Zdzisław Jackiewicz

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GLOBAL ERROR ESTIMATION IN THE NUMERICAL SOLUTION OF RETARDED DIFFERENTIAL EQUATIONS BY EULER'S METHOD

ZDZISLAW JACKIEWICZ

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1. INTRODUCTION

Consider the initial-value problem for the system of retarded ordinary differential equations

$$(1)\qquad \begin{aligned} y_i'(t) &= f_i(\bar{y}(\bar{\alpha}(t)))\,, & t \in [a, b]\,,\\ y_i(t) &= g_i(t)\,, & t \in [\alpha, a]\,, \end{aligned}$$

$i = 1, 2, \ldots, s$, where s is a positive integer. Here $\alpha \leqq a < b$, g_i are specified initial functions and

$$\bar{y}(\bar{\alpha}(t)) = (y_1(\alpha_{1,1}(t)), \ldots, y_1(\alpha_{1,k_1}(t)), \ldots, y_s(\alpha_{s,1}(t)), \ldots, y_s(\alpha_{s,k_s}(t)))\,.$$

Putting $y = [y_1, \ldots, y_s]^{\mathsf{T}}$, $y' = [y_1', \ldots, y_s']^{\mathsf{T}}$, $f = [f_1, \ldots, f_s]^{\mathsf{T}}$, $g = [g_1, \ldots, g_s]^{\mathsf{T}}$, where T stands for transposition, we can rewrite (1) in the vector form:

$$(1')\qquad \begin{aligned} y'(t) &= f(\bar{y}(\bar{\alpha}(t)))\,, & t \in [a, b]\,,\\ y(t) &= g(t)\,, & t \in [\alpha, a]\,. \end{aligned}$$

For $x \in R^q$ denote by $\|x\|$ the maximum norm. We assume the following:

H_1. The function f: $R^K \to R^s$, $K = k_1 + k_2 + \ldots + k_s$, is of class C^1 and there exists a constant $M < \infty$ such that

$$\|f(u)\| \leqq M\,, \quad \|f(u) - f(v)\| \leqq M\|u - v\|\,,$$
$$\|Df(u)\| \leqq M\,, \quad \|Df(u) - Df(v)\| \leqq M\|u - v\|$$

for $u, v \in R^K$.

H_2. The functions $\alpha_{i,j} : [a, b] \to [\alpha, b]$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, k_i$, are Lipschitz-continuous with constant $Q < \infty$, i.e.,

$$|\alpha_{i,j}(t_1) - \alpha_{i,j}(t_2)| \leqq Q|t_1 - t_2|$$

for $t_1, t_2 \in [a, b]$.

Let a fixed $h \in (0, h_0]$, $h_0 > 0$ be given. To compute an approximate solution $y_h : [\alpha, b] \to R^s$ consider Euler's method defined by

$$y_h(t_n + rh) = y_h(t_n) + rh\, f(\bar{y}_h(\bar{\alpha}(t_n))) ,$$
$$y_h(t) = g_h(t) , \quad t \in [\alpha, a] , \tag{2}$$

$n = 0, 1, \ldots, N - 1$, $r \in [0, 1]$, $Nh = b - a$, $t_n = a + nh$. Here g_h is some continuous approximation to the initial function g.

To obtain an estimate of the global error $e_h(t) = y_h(t) - y(t)$ we use the method of Zadunaisky (see [8], [7]). This method consists in the following. We construct the pseudo-problem

$$u'(t) = f(\bar{u}(\bar{\alpha}(t))) + d_h(t) , \quad t \in [a, b] ,$$
$$u(t) = g(t) , \quad t \in [\alpha, a] , \tag{3}$$

in such a way that the exact solution u of this problem is known in advance and the defect function d_h is "small". This construction will be described in § 2. Denote by e_h^* the global error committed in the numerical solution of (3) by (2). Then, under certain conditions, e_h^* is a good estimate of e_h. This result is stated in § 2 and its proof is given in § 3. In § 4 some numerical examples are given.

