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ON SPECTRAL BANDWIDTH OF A STATIONARY RANDOM PROCESS

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1. INTRODUCTION

Technical applications of the theory of stationary random processes use numerical characteristics to express the spectral bandwidth of a process. One of these characteristics is the so-called irregularity coefficient. In this paper the basic properties of this coefficient are investigated and the application to the dichotomic classification into narrow-band or wide-band processes is considered. The behaviour of the irregularity coefficient is interesting especially in connection with sufficiently wide classes of stationary processes whose spectral densities are encountered both in theory and in practice. Two classes most often used have been chosen here and we shall be concerned with their properties from the point of view of spectral bandwidth.

2. THE IRREGULARITY COEFFICIENT

Let us consider a real continuous stationary process $\{X_t, t \geqq 0\}$ (briefly, stationary process) with the standardized spectral density $s(\omega)$, i.e.

$$\int_{-\infty}^{\infty} s(\omega)\,\mathrm{d}\omega = 1\,,$$

and the standardized spectral moments

$$\lambda_i = \int_{-\infty}^{\infty} \omega^i\, s(\omega)\,\mathrm{d}\omega\,, \quad i = 0, 1, 2, \ldots.$$

To describe the spectral structure of a stationary process the following numerical characteristic will be used

$$I = \frac{\lambda_2}{\sqrt{\lambda_4}}\,,$$

which is called the irregularity coefficient. This characteristic is also encountered in different forms and under different names (see [1], [2], [3]). The irregularity coefficient satisfies the following inequalities given in

Theorem 1. *For an arbitrary stationary process X_t the irregularity coefficient lies within the interval*

$$0 \leqq I \leqq 1 .$$

For a compound stationary process

$$X_t = \sum_{j=1}^{n} X_j(t)$$

with uncorrelated stationary processes $X_j(t)$ its irregularity coefficient I is equal at most to the largest of all irregularity coefficients I_j of its components $X_j(t)$, i.e.

$$I \leqq \max(I_1, I_2, \ldots, I_n) .$$

Proof. As the spectral moments λ_2 and λ_4 are nonnegative, $I \geqq 0$. The moment λ_i may be interpreted in the form (E denotes the mean value operator)

$$\lambda_i = E\omega^i ,$$

thus according to Schwarz' inequality

$$E\omega^2 \leqq (E1^2 E\omega^4)^{1/2} = \sqrt{(E\omega^4)} ,$$

or $I \leqq 1$.

Let us prove the second part of the theorem. Each function $f(x)$ convex within an arbitrary interval satisfies the inequality

$$f(\sum_{j=1}^{n} p_j x_j) \leqq \sum_{j=1}^{n} p_j f(x_j) , \quad p_j > 0 , \quad \sum_{j=1}^{n} p_j = 1 ,$$

and there always exists $m \in \{1, 2, \ldots, n\}$ such that

$$I_j = \frac{\lambda_{2,j}}{\sqrt{\lambda_{4,j}}} \leqq I_m = \frac{\lambda_{2,m}}{\sqrt{\lambda_{4,m}}} , \quad j = 1, 2, \ldots, n ,$$

where $\lambda_{i,j}$ is the i-th standardized spectral moment of the stationary process $X_j(t)$. Therefore we have

$$(\sum_{j=1}^{n} p_j \lambda_{2,j})^2 \leqq \sum_{j=1}^{n} p_j \lambda_{2,j}^2 \leqq \frac{\lambda_{2,m}^2}{\lambda_{4,m}} \sum_{j=1}^{n} p_j \lambda_{4,j} ,$$

and because

$$I^2 = \frac{(\sum_{j=1}^{n} p_j \lambda_{2,j})^2}{\sum_{j=1}^{n} p_j \lambda_{4,j}} , \quad p_j = \frac{D^2 X_j(t)}{D^2 X_t} ,$$

we conclude the inequality $I \leqq I_m$. □

With the aid of the irregularity coefficient the "sharp" boundary foı dichotomic classification of the stationary process into narrow-band and wide-band processes may be obtained.

