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Some fast finite-difference solvers for two-dimensional evolutionary equations on special domains

SOME FAST FINITE-DIFFERENCE SOLVERS FOR TWO-DIMENSIONAL EVOLUTIONARY EQUATIONS ON SPECIAL DOMAINS

Ta Van Dinh

(Received April 20, 1979)

Our aim is to prove the existence of asymptotic error expansions to the Peaceman-Rachford finite-difference scheme for the first boundary value problem of the two-dimensional evolutionary equation on special domains. These expansions lead, by Richardson extrapolation, to fast finite-difference solvers for the problems mentioned [1].

1. UNIFORM AND NEARLY UNIFORM DOMAINS

Let D be a bounded domain in the (x, y)-plane with a boundary G. For some real numbers x_0, y_0 let us consider a uniform grid over the (x, y)-plane:

$$(1) \qquad (x_i, y_j), \quad x_i = x_0 + ih, \quad h = \text{const} > 0,$$
$$y_j = y_0 + jk, \quad k = \text{const} > 0,$$
$$0 < \text{const} < h/k < \text{const}.$$

The domain D will be called uniform if there exist two values x_0, y_0 and two sequences of positive numbers $\{h\}$ and $\{k\}$ tending simultaneously to zero such that the grid lines $x = x_i$ and $y = y_j$ cut the boundary G only at points of the form (x_m, y_n). Then the points (1) cover D with a uniform grid which consists of the set D_h of interior grid points which belong to the interior of D and the set G_h of boundary grid points lying just on G. The domain D will be called nearly uniform if there exist four real numbers a, b, c, d, a sequence of positive numbers $\{h\}$ tending to zero and two strictly increasing and smooth functions $x(t)$ $(a \leq t \leq c)$, $y(t)$ $(b \leq t \leq d)$ such that D lies in the rectangle $x(a) \leq x \leq x(c)$, $y(b) \leq y \leq y(d)$ and the lines $x = x_i = x(a + ih)$ and $y \cdot = y_j = y(b + jh)$, i, j integers, cut the boundary G only at the points of the form (x_m, y_n). So we can cover D with a grid (x_i, y_j), $x_i =$

$= x(a + ih)$, $y_j = y(b + jh)$, $i, j = 0, 1, 2, 3, \ldots$, which consists of the set D_h of interior grid points which belong to the interior of D and the set G_h of boundary grid points lying just on G. This grid is not uniform but depends uniformly on one parameter h.

2. THE DIFFERENTIAL PROBLEM

On a uniform domain D consider the differential problem

$$(2) \qquad \frac{\partial u}{\partial t} - Lu = f(x, y, t), \quad (x, y) \in D, \quad t \in (0, 1],$$

$$u(x, y, t) = g(x, y, t), \quad (x, y) \in G, \quad t \in (0, 1],$$

$$u(x, y, 0) = g^0(x, y), \quad (x, y) \in D,$$

where

$$Lu = \frac{\partial}{\partial x}\left(p(x, y, t)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x, y, t)\frac{\partial u}{\partial y}\right) - c(x, y, t) u,$$

p, q, c, f, g, g^0 being given smooth enough functions with $p \geqq p_0 = \text{const} > 0$, $q \geqq q_0 = \text{const} > 0$, $c \geqq 0$.

3. THE DISCRETE PROBLEM

We cover D with a uniform grid as described above. Furthermore we consider the points

$$t_m = ml, \quad l = 1/M, \quad M \text{ a positive inger}, \quad m = 0, \ldots, M.$$

For each discrete function v defined on $D_h \times \{t_m\}$ we use the notation

$$v_{ij}^m = v(x_i, y_j, t_m), \quad v^m = v(\cdot, \cdot, t_m),$$

and the discrete operators

$$v_{\bar{t}ij}^m = \frac{v_{ij}^m - v_{ij}^{m-1}}{l},$$

$$L_{1h}^m v_{ij}^m = [p(x_i + 0{\cdot}5h, y_j, t_m)(v_{i+1j}^m - v_{ij}^m) -$$
$$- p(x_i - 0{\cdot}5h, y_j, t_m)(v_{ij}^m - v_{i-1j}^m)]/h^2 - \tfrac{1}{2}c(x_i, y_j, t_m) v_{ij}^m;$$

$$L_{2h}^m v_{ij}^m = [q(x_i, y_j + 0{\cdot}5k, t_m)(v_{ij+1}^m - v_{ij}^m) -$$
$$- q(x_i, y_j - 0{\cdot}5k, t_m)(v_{ij}^m - v_{ij-1}^m)]/k^2 - \tfrac{1}{2}c(x_i, y_j, t_m) v_{ij}^m.$$