2. GLOBAL ERROR ESTIMATION

Assume that N is even and consider a piecewise polynomial interpolation of degree two to the numerical solutions $\{y_{i,h}(t_n)\}_{n=0}^N$, $i = 1, 2, \ldots, s$. In vector notation this can be written as

$$P(t) = P^m(t) = a_0^m + (t - t_{2m})\left(a_1^m + (t - t_{2m+1})\, a_2^m\right) , \quad t \in [t_{2m}, t_{2m+2}] .$$

Here, a_j^m, $j = 0, 1, 2$, are divided differences given by

$$a_0^m = [t_{2m}; y_h] = y_h(t_{2m}) ,$$
$$a_1^m = [t_{2m}, t_{2m+1}; y_h] = f(\bar{y}_h(\bar{\alpha}(t_{2m}))) ,$$
$$a_2^m = [t_{2m}, t_{2m+1}, t_{2m+2}; y_h] = \frac{1}{2h}\left[f(\bar{y}_h(\bar{\alpha}(t_{2m+1}))) - f(\bar{y}_h(\bar{\alpha}(t_{2m})))\right] .$$

Consider now the pseudo-problem defined by

$$u'(t) = f(\bar{u}(\bar{\alpha}(t))) + d_h(t) , \quad t \in [t_{2m}, t_{2m+2}) ,$$
$$u(t) = g(t) , \quad t \in [\alpha, a] , \tag{4}$$

where

$$d_h(t) = P'(t) - f(\bar{P}(\bar{\alpha}(t))) , \quad t \in [t_{2m}, t_{2m+2}) .$$

By $u'(t_{2m})$ and $P'(t_{2m})$ we mean the right hand side derivatives. It is obvious that P

is the continuous solution of this problem. The method (2) applied to (4) takes the form

$$u_h(t_n + rh) = u_h(t_n) + rh[f(\bar{u}_h(\bar{\alpha}(t_n))) + d_h(t_n)],$$
$$u_h(t) = g_h(t), \quad t \in [\alpha, a],$$

$n = 0, 1, \ldots, N - 1$, $r \in [0, 1]$. Put $e_h^*(t) = u_h(t) - P(t)$. We have the following.

Theorem. *Assume that H_1 and H_2 hold. Then $e_h(t) = e_h^*(t) + 0(h^2)$ as $h \to 0$.*

This theorem generalizes some of the results obtained by Frank [2] and Frank/Ueberhuber [3] for ordinary differential equations. In [6] a similar result was obtained for Volterra integro-differential equations. The proof of this theorem is given in the next section and, as in [6], consists in checking if the method (2) possesses the "property (E)" defined by Stetter [7] (see also [8]).

3. THE PROOF OF THEOREM

We assume throughout this section that the conditions H_1 and H_2 are fulfilled and that N is even. Similarly as in [2] the proof is divided into a sequence of Lemmas.

Lemma 1. *There exists a constant $A < \infty$ independent of m and h such that $\|a_j^m\| \leqq A$ for $m = 0, 1, \ldots, N/2 - 1$; $j = 0, 1, 2$.*

Proof. The proof for $j = 0$ and $j = 1$ is obvious. For $j = 2$, using H_1, we obtain

$$\|a_2^m\| \leqq \frac{M}{2h} \|\bar{y}_h(\bar{\alpha}(t_{2m+1})) - \bar{y}_h(\bar{\alpha}(t_{2m}))\| .$$

It is easy to see that the function y_h is Lipschitz-continuous with constant M. This yields

$$\|a_2^m\| \leqq \frac{M^2}{2h} \|\bar{\alpha}(t_{2m+1}) - \bar{\alpha}(t_{2m})\| \leqq \frac{M^2 Q}{2h} |t_{2m+1} - t_{2m}| = \tfrac{1}{2} M^2 Q .$$

Here, $\bar{\alpha}(t) = (\alpha_{1,1}(t), \ldots, \alpha_{1,k_1}(t), \ldots, \alpha_{s,1}(t), \ldots, \alpha_{s,k_s}(t))$.