Definition 1. *Let us choose a number I^0 near to one (according to convention). The stationary process will be called a narrow-band one if the irregularity coefficient $I \in \langle I^0, 1\rangle$ and a wide-band one if $I \in (0, I^0)$. We shall say that I^0 is the boundary of the spectral bandwidth of the process.* □

For a normal process the boundary I^0 may be determined by using the distribution function of a local maximum of the process. Let us consider, without loss of generality, a standardized process ($EX_t = 0$, $D^2X_t = 1$), for which the distribution function of its local maximum [1]

$$G(u) = F\left(\frac{u}{W}\right) - Ie^{-u^2/2}F\left(\frac{I}{W}u\right), \quad u \in R_1,$$

where $W = \sqrt{(1 - I^2)}$ and F is the distribution function of the normal distribution $N(0, 1)$. The value of the distribution function G at the point 0

$$G^0 = G(0) = \tfrac{1}{2}(1 - I^0), \quad I = I^0,$$

gives the probability of a negative local maxima of the stationary process, while $G^0(I^0)$ is a decreasing function. Small probability G^0 characterizes the narrow-band process (see Section 4) and if we choose, according to convention, for instance $G^0 = 0{\cdot}025$, the boundary $I^0 = 0{\cdot}95$ is obtained.

3. THE CLASSES OF STATIONARY PROCESSES

The behaviour of the irregularity coefficient is of interest especially in connection with sufficiently wide classes of stationary processes which are widely applied in technical practice.

Definition 2. *A stationary process X_t belongs to the class C_1 if its standardized spectral density is of the form*

$$(1) \qquad s(\omega) = \tfrac{1}{2}[v(-\omega - \omega_0) + v(\omega - \omega_0)], \quad \omega \in R_1, \quad \omega_0 \geqq 0,$$

where ω_0 is a constant and the generating density $v(\omega)$ is a unimodal function around the point $\omega = \omega^0$ with the properties of a probability density on R_1. The stationary process X_t belongs to the class C_1^0, if it belongs to the class C_1 and the function $v(\omega)$ is even (then $\omega^0 = 0$). Finally, the stationary process X_t belongs to the class $C_n(C_n^0)$ if it is a compound process with n components $X_j(t)$ of the class $C_1(C_1^0)$. □

Further characteristics of the process X_t of the classes C_1 and C_1^0 are given in

Theorem 2. *Let X_t be a stationary process of C_1. Then its correlation function is*

$$k(\tau) = a(\tau)\cos(\omega_0\tau + b(\tau)), \tag{2}$$

where

$$a(\tau) = (c^2(\tau) + d^2(\tau))^{1/2},$$

$$b(\tau) = \operatorname{arc\,tg}\frac{d(\tau)}{c(\tau)},$$

$$c(\tau) = \int_{-\infty}^{\infty} v(\omega)\cos\omega\tau\,d\omega,$$

$$d(\tau) = \int_{-\infty}^{\infty} v(\omega)\sin\omega\tau\,d\omega,$$

and $a(\tau)$ is itself a correlation function such that

$$\lim_{\tau\to\pm\infty} a(\tau) = 0. \tag{3}$$

The standardized spectral moment is

$$\lambda_i = \sum_{m=0}^{i}\binom{i}{m}\omega_0^m\mu_{i-m},$$

where

$$\mu_i = \int_{-\infty}^{\infty}\omega^i v(\omega)\,d\omega.$$

Let $X_t \in C_1^0$. Its correlation function then is

$$k(\tau) = c(\tau)\cos\omega_0\tau,$$

the standardized spectral moment

$$\lambda_i = \sum_{m=0}^{i/2}\binom{i}{2m}\omega_0^{2m}\mu_{i-2m},$$

and the irregularity coefficient

$$I = \frac{\varkappa^2 + 1}{(\gamma\varkappa^4 + 6\varkappa^2 + 1)^{1/2}}, \tag{4}$$

where

$$\varkappa = \frac{\sqrt{\mu_2}}{\omega_0} \quad \textit{and} \quad \gamma = \frac{\mu_4}{\mu_2^2} \quad (\varkappa \geqq 0,\ \gamma \geqq 1)$$

are called the spectral variation coefficient and the spectral excess coefficient, respectively.