The discrete problem is of the Peaceman-Rachford type:

$$(3) \qquad v_{\bar{t}ij}^{2n+1} - L_{1h}^{2n+1}v_{ij}^{2n+1} - L_{2h}^{2n}v_{ij}^{2n} = f(x_i, y_j, t_{2n+1}),$$

$$v_{\bar{t}ij}^{2n+2} - L_{1h}^{2n+1} v_{ij}^{2n+1} - L_{2h}^{2n+2} v_{ij}^{2n+2} = f(x_i, y_j, t_{2n+1}), \quad (i,j) \in D_h,$$

$$v_{ij}^m = g(x_i, y_j, t_m), \quad (i,j) \in G_h,$$

$$v_{ij}^0 = g^0(x_i, y_j), \quad (i,j) \in D_h;$$

here and in what follows we write (i, j) instead of (x_i, y_j).

4. MAIN RESULT

Theorem 1. *Assume that*

1) *the problem* (2) *has a unique solution*

$$u(x, y, t) \in C^7(\bar{D} \times [0, 1]),$$

2) *each problem*

$$\frac{\partial w_i}{\partial t} - L w_i = F_i(x, y, t), \quad (x, y) \in D, \quad t \in (0, 1],$$

$$w_i(x, y, t) = 0, \quad (x, y) \in G, \quad t \in (0, 1],$$

$$w_i(x, y, 0) = 0, \quad (x, y) \in D,$$

$$F_1 = \frac{1}{24}\left[\frac{\partial}{\partial x}\left(p\frac{\partial^3 u}{\partial x^3}\right) + \frac{\partial^2}{\partial x^2} L_1 u\right], \quad L_1 u = \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right),$$

$$F_2 = \frac{1}{24}\left[\frac{\partial}{\partial y}\left(q\frac{\partial^3 u}{\partial y^3}\right) + \frac{\partial^2}{\partial y^2} L_2 u\right], \quad L_2 u = \frac{\partial}{\partial y}\left(q\frac{\partial u}{\partial y}\right),$$

$$F_3 = -u'''/6 + (L_1 + L_2) u''/2 - L_1 L_2 u' - (cu)''/2,$$

$$u' = \frac{\partial u}{\partial t},$$

has a unique solution and $w_i \in C^5(\bar{D} \times [0, 1])$,

3) $0 < \text{const} < 1/h < \text{const}, \ 0 < \text{const} < 1/k < \text{const}.$

Then the discrete problem has a unique solution v_{ij}^m *and we have*

$$v_{ij}^{2n} - u(x_i, y_j, t_{2n}) = h^2 w_1(x_i, y_j, t_{2n}) + k^2 w_2(x_i, y_j, t_{2n}) + l^2 w_3(x_i, y_j, t_{2n}) +$$
$$+ O(h^4 + k^4 + l^4).$$

Proof. We put

$$z = v - u - h^2 w_1 - k^2 w_2 - l^2 w_3.$$

Then we have

(4) $\qquad z_{\bar{t}ij}^{2n+1} - L_{1h}^{2n+1} z_{ij}^{2n+1} - L_{2h}^{2n} z_{ij}^{2n} = \varphi_{ij}^{2n+1}, \quad (i,j) \in D_h,$

$\qquad z_{\bar{t}ij}^{2n+2} - L_{1h}^{2n+1} z_{ij}^{2n+1} - L_{2h}^{2n+2} z_{ij}^{2n+2} = \psi_{ij}^{2n+1}, \quad (i,j) \in D_h,$