Lemma 2. $\|d_h(t)\| = 0(h)$ *as $h \to 0$ for $t \in [a, b]$.*

Proof. For $t \in [t_{2m}, t_{2m+2})$ we get

$$d_h(t) = (P^m)'(t) - f(\bar{P}^m(\bar{\alpha}(t))) =$$
$$= a_1^m + a_2^m[(t - t_{2m}) + (t - t_{2m+1})] - f(\bar{y}_h(\bar{\alpha}(t_{2m})) + \bar{P}^m(\bar{\alpha}(t)) - \bar{y}_h(\bar{\alpha}(t_{2m}))) =$$
$$= a_2^m[(t - t_{2m}) + (t - t_{2m+1})] - D f(\eta(t)) (\bar{P}^m(\bar{\alpha}(t)) - \bar{y}_h(\bar{\alpha}(t_{2m}))) ,$$

where $\eta(t) \in R^K$ lies between $\bar{P}(\bar{\alpha}(t))$ and $\bar{y}(\bar{\alpha}(t_{2m}))$. In view of Lemma 1 and H_1

we obtain

$$\|d_h(t)\| \leqq 2Ah + M\|\bar{P}^m(\bar{\alpha}(t)) - \bar{y}_h(\bar{\alpha}(t_{2m}))\| .$$

We have to estimate the quantities $|P_i(\alpha_{i,j}(t)) - y_{i,h}(\alpha_{i,j}(t_{2m}))|$ for $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, k_i$. For any i, j, $\alpha_{i,j}(t) \in [t_{2v}, t_{2v+2}]$ for some $v = v(i, j) \leqq m$. We have

$$|P_i(\alpha_{i,j}(t)) - y_{i,h}(\alpha_{i,j}(t_{2m}))| =$$
$$= |a^v_{i,0} + (\alpha_{i,j}(t) - t_{2v})(a^v_{i,1} + (\alpha_{i,j}(t) - t_{2v+1})\, a^v_{i,2}) - y_{i,h}(\alpha_{i,j}(t_{2m}))| \leqq$$
$$\leqq |y_{i,h}(t_{2v}) - y_{i,h}(\alpha_{i,j}(t_{2m}))| + 2h(A + Ah) \leqq 2hM + 2hA + 0(h^2) .$$

Finally,

$$\|\bar{P}^m(\bar{\alpha}(t)) - \bar{y}_h(\bar{\alpha}(t_{2m}))\| = 0(h) \quad \text{and} \quad \|d_h(t)\| = 0(h) \quad \text{as} \quad h \to 0 .$$

Lemma 3. *Denote by e the solution of the problem*

$$(5) \qquad \begin{aligned} e'(t) &= D f(\bar{y}(\alpha(t)))\, \bar{e}(\bar{\alpha}(t)) - \tfrac{1}{2} y''(t) , & t \in [a, b] , \\ e(t) &= 0 , & t \in [\alpha, a] , \end{aligned}$$

where y is the solution of (1). *Then* $e_h(t_n + rh) = he(t_n + rh) + 0(h^2)$ *as* $h \to 0$.

Proof. Define the local error $\mu(t_n, r, h)$ of the method (2) at the point $t_n + rh$ by

$$(6) \qquad y(t_n + rh) = y(t_n) + rh f(\bar{y}(\bar{\alpha}(t_n))) + \mu(t_n, r, h) ,$$

$n = 0, 1, \ldots, N - 1$, $r \in [0, 1]$. After simple calculations we obtain

$$\mu(t_n, r, h) = y''(t_n) \frac{r^2 h^2}{2} + 0(h^3) \quad \text{as} \quad h \to 0 .$$

Subtracting (6) from (2) we get

$$e_h(t_n + rh) = e_h(t_n) + rh[f(\bar{y}_h(\bar{\alpha}(t_n))) - f(\bar{y}(\bar{\alpha}(t_n)))] - \tfrac{1}{2} r^2 h^2\, y''(t_n) + 0(h^3) .$$