Proof. Inserting (1) into the formula

$$k(\tau) = \int_{-\infty}^{\infty} s(\omega)\, e^{i\omega\tau}\, d\omega$$

we get

$$k(\tau) = \int_{-\infty}^{\infty} v(\omega) \cos(\omega + \omega_0)\, \tau\, d\omega = c(\tau) \cos \omega_0 \tau - d(\tau) \sin \omega_0 \tau$$

and thus (2). It is easy to verify that $a(\tau)$ is a correlation function.

Using the second mean value theorem of the integral calculus, we see that

$$\int_{-\infty}^{\omega^0} v(\omega) \cos \omega\tau\, d\tau = v(\omega^0) \int_{A_1}^{\omega^0} \cos \omega\tau\, d\tau\,,$$

$$\int_{\omega^0}^{\infty} v(\omega) \cos \omega\tau\, d\tau = v(\omega^0) \int_{\omega^0}^{A_2} \cos \omega\tau\, d\tau\,,$$

where the points A_1, A_2 fulfil $A_1 \in (-\infty, \omega^0)$ and $A_2 \in (\omega^0, \infty)$. These relations and the inequality (B_1, B_2 are arbitrary constants)

$$\left| \int_{B_1}^{B_2} \cos \omega\tau\, d\omega \right| \leqq \frac{2}{|\tau|}$$

then imply

$$|c(\tau)| = v(\omega^0) \left| \int_{A_1}^{A_2} \cos \omega\tau\, d\tau \right| \leqq \frac{2}{|\tau|}\, v(\omega^0)\,,$$

and in a similar manner,

$$|d(\tau)| = v(\omega^0) \left| \int_{A_1}^{A_2} \sin \omega\tau\, d\tau \right| \leqq \frac{2}{|\tau|}\, v(\omega^0)\,.$$

Both these inequalities imply the limit (3). Formulas for the remaining characteristics in the theorem may be verified immediately by means of (1) or by considering the fact that $v(\omega)$ is even. □

Note that the characteristics of a compoud stationary process X_t of the classes C_n and C_n^0 will be obtained from the known relations

$$k(\tau) = \sum_{j=1}^{n} p_j\, k_j(\tau)\,, \quad s(\omega) = \sum_{j=1}^{n} p_j\, s_j(\omega)\,, \quad \lambda_i = \sum_{j=1}^{n} p_j \lambda_{i,j}\,,$$

where

$$p_j = \frac{D^2 X_j(t)}{D^2 X_t}\,, \quad \sum_{j=1}^{n} p_j = 1\,,$$

and the characteristics with the index j correspond to the component $X_j(t)$.

4. SPECTRAL BANDWIDTH OF THE PROCESS OF THE CLASSES C_1^0 AND C_2^0

Both in theory and in applications the stationary processes of the classes C_1^0 and C_2^0 are most often encountered. Let us therefore consider their properties from the point of view of the spectral bandwidth.

1. Let $X_t \in C_1^0$ and let the generating density be $v(\omega) = \delta(\omega)$, $\delta(\omega)$ being the Dirac delta function. Then the spectral variation coefficient $\varkappa = 0$ and the process X_t has, according to (4), the irregularity coefficient $I = 1$. Such a process is said to be an *ideal narrow-band* one. An ideal narrow-band process is the harmonic process

$$X_t = A \cos(\omega_0 t + \Phi)$$

with a constant frequency $\omega_0 > 0$, with a random amplitude $A > 0$ and with a uniformly distributed random phase in the interval $(0, 2\pi)$ independent of A.

