhood $\mathcal{O}_1$ of σ_0 such that

$$(3.3) \qquad P_1, P_2 \in \Pi\,, \quad P_1\sigma_0, P_2\sigma_0 \in \mathcal{O}_1 \Rightarrow |\alpha(P_1, \sigma_0) - \alpha(P_2, \sigma_0)| < \varepsilon/2\,.$$

We now consider two complementary cases:

(i) there exists no $P \in \Pi$ with $P\sigma_0 \in \mathcal{O}_1$;
(ii) there exists a $P_0 \in \Pi$ with $P_0\sigma_0 \in \mathcal{O}_1$.

In case (i) there obviously is a neighborhood of σ_0 such that

$$P \in \Pi\,, \quad P\sigma_0 \in \mathcal{O} \Rightarrow |a(P, \sigma_0)| < \varepsilon\,, \tag{3.4}$$

namely, $\mathcal{O} = \mathcal{O}_1$, since for such an $\mathcal{O}$ no P satisfying the hypothesis of the implication (3.4) exists.

In case (ii) we choose some $P_0 \in \Pi$ with

$$P_0\sigma_0 \in \mathcal{O}_1\,. \tag{3.5}$$

By the continuity of ϱ_{P_0} and $a(P_0, \cdot)$ there exists a neighborhood $\mathcal{O}$ of σ_0 such that

$$P_0\mathcal{O} \subset \mathcal{O}_1 \tag{3.6}$$

and

$$a(P_0, \mathcal{O}) \subset \left(a(P_0, \sigma_0) - \frac{\varepsilon}{2},\ a(P_0, \sigma_0) + \frac{\varepsilon}{2}\right). \tag{3.7}$$

The proof will be complete if one shows that

$$|a(P, \sigma_0)| < \varepsilon \tag{3.8}$$

for all $P \in \Pi$ with $P\sigma_0 \in \mathcal{O}$. Accordingly, let $P \in \Pi$ satisfy $P\sigma_0 \in \mathcal{O}$. Then by (3.6) and (3.7) one has

$$P_0P\sigma_0 \in \mathcal{O}_1 \tag{3.9}$$

and

$$|a(P_0, P\sigma_0) - a(P_0, \sigma_0)| < \varepsilon/2\,. \tag{3.10}$$

By (3.5) and (3.9) the processes $P_1 := P_0$ and $P_2 := P_0P$ satisfy the relations

$$P_1\sigma_0,\ P_2\sigma_0 \in \mathcal{O}_1$$

and hence the implication (3.3) yields

$$|a(P_0, \sigma_0) - a(P_0P, \sigma_0)| < \varepsilon/2$$

which in view of the additivity of a may be rewritten as

$$|a(P_0, \sigma_0) - a(P_0, P\sigma_0) - a(P, \sigma_0)| < \varepsilon/2\,. \tag{3.11}$$

But (3.10) and (3.11) yield (3.8). To summarize, we have found, for each $\varepsilon > 0$ and in both cases (i), (ii), a neighborhood $\mathcal{O}$ of σ_0 such that the implication (3.1) holds. The proof is complete.

Proposition 3.2. *Let a be an action for a system (Σ, Π) and let σ_0 be a state. Then a is path-independent at σ_0 if and only if a has the conservation property at σ_0.*

Proof. That the path-independence of a at σ_0 implies the conservation property of a at σ_0 is the assertion of Proposition 3.1.

We now prove that, for systems, also the converse is true. Hence, suppose that a has the conservation property at σ_0, and let $\sigma \in \Sigma$ and $\varepsilon > 0$. The conservation property implies that there is a neighborhood $\mathcal{O}_0$ of σ_0 such that

$$(3.12) \qquad P \in \Pi, \quad P\sigma_0 \in \mathcal{O}_0 \Rightarrow |a(P, \sigma_0)| < \varepsilon/4 .$$

By the accessibility axiom for systems, i.e., by the property $[\mathrm{I}^+]$, there exists a process $\bar{P}$ such that $\bar{P}\sigma \in \mathcal{O}_0$.