Routine manipulations yield

$$e_h(t_n + rh) = e_h(t_n) + rh[D f(\bar{y}(\bar{\alpha}(t_n)))\, \bar{e}_h(\bar{\alpha}(t_n)) +$$
$$+ \tfrac{1}{2} D^2 f(\xi)\, (\bar{e}_h(\bar{\alpha}(t_n)), \bar{e}_h(\bar{\alpha}(t_n)))] - \tfrac{1}{2} r^2 h^2\, y''(t_n) + 0(h^3) =$$
$$= e_h(t_n) + rhD f(\bar{y}(\bar{\alpha}(t_n)))\, \bar{e}_h(\bar{\alpha}(t_n)) - \tfrac{1}{2} r^2 h^2\, y''(t_n) + 0(h^3) .$$

Let $\tilde{e_h}(t_n + rh) = e_h(t_n + rh)/h$. Then

$$(7) \qquad \tilde{e_h}(t_n + rh) = \tilde{e_h}(t_n) + rh[D f(\bar{y}(\bar{\alpha}(t_n)))\, \bar{\tilde{e}}_h(\bar{\alpha}(t_n)) - \tfrac{1}{2} r\, y''(t_n)] + 0(h^2) .$$

Putting $\tilde{e_h}(t) = 0$ for $t \in [\alpha, a]$ we can look at (6) as the result of applying to the equation (5) some numerical method with additional error of order two. Similarly as in [6] it is easy to check that this method is consistent with order one. Consequently, it follows from Theorem 5 of [5] that $\|\tilde{e_h}(t_n + rh) - e(t_n + rh)\| = 0(h)$ as $h \to 0$ or $e_h(t_n + rh) = h\, e(t_n + rh) + 0(h^2)$, which is our claim.

Lemma 4. *Denote by e^* the continuous solution of the problem*

$$(8) \qquad \begin{aligned} (e^*)'(t) &= Df(\bar{P}(\bar{\alpha}(t)))\,\bar{e}^*(\bar{\alpha}(t)) - \tfrac{1}{2}P''(t)\,, & t \in [t_{2m}, t_{2m+2})\,,\\ e^*(t) &= 0\,, & t \in [\alpha, a]\,, \end{aligned}$$

$m = 0, 1, \ldots, N/2 - 1$, *where* P *is the solution of* (4). *Then* $e_h^*(t_n + rh) = h\,e^*(t_n + rh) + 0(h^2)$ *as* $h \to 0$ *for* $n = 0, 1, \ldots, \mathrm{N} - 1$, $r \in [0, 1]$.

Proof. The proof of this lemma is similar to that of Lemma 3 is therefore omitted. Compare with Lemma 7 in [6].

The next lemma is a generalization of Gronwall's inequality.

Lemma 5. *Assume that* $w_i(t) \geqq 0$, $i = 1, 2, \ldots, s$, $t \in [\alpha, a]$ *and*

$$w_i(t) \leqq B \int_a^t \sum_{i=1}^{s} \sum_{j=1}^{k_i} w_i(\alpha_{i,j}(x))\,\mathrm{d}x + C\,, \quad t \in [a, b]\,,$$

where B *and* C *are nonnegative constants. Then*

$$w_i(t) \leqq C \exp\left(BK(t - a)\right), \quad t \in [a, b]\,.$$

Proof. It follows from the theory of integral inequalities that $w_i(t) \leqq W_i(t)$, $t \in [\alpha, b]$, where W_i are functions satisfying the equations

$$\begin{aligned} W_i(t) &= B \int_a^t \sum_{i=1}^{s} \sum_{j=1}^{k_i} W_i(\alpha_{i,j}(x))\,\mathrm{d}x + C\,, & t \in [a, b]\,,\\ W_i(t) &= w_i(t)\,, & t \in [\alpha, a]\,. \end{aligned}$$

It is easy to see that the functions W_i are nondecreasing for $t \in [a, b]$. This yields

$$W_i(t) \leqq B \int_a^t \sum_{i=1}^{s} \sum_{j=1}^{k_i} W_i(x)\,\mathrm{d}x + C = B \int_a^t \sum_{i=1}^{s} k_i\, W_i(x)\,\mathrm{d}x + C\,, \quad t \in [a, b]\,.$$

Now, after simple calculations, the result follows from Gronwall's inequality.

