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Aplikace matematiky, Vol. 27 (1982), No. 2, 146–149

Persistent URL: http://dml.cz/dmlcz/103954

PRODUCTIVITY OF ACTIVITIES IN THE OPTIMAL ALLOCATION OF ONE RESOURCE

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(Received September 29, 1980)

The problem of optimal allocation of one resource is formulated as follows:

$$(1) \qquad \max \{ \sum_{k=1}^{n} f_k(x_k) \mid \sum_{k=1}^{n} x_k = B \,, \; x_k \geqq 0 \; \forall k \} \,,$$

where B is the amount of resource to be allocated and f_k is the return function of the k-th activity ($k = 1, \ldots, n$). Problems of this form arise in marketing, capital budgeting, portfolio-selection problems etc., see [1], [2], [3]. In this paper, we introduce the notion of productivity of activities in the optimal solution of (1), give its characterization in terms of the f_k's and their derivatives and examine three special types of return functions.

Assume that the return functions are defined over $[0, \infty)$ and that for each $B \geqq 0$ the problem (1) has a unique optimal solution $x^*(B) = (x_k^*(B))_{k=1}^n$ (this is e.g. the case of continuously differentiable and strictly concave return functions, as proved below). If $x_k^*(B) > 0$ for some k and B, then the number

$$p_k(B) = \frac{f_k(x_k^*(B))}{x_k^*(B)}$$

can be considered the productivity of the k-th activity at the resource level of B, since its value is equal to the average return corresponding to one allocated unit of resource. In order to be able to compare the productivities of activities independently of B, we introduce the following definition: we say that the i-th activity is more productive than the j-th one if $p_i(B) > p_j(B)$ for any $B > 0$ with $x_i(B) > 0$, $x_j(B) > 0$; and we say that the i-th and the j-th activities are equally productive if $p_i(B) = p_j(B)$ for any such B. Obviously, two activities need not be comparable in the given sense; but under certain assumptions, a simple criterion of comparability can be formulated in terms of return functions f_k and their derivatives f_k' ($k = 1, \ldots, n$).

Theorem. *Let the return functions be continuously differentiable and strictly concave in* $[0, \infty)$ *and let they have a common value of* $\lim_{x_k \to \infty} f_k'(x_k)$ *(finite or infinite). Then, for any i, j we have:*

(i) *the i-th activity is more productive than the j-th one if and only if* $f_i'(x_i) = f_j'(x_j)$ *implies* $f_i(x_i)/x_i > f_j(x_j)/x_j$ *for any positive* x_i, x_j,

(ii) *the i-th and the j-th activities are equally productive if and only if* $f_i'(x_i) = f_j'(x_j)$ *implies* $f_i(x_i)/x_i = f_j(x_j)/x_j$ *for any positive* x_i, x_j.

Proof. The Kuhn-Tucker conditions [4] applied to (1) give that a nonnegative x^* satisfying $\sum_{k=1}^{n} x_k^* = B$ is an optimal solution to (1) if and only if there is a K such that $f_k'(x_k^*) \leqq K$ $(k = 1, \ldots, n)$ and $f_k'(x_k^*) = K$ if $x_k^* > 0$. Hence for each $B \geqq 0$ the problem (1) has a unique optimal solution $x^*(B)$. We shall prove the assertion (i) only because the proof of (ii) is analogous. To prove the "if" part of (i), consider a B with $x_i^*(B) > 0$, $x_j^*(B) > 0$. Then the above conditions give $f_i'(x_i^*(B)) = f_j'(x_j^*(B))$ which along with the assumption implies

$$p_i(B) = \frac{f_i(x_i^*(B))}{x_i^*(B)} > \frac{f_j(x_j^*(B))}{x_j^*(B)} = p_j(B)\,,$$