By the continuity of $\varrho_{\bar{P}}$ and $a(\bar{P}, \cdot)$, there exists a neighborhood $\mathcal{O}$ of σ such that

$$(3.13) \qquad \bar{P}\mathcal{O} \subset \mathcal{O}_0$$

and

$$(3.14) \qquad a(\bar{P}, \mathcal{O}) \subset \left(a(\bar{P}, \sigma) - \tfrac{1}{4}\varepsilon,\ a(\bar{P}, \sigma) + \tfrac{1}{4}\varepsilon\right).$$

The proof will be complete if one shows that

$$(3.15) \qquad P_1, P_2 \in \Pi, \quad P_1\sigma_0, P_2\sigma_0 \in \mathcal{O} \Rightarrow |a(P_1, \sigma_0) - a(P_2, \sigma_0)| < \varepsilon .$$

Accordingly, let $P_1, P_2 \in \Pi$ satisfy

$$(3.16) \qquad P_1\sigma_0, P_2\sigma_0 \in \mathcal{O} .$$

Then in view of (3.13) the processes $\bar{P}P_1$, $\bar{P}P_2$ satisfy

$$\bar{P}P_1\sigma_0, \bar{P}P_2\sigma_0 \in \mathcal{O}_0$$

and so by (3.12)

$$|a(\bar{P}P_1, \sigma_0)| < \varepsilon/4 ,$$
$$|a(\bar{P}P_2, \sigma_0)| < \varepsilon/4 .$$

In view of the additivity of a this may be rewritten as

$$|a(\bar{P}, P_1\sigma_0) + a(P_1, \sigma_0)| < \varepsilon/4 ,$$
$$|a(\bar{P}, P_2\sigma_0) + a(P_2, \sigma_0)| < \varepsilon/4 .$$

Moreover, (3.16) and (3.14) yield

$$|a(\bar{P}, \sigma) - a(\bar{P}, P_1\sigma_0)| < \varepsilon/4 ,$$
$$|a(\bar{P}, \sigma) - a(\bar{P}, P_2\sigma_0)| < \varepsilon/4 .$$

Eliminating $a(\bar{P}, P_1\sigma_0)$ and $a(\bar{P}, P_2\sigma_0)$ from the last four inequalities yields

$$|a(\bar{P}, \sigma) - a(P_1, \sigma_0)| < \varepsilon/2 ,$$
$$|a(\bar{P}, \sigma) - a(P_2, \sigma_0)| < \varepsilon/2$$

and these two inequalities yield (3.15). The proof is complete.

Definition 4.1.[2]) Let a be an action for a semi-system (Σ, Π). A real valued function A is called a *potential* for a if

(i) the domain of A is a dense subset $\mathscr{A}$ of Σ, and
(ii) whenever σ_1 and σ_2 are in $\mathscr{A}$, there is, for each $\varepsilon > 0$, a neighborhood $\mathscr{O}$ of σ_2 such that

$$(4.1) \qquad P \in \Pi\,, \quad P\sigma_1 \in \mathscr{O} \Rightarrow |A(\sigma_2) - A(\sigma_1) - a(P, \sigma_1)| < \varepsilon\,.$$

Proposition 4.1.[3]) *If an action a for a semi-system (Σ, Π) has a potential, then it has the conservation property at every state in the domain of A.*

Proof. If a is an action with a potential A, and if σ_0 is a state in the domain of A, then by applying item (ii) of Definition 4.1 to states $\sigma_1 = \sigma_2 = \sigma_0$ one finds, for each $\varepsilon > 0$, a neighborhood $\mathscr{O}$ of σ_0 such that (4.1) holds; since $\sigma_1 = \sigma_2 = \sigma_0$, the inequality in (4.1) reduces to

$$|a(P, \sigma_0)| < \varepsilon$$

and the proof is complete.

Proposition 4.1 shows that there is an immediate relation between the existence of a potential and the conservation property. Unless the potential is defined on the whole of Σ, no such direct relation exists between the existence of a potential and the path-independence of an action.

Proposition 4.2. *If an action a for a semi-system (Σ, Π) has a potential which is defined on the whole of Σ, then a is path-independent at every state.*

Proof. Let A be an everywhere defined potential for the action a, and let σ_0 and σ be states and $\varepsilon > 0$. By the definition of a potential, there exists a neighborhood $\mathscr{O}$ of σ such that

$$(4.2) \qquad P \in \Pi\,, \quad P\sigma_0 \in \mathscr{O} \Rightarrow |A(\sigma) - A(\sigma_0) - a(P, \sigma_0)| < \varepsilon/2\,.$$

To complete the proof, it suffices to show that

$$(4.3) \qquad |a(P_1, \sigma_0) - a(P_2, \sigma_0)| < \varepsilon$$

for each pair P_1, P_2 of processes satisfying

$$(4.4) \qquad P_1\sigma_0, P_2\sigma_0 \in \mathscr{O}\,.$$

But if P_1, P_2 satisfy (4.4), then (4.2) implies that

$$|A(\sigma) - A(\sigma_0) - a(P_1, \sigma_0)| < \varepsilon/2\,, \quad |A(\sigma) - A(\sigma_0) - a(P_2, \sigma_0)| < \varepsilon/2$$

and the last two inequalities yield (4.3), which completes the proof.