Lemma 6. $\|y(t) - P(t)\| = 0(h)$ *and* $\|y'(t) - P'(t)\| = 0(h)$ *as* $h \to 0$ *for* $t \in [a, b]$.

Proof. Integrating (1′) and (4) we obtain

$$y(t) = y(a) + \int_a^t f(\bar{y}(\bar{\alpha}(x)))\,\mathrm{d}x\,, \qquad t \in [a,b]\,,$$

$$P(t) = P(a) + \int_a^t f(\bar{P}(\bar{\alpha}(x)))\,\mathrm{d}x + \int_a^t d_h(x)\,\mathrm{d}x\,, \quad t \in [a, b]\,.$$

Subtracting these equations and using H_1 we get

$$|y_i(t) - P_i(t)| \leqq \int_a^t M \sum_{i=1}^{s} \sum_{j=1}^{k_i} |y_i(\alpha_{i,j}(x)) - P_i(\alpha_{i,j}(x))|\,\mathrm{d}x + C\,,$$

where $C = (b - a) \sup \{\|d_h(x)\| : x \in [a, b]\}$. Putting $w_i(t) = |y_i(t) - P_i(t)|$, we obtain from Lemma 5 that

$$w_i(t) \leqq C \exp (MK(b - a)),$$

$i = 1, 2, \ldots, s$. This proves the first part of the lemma. The second part follows from the inequality

$$\|y'(t) - P'(t)\| \leqq M\|\bar{y}(\bar{\alpha}(t)) - \bar{P}(\bar{\alpha}(t))\| + \|d_h(t)\|, \quad t \in [a,b].$$

Lemma 7. $e^*(t) = e(t) + 0(h)$ *as* $h \to 0$ *for* $t \in [a, b]$.

Proof. Integrating (5) and (8) and subtracting the resulting equations we obtain

$$|e_i(t) - e_i^*(t)| \leqq \int_a^t |D f_i(\bar{y}(\bar{\alpha}(x)))\, \bar{e}(\bar{\alpha}(x)) - D f_i(\bar{P}(\bar{\alpha}(x)))\, \bar{e}^*(\bar{\alpha}(x))| \,\mathrm{d}x +$$

$$+ \tfrac{1}{2}(|y_i'(t) - P_i'(t)| + |y_i'(a) - P_i'(a)|), \quad t \in [a, b].$$

Putting $E = \sup \{\|\bar{e}^*(\bar{\alpha}(x))\| : x \in [a, b]\}$ we get

$$\begin{aligned} &|D f_i(\bar{y}(\bar{\alpha}(x)))\, \bar{e}(\bar{\alpha}(x)) - D f_i(\bar{P}(\bar{\alpha}(x)))\, \bar{e}^*(\bar{\alpha}(x))| \leqq \\ &\leqq |D f_i(\bar{y}(\bar{\alpha}(x)))\, \bar{e}(\bar{\alpha}(x)) - D f_i(\bar{y}(\bar{\alpha}(x)))\, \bar{e}^*(\bar{\alpha}(x))| + \\ &+ |D f_i(\bar{y}(\bar{\alpha}(x)))\, \bar{e}^*(\bar{\alpha}(x)) - D f_i(\bar{P}(\bar{\alpha}(x)))\, \bar{e}^*(\bar{\alpha}(x))| \leqq \\ &\leqq M\|\bar{e}(\bar{\alpha}(x)) - \bar{e}^*(\bar{\alpha}(x))\| + ME\|\bar{y}(\bar{\alpha}(x)) - \bar{P}(\bar{\alpha}(x))\|. \end{aligned}$$

Hence, in view of Lemma 6,

$$|e_i(t) - e_i^*(t)| \leqq M \int_a^t \sum_{i=1}^{s} \sum_{j=1}^{k_i} |e_i(\alpha_{i,j}(x)) - e_i^*(\alpha_{i,j}(x))| \,\mathrm{d}x + 0(h)$$

as $h \to 0$. Now the desired conclusion follows from Lemma 5.