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$$z_{ij}^m = 0 , \quad (i,j) \in G_h ,$$
$$z_{ij}^0 = 0 , \quad (i,j) \in D_h ,$$

where, by Taylor's formula, we have

$$(5) \quad \Phi_{ij}^{2n+1} \equiv \varphi_{ij}^{2n+1} + \psi_{ij}^{2n+1} - l^2 L_{1h}^{2n+1} \frac{\psi_{ij}^{2n+1} - \varphi_{ij}^{2n+1}}{l} = O(h^4 + k^4 + l^4) .$$

We consider the finite dimensional space H of discrete functions defined on $D_h \cup G_h$ and equal to zero on G_h. We introduce a scalar product and a norm in H:

$$(v, w) = \sum_{(i,j)\in D_h} v_{ij} w_{ij} hk , \quad \|v\| = ((v, v))^{1/2} .$$

We define two linear operators A_s^m $(s = 1, 2)$ by putting

$$(A_s^m v^m)_{ij} = - L_{sh}^m v_{ij}^m , \quad v^m \in H .$$

It is clear that $z^m \in H$ and the discrete problem (4) can be rewritten as

$$(z^{2n+1} - z^{2n})/l + A_1^{2n+1} z^{2n+1} + A_2^{2n} z^{2n} = \varphi^{2n+1} ,$$
$$(z^{2n+2} - z^{2n+1})/l + A_1^{2n+1} z^{2n+1} + A_2^{2n+2} z^{2n+2} = \psi^{2n+1} ,$$
$$z^0 = 0 .$$

Hence we have

$$(6) \qquad (E + lA_1^{2n+1}) z^{2n+1} = (E - lA_2^{2n}) z^{2n} + l\varphi^{2n+1} ,$$
$$(E + lA_2^{2n+2}) z^{2n+2} = (E - lA_1^{2n+1}) z^{2n+1} + l\psi^{2n+1} ,$$

E being the unit operator in H. It is clear that A_1^m and A_2^m are positive definite so that $(E + lA_s^m)^{-1}$ and $T_s^m = (E - lA_s^m)(E + lA_s^m)^{-1}$ always exist (for $s = 1, 2$) and

$$(7) \qquad \qquad \|(E + lA_s^m)^{-1}\| \leqq 1 , \quad \|T_s^m\| \leqq 1 .$$

Now we deduce from (6)

$$(E + lA_2^{2n+2}) z^{2n+2} = (E - lA_1^{2n+1})(E + lA_1^{2n+1})^{-1}(E - lA_2^{2n}) z^{2n} +$$
$$+ l\psi^{2n+1} + (E - lA_1^{2m+1})(E + lA_1^{2n+1})^{-1} l\varphi^{2n+1} .$$

Then we can write

$$(E + lA_2^{2n+2}) z^{2n+2} = T_1^{2n+1} T_2^{2n}(E + lA_2^{2n}) z^{2n} +$$
$$+ l(E + lA_1^{2n+1})^{-1} \left[(E + lA_1^{2n+1}) \psi^{2n+1} + (E - lA_1^{2n+1}) \varphi^{2n+1} \right] .$$

Thus we have

$$(8) \qquad (E + lA_2^{2n+2}) z^{2n+2} = T_1^{2n+1} T_2^{2n}(E + lA_2^{2n}) z^{2n} + (E + lA_1^{2n+1})^{-1} l\Phi^{2n+1} ,$$

where Φ^m have been determined by (5). We define a new norm

$$\||z^m\||_{(m)} = \|(E + lA_2^m) z^m\| .$$

Hence, from (8) and (7) we have

$$\left\||z^{2n+2}|\right\|_{(2n+2)} \leqq \left\||z^{2n}|\right\|_{(2n)} + l\left\|\Phi^{2n+1}\right\| .$$

We deduce

$$\left\||z^{2n}|\right\|_{(2n)} \leqq \left\||z^{0}|\right\|_{(0)} + \sum_{r=0}^{n-1} l\left\|\Phi^{2r+1}\right\| .$$

Then the theorem follows from $z^0 = 0$ and the estimate (5).