2. Let $X_t \in C_1^0$ be a narrow-band process. In this case X_t may be approximated by the relation [3]

$$X_t = A_t \cos(\omega_0 t + \Phi_t),$$

where $\omega_0 > 0$ is the constant frequency while the envelope $A_t > 0$ and the uniformly distributed phase Φ_t in the interval $(0, 2\pi)$ are independent processes that change slowly with time. A narrow-band process expressed in this way is called a quasi-harmonic one.

3. Let $X_t \in C_1^0$ with the spectral excess coefficient $\gamma \leqq 3$. For example, such is the process with the spectral density generated by the uniform, triangular or normal density $v(\omega)$ provided $\gamma = 9/5$, $12/5$ and 3, respectively. It is easy to see that for $\varkappa \geqq 0$ and $\gamma \leqq 3$ the irregularity coefficient $1 \geqq I \geqq 1/\sqrt{3} \doteq 0{\cdot}577$. Thus the process considered here is never an extremely wide-band one.

4. Because of the shape of the spectral density the ideal narrow-band process as well as the white noise are in many ways "extremes". This is not true for the white noise from the point of view of spectral bandwidth. Every limited white noise for $\omega_0 = 0$ and for an arbitrary uniform generating density $v(\omega)$ has $I = \sqrt{5/9} \doteq 0{\cdot}745$. Thus the white noise itself has the same value of I. It is then only a moderately wide-band process, and so not even an "extreme" in the class C_1^0 for $\gamma \leqq 3$.

5. Extremely wide-band processes may be found in the class C_2^0. Let us denote by a simple index, or by the second one in a double index, the correspondence to the component $X_j(t)$ of the compound process X_t. Then the process $X_t \in C_2^0$ has the irregularity coefficient

$$I = \frac{pq^2(\varkappa_1^2 + 1) + (1 - p)(\varkappa_2^2 + 1)}{[pq^4(\gamma_1\varkappa_1^4 + 6\varkappa_1^2 + 1) + (1 - p)(\gamma_2\varkappa_2^4 + 6\varkappa_2^2 + 1)]^{1/2}}, \tag{5}$$

where

$$\varkappa_j = \frac{\sqrt{\mu_{2,j}}}{\omega_j}, \quad \gamma_j = \frac{\mu_{4,j}}{\mu_{2,j}^2} \quad \left(\mu_{i,j} = \int_{-\infty}^{\infty} \omega^i\, v_j(\omega)\, \mathrm{d}\omega\right)$$

are the spectral variation coefficient and the spectral excess coefficient of the component $X_j(t)$, respectively, and

$$0 < p < 1 \,, \quad q = \frac{\omega_1}{\omega_2} \,, \quad 0 < q < 1 \,,$$

where ω_j is a constant in the standardized spectral density $s_j(\omega)$. Because $q \to 0$ implies $I \to (1 - p)^{1/2} I_2, I$ takes all the values within the interval $(0, I_2)$ and thus the class C_2^0 indeed contains extremely wide-band processes.

6. An *ideal wide-band* process $X_t \in C_2^0$ with $I = 0$ is not realizable (similarly as the white noise) since for finite $\varkappa_j$ and γ_j it follows immediately from (5) that $I > 0$.

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Souhrn

O SPEKTRÁLNÍ PÁSMOVOSTI STACIONÁRNÍHO NÁHODNÉHO PROCESU

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Součinitel nepravidelnosti je číselnou charakteristikou spektrální pásmovosti stacionárního náhodného procesu. Vyšetřují se jeho základní vlastnosti a uvádí se jeho použití pro dichotomické třídění procesu na úzkopásmový a širokopásmový proces. Chování součinitele nepravidelnosti se dále analýzuje pro dostatečně široké třídy stacionárních procesů s jejichž spektrálními hustotami se setkáváme jak v teorii tak v aplikacích.

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