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[2]) Cf. [1].
[3]) Cf. [1], Theorems 3.2 and 4.3. The present proof is the same as that given in [1].

The following remark shows that the condition (ii) in the definition of a potential can be given a more classical form when one knows a priori that A is defined everywhere and is continuous.

Remark 4.1. *Let α be an action for a semi-system (Σ, Π) and let A be a real-valued function defined and continuous on Σ. If A satisfies*

$$A(P\sigma) - A(\sigma) = \alpha(P, \sigma)$$

for all $(P, \sigma) \in \Pi \,\square\, \Sigma$, then A is a potential for α.

Proof. Let σ_1, σ_2 be two states and $\varepsilon > 0$. By the continuity of A at σ_2 there exists a neighborhood $\mathcal{O}$ of σ_2 such that

$$A(\mathcal{O}) := \{A(\sigma): \sigma \in \mathcal{O}\} \subset (A(\sigma_2) - \varepsilon,\ A(\sigma_2) + \varepsilon)\,. \tag{4.5}$$

We now prove that with this neighborhood $\mathcal{O}$ of σ_2 the implication (4.1) in item (ii) of the definition of a potential is valid. Accordingly, let $P \in \Pi$ satisfy

$$P\sigma_1 \in \mathcal{O}\,. \tag{4.6}$$

By the hypothesis of the remark,

$$A(P\sigma_1) - A(\sigma_1) = \alpha(P, \sigma_1) \tag{4.7}$$

while by (4.6) and (4.5)

$$|A(P\sigma_1) - A(\sigma_2)| < \varepsilon\,. \tag{4.8}$$

By eliminating $A(P\sigma_1)$ in (4.8) by (4.7) one obtains the inequality in (4.1), and the proof is complete.

The main results of this note are contained in the following two theorems.

Theorem 4.1. *Let α be an action for a semi-system (Σ, Π). Then the following three conditions on an action α are equivalent:*

(i) *α is path-independent at least at one base state;*
(ii) *α is path-independent at each state;*
(iii) *there exists an everywhere defined, continuous potential A_0 for α.*

Moreover, if A is a potential for α whose domain $\mathcal{A}$ contains a base state, then A differs from A_0 by a constant on $\mathcal{A}$, i.e., there exists a $c \in \mathbb{R}$ such that

$$A(\sigma) = A_0(\sigma) + c$$

for all $\sigma \in \mathcal{A}$. Consequently, every potential whose domain contains a base state is continuous.

For systems the path-independence is equivalent to the conservation property and every state is a base state; the above theorem then takes on a simpler form:

Theorem 4.2.[4]) *Let α be an action for a system (Σ, Π). Then the following conditions on an action are equivalent:*

(i) *α has the conservation property at one state;*

(ii) *α has the conservation property at every state;*

(iii) *there exists an everywhere defined, continuous potential A_0 for α.*

Moreover, if A is a potential for α with domain $\mathscr{A}$, then A differs from A_0 by a constant on $\mathscr{A}$, i.e., there exists a $c \in \mathbb{R}$ such that

$$A(\sigma) = A_0(\sigma) + c \tag{4.9}$$

for all $\sigma \in \mathscr{A}$; consequently, every potential for $\mathscr{A}$ is continuous.

Proof of Theorem 4.1. We first establish the equivalence of conditions (i), (ii), and (iii) of Theorem 4.1. The implication (iii) ⇒ (ii) follows from Proposition 4.2, and the implication (ii) ⇒ (i) is trivial in view of the fact that each semi-system has at least one base state. Hence the proof of (i) ⇒ (iii) will establish the equivalence of (i), (ii) and (iii). Accordingly, assume that α is path-independent at a base state σ_0. Our aim is to construct an everywhere defined, continuous potential for α.