5. NOTE ABOUT THE NEARLY UNIFORM DOMAINS

The previous result is still available when D is nearly uniform domain. Considering the grid depending uniformly on one parameter h described in Section 1, we put

$$h_i = x_i - x_{i-1} , \quad k_j = y_j - y_{j-1} ,$$

$$L_{1h}^m v_{ij}^m = \left(2/(h_i + h_{i+1})\right)\left[p(x_i + 0.5h_{i+1}, y_j, t_m)(v_{i+1j}^m - v_{ij}^m)/h_{i+1} - \right.$$
$$\left. - p(x_i - 0.5h_i, y_j, t_m)(v_{ij}^m - v_{i-1j}^m)/h_i\right] - \tfrac{1}{2}c(x_i, y_j, t_m) v_{ij}^m ,$$

$$L_{2h}^m v_{ij}^m = \left(2/(k_j + k_{j+1})\right)\left[q(x_i, y_j + 0.5k_{i+1}, t_m)(v_{ij+1}^m - v_{ij}^m)/k_{j+1} - \right.$$
$$\left. - q(x_i, y_j - 0.5k_j, t_m)(v_{ij}^m - v_{ij-1}^m)/k_j\right] - \tfrac{1}{2}c(x_i, y_j, t_m) v_{ij}^m .$$

Then the discrete problem takes the form (3) with the operators L_{sh}^m just introduced. The asymptotic error expansion takes the form

$$v_{ij}^{2n} - u(x_i, y_j, t_{2n}) = h^2 w_1(x_i, y_j, t_{2n}) + l^2 w_2(x_i, y_j, t_{2n}) + O(h^4 + l^4) ,$$

if in addition to the assumptions Theorem 1 we assume that

$$x(t) \in C^4([a, c]) , \quad y(t) \in C^4([b, d]) .$$

6. A NUMERICAL EXAMPLE

Let D be the square $0 < x < 1, 0 < y < 1$. We consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = (1 + 2\pi^2 t)\sin \pi x \cdot \sin \pi y , \quad (x, y) \in D ,$$

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0 ,$$

$$u(x, y, 0) = 0 , \quad (x, y) \in D .$$

Its solution is $u(x, y, t) = t \cdot \sin \pi x \cdot \sin \pi y$. In the square D we consider the uniform grid

$$D_h = \{(x_i, y_j)\} , \quad x_i = ih , \quad y_j = jh , \quad h = 1/N ,$$

N being an integer > 0, $i, j = 0, \ldots, N$, and the points

$$t_m = ml, \quad l = 1/M, \quad M \text{ being an integer } > 0, \quad m = 0, \ldots, M.$$

The discrete problem is that described in Section 3. We assume that $1/h = \text{const} > 0$ and denote the approximate value of $u(x_P, y_P, 1)$ calculated on this grid at a grid point $P \in D_h$ and at the moment $t = 1$ by $v(P; h)$. Theorem 1 with $1 = \text{const} \cdot h$ yields

$$v(P; h; h/2) \equiv \tfrac{4}{3}v(P; h/2) - \tfrac{1}{3}v(P; h) = u(x_P, y_P, 1) + O(h^4)$$

at each point P common to two grids with grid spacings h and $h/2$. The numerical results in the case $l/h = \tfrac{1}{2}$ at the point $P(\tfrac{1}{2}, \tfrac{1}{2})$ for $t = 1$ are presented in Table 1.

Table 1

h/l	$1/h$	$v(P; h)$	$v(P; h; h/2)$
2	2	1·1419679	0·9939090
2	4	1·0309237	0·9996481
2	8	1·0074670	0·9999785
2	16	1·0018506	

Reference

[1] *O. B. Widlund:* Some recent applications of asymptotic error expansions to finite difference schemes. Proc. Roy. Soc. London, A 323, N. 1553, 1971, 167—177.

Souhrn

RYCHLÉ ŘEŠENÍ DVOUDIMENZIONÁLNÍ EVOLUČNÍ ROVNICE NA SPECIÁLNÍ OBLASTI METODOU KONEČNÝCH DIFERENCÍ

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Autor dokazuje existenci mnohoparametrického asymptotického rozvoje pro chybu obvyklého pětibodového diferenčního schématu pro první okrajovou úlohu pro dvoudimenzionální evoluční rovnici na jistých speciálních (tzv. uniformních) oblastech. Tento rozvoj dává s použitím Richardsonovy extrapolace jednoduchý způsob zrychlení konvergence dané metody. Postup je ilustrován na numerickém příkladě.

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