hence the i-th activity is more productive than the j-th one. To prove the "only if" part of (i), take positive x_i, x_j, $i \neq j$, satisfying $f_i'(x_i) = f_j'(x_j)$. Denote $K = f_i'(x_i)$ and define x_k^* $(k = 1, \ldots, n)$ as follows: if $f_k'(0) > K$, let x_k^* be the solution of the equation $f_k'(x_k) = K$ (which exists uniquely because $f_k'(0) > K > \lim_{x_k \to \infty} f_k'(x_k)$); if $f_k'(0) \leqq K$ put $x_k^* = 0$. Take $B^* = \sum_{k=1}^{n} x_k^*$. Then it can be easily seen that $x^* = (x_k^*)$ satisfies the Kuhn-Tucker conditions for the problem (1) with $B = B^*$ and that $x_i^* = x_i$, $x_j^* = x_j$. Since $p_i(B^*) > p_j(B^*)$ due to the assumption, we obtain $\frac{f_i(x_i)}{x_i} > \frac{f_j(x_j)}{x_j}$, Q.E.D.

We shall apply this result to three types of return functions:

(a) $$f_k(x_k) = s_k \ln(1 + m_k x_k) \quad (k = 1, \ldots, n)\,,$$

(b) $$f_k(x_k) = s_k(1 - e^{-m_k x_k}) \quad (k = 1, \ldots, n)\,,$$

(c) $$f_k(x_k) = s_k x_k - m_k x_k^2 \quad (k = 1, \ldots, n)\,,$$

where the parameters s_k and m_k are always assumed to be positive. The return functions of these types were studied by Luss and Gupta [2]; the return function examined by Charnes and Cooper [1] is a special case of (b).

Corollary. *Let the functions* $f_k(x_k)$ *take on any of the forms* (a)–(c). *Then, for any i, j, we have:*

(i) *if* $f_i'(0) > f_j'(0)$, *then the i-th activity is more productive than the j-th one,*
(ii) *if* $f_i'(0) = f_j'(0)$, *then the i-th and the j-th activities are equally productive.*

Proof. Because of the similarity of proofs, we shall consider the case (a) only. The return functions obviously satisfy the assumptions of Theorem. Let $x_i > 0$, $x_j > 0$, $f_i'(x_i) = f_j'(x_j)$, $i \neq j$ (the case $i = j$ is trivial). Denote $b_i = f_i'(0)$, $b_j = f_j'(0)$, $K = f_i'(x_i)$, so that $0 < K < \min\{b_i, b_j\}$. After expressing x_i and x_j with K, we obtain

$$\frac{f_i(x_i)}{x_i} = \varphi_K(b_i)$$

and

$$\frac{f_j(x_j)}{x_j} = \varphi_K(b_j),$$

where

$$\varphi_K(x) = Kx\,\frac{\ln x - \ln K}{x - K}.$$

Hence if $b_i = b_j$, then $f_i(x_i)/x_i = f_j(x_j)/x_j$, which proves (ii). To prove (i), it suffices to show that $\varphi_K(x)$ is strictly increasing in (K, ∞), since then $b_i > b_j$ will imply $f_i(x_i)/x_i = \varphi_K(b_i) > \varphi_K(b_j) = f_j(x_j)/x_j$ which due to the assertion (i) of Theorem will complete the proof. We have

$$\varphi_K'(x) = \frac{\psi_K(x)}{(x - K)^2},$$

where $\psi_K(x) = K(x - K) - K^2(\ln x - \ln K)$. Since $\psi_K(K) = 0$ and $\psi_K'(x) = K(x - K)/x > 0$ for $x > K$, the function $\psi_K(x)$ is positive in (K, ∞), hence $\varphi_K(x)$ is strictly increasing in (K, ∞), Q.E.D.

This is a little surprising result showing that if all the activities have return functions of one of the types (a)—(c), then they are comparable with one another as to their productivity and the result depends only on the initial values of the derivatives of the return functions.

References

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Souhrn

PRODUKTIVITA ČINNOSTÍ PŘI OPTIMÁLNÍ ALOKACI JEDNOHO ZDROJE

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V článku je zaveden jistý způsob srovnávání produktivit činností v optimálních řešeních problému alokace jednoho zdroje

$$\max \{ \sum_{k=1}^{n} f_k(x_k) \mid \sum_{k=1}^{n} x_k = B,\ x_k \geqq 0\ \forall k \} .$$

Je uvedena nutná a postačující podmínka srovnatelnosti v daném smyslu a pro tři speciální typy funkcí f_k (zkoumané již dříve) je odvozeno jednoduché kritérium srovnatelnosti.

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