Let σ be a state, and introduce the following notation: $\mathfrak{S}(\sigma)$ denotes the set of all neighborhoods of σ,

$$\mathfrak{S}(\sigma) := \{\mathcal{O} \subset \Sigma \colon \mathcal{O} \text{ open}, \ \sigma \in \mathcal{O}\} \ ;$$

if $\mathcal{O}$ is an open subset of Σ, then $\alpha\{\sigma_0 \to \mathcal{O}\}$ denotes the set of numbers $\alpha(P, \sigma_0)$ obtained by letting P vary over the processes whose induced transformations take σ_0 into $\mathcal{O}$, i.e.

$$\alpha\{\sigma_0 \to \mathcal{O}\} := \{\alpha(P, \sigma_0) \colon P \in \Pi, \ P\sigma_0 \in \mathcal{O}\} \ ; \ ^{5})$$

$\mathfrak{A}(\sigma)$ denotes the intersection

$$\mathfrak{A}(\sigma) = \bigcap_{\mathcal{O} \in \mathfrak{S}(\sigma)} \overline{\alpha\{\sigma_0 \to \mathcal{O}\}} \ , \tag{4.10}$$

where the superposed bar denotes the closure of the corresponding set. Note that the fact that α is path-independent at σ_0 is expressed in terms of the sets $\alpha\{\sigma_0 \to \mathcal{O}\}$ as follows: for each $\sigma \in \Sigma$ and each $\varepsilon > 0$ there exists a neighborhood $\mathcal{O} \in \mathfrak{S}(\sigma)$ of σ such that

$$\operatorname{diam} \alpha\{\sigma_0 \to \mathcal{O}\} \leqq \varepsilon \ , \tag{4.11}$$

where diam M denotes the diameter of a set $M \subset \mathbb{R}$,

$$\operatorname{diam} M = \sup \{|x - y| \colon x, y \in M\} \ .$$

[4]) Cf. Theorems 4.1—4.5 of [1]. Instead of the implications (i) ⇒ (ii) & (iii) of the present theorem a weaker result is proved in [1] saying that the conservation property at one state implies the conservation property on a dense set of states and the existence of a densely defined continuous potential.

[5]) See Coleman & Owen [1].

We now prove that $\mathfrak{A}(\sigma)$ consists of exactly one point. To prove that $\mathfrak{A}(\sigma)$ is non-empty, note that the path-independence (4.11) implies in particular that there exists a neighborhood $\mathcal{O}_1 \in \mathfrak{S}(\sigma)$ of σ such that

$$\text{diam}\, a\{\sigma_0 \to \mathcal{O}_1\} \leqq 1 \,. \tag{4.12}$$

We denote by $\mathfrak{S}_1(\sigma)$ the set of those neighborhoods of σ which are contained in $\mathcal{O}_1$, i.e.

$$\mathfrak{S}_1(\sigma) := \{\mathcal{O} \in \mathfrak{S}(\sigma) \colon \mathcal{O} \subset \mathcal{O}_1\} \,.$$

Obviously

$$\overline{a\{\sigma_0 \to \mathcal{O}\}} \subset \overline{a\{\sigma_0 \to \mathcal{O}_1\}} \tag{4.13}$$

for all $\mathcal{O} \in \mathfrak{S}_1(\sigma)$ and hence (4.10) implies

$$\mathfrak{A}(\sigma) = \bigcap_{\mathcal{O} \in \mathfrak{S}_1(\sigma)} \overline{a\{\sigma_0 \to \mathcal{O}\}} \,. \tag{4.14}$$

Now by (4.12) the set $a\{\sigma_0 \to \mathcal{O}_1\}$ is bounded; hence $\overline{a\{\sigma_0 \to \mathcal{O}_1\}}$ is closed and bounded and thus compact; by (4.13) also all the sets

$$\overline{a\{\sigma_0 \to \mathcal{O}\}}\,, \quad \mathcal{O} \in \mathfrak{S}_1(\sigma)\,,$$

are compact. Suppose that $\mathfrak{A}(\sigma)$ is empty. This means, in view of (4.14), that the family

$$\overline{a\{\sigma_0 \to \mathcal{O}\}}\,, \quad \mathcal{O} \in \mathfrak{S}_1(\sigma)\,,$$

of compact subsets of a compact space

$$\overline{a\{\sigma_0 \to \mathcal{O}_1\}}$$

has empty intersection. By the finite-intersection property of compact spaces[6]), there exists a finite sequence $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ of neighborhoods from $\mathfrak{S}_1(\sigma)$ such that

$$\bigcap_{i=1}^{n} \overline{a\{\sigma_0 \to \mathcal{O}_i\}} = \emptyset \,. \tag{4.15}$$