Proof of Theorem. The theorem follows immediately from Lemmas 3, 4, and 7. Compare also the proof of Theorem 2 in [6].

4. NUMERICAL EXAMPLES

Example 1 (Hill [4]).

$$y'(t) = -[y(t/(1 + 2t)^2)]^{(1+2t)^2}, \quad t \in [0, 1],$$

$$y(0) = 1.$$

The exact solution is $y(t) = -\exp(t)$.

Example 2 (Bellman, Buell, Kalaba [1]).

$$y'(t) = -y(t - \exp(-t) - 1) + [\cos(t) + \sin(t - \exp(-t) - 1)] ,$$
$$t \in [0, 1] ,$$
$$y(t) = \sin(t) , \qquad t \in [-2, 0] .$$

The solution is $y(t) = \sin(t)$.

Example 3.

$$y'(t) = -2 \tan(t/2)\, y^2(t/2) , \quad t \in [0, 1]$$
$$y(0) = 1 .$$

The exact solution is $y(t) = \cos(t)$.

Example 4.

$$y'(t) = \exp(y(\alpha(t)))/(t^2 + 4t + 3) , \quad t \in [0, 1] ,$$
$$y(t) = \ln(2 + t) , \qquad t \in [-1/2, 0] ,$$

where $\alpha(t) = t - 1/(2 + t)$. The solution is $y(t) = \ln(2 + t)$.

The results of computations are given in the tables below, where $E := e_h(b) - e_h^*(b)$. These results confirm the Theorem given in § 2.

Table 1. Results for Example 1

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-2}	0·087 289	−0·421 262	8·13
2^{-3}	0·067 509	0·028 448	2·49
2^{-4}	0·023 172	0·034 392	−0·57
2^{-5}	0·015 584	0·014 799	−0·80
2^{-6}	0·007 698	0·006 959	3·03
2^{-7}	0·003 827	0·003 384	7·25

Table 2. Results for Example 2

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-2}	−0·120 152	−0·042 020	−1·25
2^{-3}	−0·072 510	−0·048 020	−1·56
2^{-4}	−0·039 603	−0·032 947	−1·70
2^{-5}	−0·020 665	−0·018 941	−1·76
2^{-6}	−0·010 551	−0·010 114	−1·78
2^{-7}	−0·005 328	−0·005 218	−1·78

Table 3. Results for Example 3

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-2}	−0·269 199	−0·160 093	−1·74
2^{-3}	−0·126 507	−0·131 445	0·32
2^{-4}	−0·057 637	−0·063 532	1·51
2^{-5}	−0·027 118	−0·028 893	1·82
2^{-6}	−0·013 114	−0·013 578	1·90
2^{-7}	−0·006 442	−0·006 561	1·93

Table 4. Results for Example 4

h	$e_h(1)$	$e_h^*(1)$	E/h^2
2^{-2}	−0·055 766	−0·030 374	−0·41
2^{-3}	−0·030 568	−0 023 474	−0·45
2^{-4}	−0·016 092	−0·014 165	−0·49
2^{-5}	−0·008 262	−0·007 763	−0·52
2^{-6}	−0·004 180	−0·004 056	−0·51
2^{-7}	−0·002 088	−0·002 064	−0·40

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Souhrn

ODHAD GLOBÁLNÍ CHYBY NUMERICKÉHO ŘEŠENÍ ZPOŽDĚNÍ DIFERENCIÁLNÍ ROVNICE EULEROVOU METODOU

ZDZISLAW JACKIEWICZ

V článku je použita metoda Zadunaiského k odhadu globální chyby vzniklé při numerickém řešení soustavy zpožděných diferenciálních rovnic Eulerovou metodou. Je uvedeno několik numerických příkladů.

Author's address: Prof. *Zdzislaw Jackiewicz*, Department of Mathematics, University of Arkansas SE 301, Fayetteville, AR 72701, USA.