But $\mathcal{O}_0 := \bigcap_{i=1}^{n} \mathcal{O}_i$ is again a neighborhood of σ and

$$\overline{a\{\sigma_0 \to \mathcal{O}_0\}} \subset \overline{a\{\sigma_0 \to \mathcal{O}_i\}} \tag{4.16}$$

for $i = 1, 2, \ldots, n$. Since σ_0 is a base state, $\overline{a\{\sigma_0 \to \mathcal{O}_0\}}$ is non-empty,

$$\overline{a\{\sigma_0 \to \mathcal{O}_0\}} \neq \emptyset \,, \tag{4.17}$$

and (4.15), (4.16), and (4.17) establish the desired contradiction showing that $\mathfrak{A}(\sigma)$ is non-empty.

The definition of $\mathfrak{A}(\sigma)$ implies that

$$\mathfrak{A}(\sigma) \subset \overline{a\{\sigma_0 \to \mathcal{O}\}} \tag{4.18}$$

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[6]) A property dual to the finite covering property; use the finite covering property and de Morgan laws.

for each $\mathcal{O} \in \mathfrak{S}(\sigma)$. Then by (4.11) for each $\varepsilon > 0$ there exists $\mathcal{O} \in \mathfrak{S}(\sigma)$ such that

$$\operatorname{diam} \alpha\{\sigma_0 \to \mathcal{O}\} \leqq \varepsilon \tag{4.19}$$

and hence (4.18) and (4.19) yields that

$$\operatorname{diam} \mathfrak{A}(\sigma) \leqq \varepsilon$$

for each $\varepsilon > 0$, i.e.,

$$\operatorname{diam} \mathfrak{A}(\sigma) = 0$$

and so $\mathfrak{A}(\sigma)$ consists of exactly one point. We denote this point by $A_0(\sigma)$,

$$\mathfrak{A}(\sigma) = \{A_0(\sigma)\} .$$

So A_0 is a real-valued function defined on Σ. It is clear from the definition of A_0 that

$$A_0(\sigma) \in \overline{\alpha\{\sigma_0 \to \mathcal{O}\}} \tag{4.20}$$

for each $\sigma \in \Sigma$ and $\mathcal{O} \in \mathfrak{S}(\sigma)$.

We now prove that A_0 is continuous. Let σ be a state and $\varepsilon > 0$. By the path-independence of α at σ_0, i.e. by (4.11), there exists a neighborhood $\mathcal{O} \in \mathfrak{S}(\sigma)$ such that

$$\operatorname{diam} \alpha\{\sigma_0 \to \mathcal{O}\} \leqq \varepsilon . \tag{4.21}$$

We show that

$$|A_0(\sigma') - A_0(\sigma)| \leqq \varepsilon \tag{4.22}$$

for all $\sigma' \in \mathcal{O}$. Indeed, if $\sigma' \in \mathcal{O}$, then $\mathcal{O} \in \mathfrak{S}(\sigma')$ and hence applying (4.20) to the state σ' one obtains

$$A_0(\sigma') \in \overline{\alpha\{\sigma_0 \to \mathcal{O}\}} , \tag{4.23}$$

while applying (4.20) to the state σ yields

$$A_0(\sigma) \in \overline{\alpha\{\sigma_0 \to \mathcal{O}\}} . \tag{4.24}$$

Relations (4.21), (4.23), and (4.24) obviously yield (4.22) and the proof of the continuity of A_0 is complete.

Next we prove that A_0 is a potential for α. Accordingly, let σ_1 and σ_2 be two states and $\varepsilon > 0$. By the path-independence of α at σ_0, i.e. by (4.11), there exists a neighborhood $\mathcal{O}$ of σ_2 such that

$$\operatorname{diam} \alpha\{\sigma_0 \to \mathcal{O}\} < \varepsilon/3 . \tag{4.25}$$

We want to show that

$$P \in \Pi , \quad P\sigma_1 \in \mathcal{O} \Rightarrow |A_0(\sigma_2) - A_0(\sigma_1) - \alpha(P, \sigma_1)| < \varepsilon . \tag{4.26}$$

Accordingly, let $P \in \Pi$ satisfy

$$P\sigma_1 \in \mathcal{O} . \tag{4.27}$$

By the continuity of ϱ_P and $\alpha(P, \cdot)$ there exists a neighborhood $\mathcal{O}_1$ of σ_1 such that

$$P\mathcal{O}_1 \subset \mathcal{O} \tag{4.28}$$

and

$$\alpha(P, \mathcal{O}_1) \subset (\alpha(P, \sigma_1) - \varepsilon/3,\ \alpha(P, \sigma_1) + \varepsilon/3), \tag{4.29}$$

in view of (4.11) one can choose $\mathcal{O}_1$ so small that

$$\operatorname{diam} \alpha\{\sigma_0 \to \mathcal{O}_1\} < \varepsilon/3 . \tag{4.30}$$

Finally, since σ_0 is a base state, there exists a $P_0 \in \Pi$ such that

$$P_0\sigma_0 \in \mathcal{O}_1 . \tag{4.31}$$

(4.31) and (4.28) yield

$$PP_0\sigma_0 \in \mathcal{O}$$

and so

$$\alpha(P, P_0\sigma_0) + \alpha(P_0, \sigma_0) = \alpha(PP_0, \sigma_0) \in \alpha\{\sigma_0 \to \mathcal{O}\} , \tag{4.32}$$

while

$$\alpha(P_0, \sigma_0) \in \alpha\{\sigma_0 \to \mathcal{O}_1\} . \tag{4.33}$$

Now the definition of $A_0(\sigma_2)$ and $A_0(\sigma_1)$ yields

$$A_0(\sigma_2) \in \overline{\alpha\{\sigma_0 \to \mathcal{O}\}} \tag{4.34}$$

and

$$A_0(\sigma_1) \in \overline{\alpha\{\sigma_0 \to \mathcal{O}_1\}} . \tag{4.35}$$

Then (4.34), (4.32), and (4.25) yield

$$|\alpha(P, P_0\sigma_0) + \alpha(P_0, \sigma_0) - A_0(\sigma_2)| \leqq \varepsilon/3 , \tag{4.36}$$

while (4.35), (4.33), and (4.30) yield

$$|\alpha(P_0, \sigma_0) - A_0(\sigma_1)| \leqq \varepsilon/3 . \tag{4.37}$$

Finally, (4.31) and (4.29) yield

$$|\alpha(P, \sigma_1) - \alpha(P, P_0\sigma_0)| < \varepsilon/3 \tag{4.38}$$

and inequalities (4.38), (4.37), and (4.36) yield (4.26). This shows that A_0 is a potential for α. Thus, an explicit construction of a function A_0 having all the properties required in (iii) has been given. The proof of the equivalence of (i), (ii), and (iii) is now complete.

The only thing that now remains to be proved is that if $A: \mathscr{A} \to \mathbb{R}$ is a potential for a such that $\mathscr{A}$ contains a base state, then there exists a $c \in \mathbb{R}$ such that (4.9) holds. Let $\sigma_0 \in \mathscr{A}$ be a base state and set

$$c = A(\sigma_0) - A_0(\sigma_0) .$$

Then, if $\sigma \in \mathscr{A}$ and $\varepsilon > 0$, there exists a neighborhood $\mathcal{O}$ of σ such that

$$|A(\sigma) - A(\sigma_0) - a(P, \sigma_0)| < \varepsilon/2 , \tag{4.39}$$

$$|A_0(\sigma) - A_0(\sigma_0) - a(P, \sigma_0)| < \varepsilon/2 \tag{4.40}$$

for all $P \in \Pi$ with $P\sigma_0 \in \mathcal{O}$; this follows from the fact that both A and A_0 are potentials. Since σ_0 is a base state, one can be sure that there exists at least one $P \in \Pi$ with $P\sigma_0 \in \mathcal{O}$. Eliminating then $a(P, \sigma_0)$ from (4.39) and (4.40) yields

$$|A(\sigma) - A(\sigma_0) - [A_0(\sigma) - A_0(\sigma_0)]| < \varepsilon$$

and as this inequaity must be satisfied for all $\varepsilon > 0$, one has

$$A(\sigma) - A(\sigma_0) - [A_0(\sigma) - A_0(\sigma_0)] = 0$$

which, in view of the definition of c, yields (4.9). The proof of the theorem is complete.

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Souhrn

KONZERVATIVNÍ AKCE

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Článek se zabývá akcemi na termodynamických systémech. Je dokázáno, že akce, která je konzervativní v jednom stavu, je konzervativní v každém stavu a má všude definovaný spojitý potenciál. Je dokázán analogický výsledek pro semi-systémy.

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